Polynomial p-groups and their connections with fusion systems and finite simple groups

Chris Parker

University of Birmingham

AGTA Workshop - Reinhold Baer Prize 2024
October 8th

Motivation

Theorem 1

The finite non-abelian simple groups are members of the following list.

1. *The alternating groups.*
2. *The groups of Lie type (automorphism groups of buildings).*
3. *The sporadic simple groups.*

From here on p denotes a prime number.
Brian, Francesco.

Amazing fact

It is very rare that a p-subgroup can be the Sylow p-subgroup of a finite simple group. There's something special about such p-groups.

Example 2

There are 10494213 groups of order 2^9 and only about 10 of them are Sylow 2-subgroups of a simple groups.

Hence such 2-groups are very rare and interesting groups. The same observation applies to p-groups in general.

Question 3

Is there a way we could know that being a Sylow p-subgroup of a simple group is unusual?

Examples of groups with isomorphic Sylow subgroups

It's very frustrating because many groups have isomorphic Sylow 2-subgroups.

Example 4

Suppose that $q = p^a$ and $r = s^b$ are powers of the primes p and s respectively.
If $(q-1)_2 = (r-1)_2$, then the simple groups $\mathrm{PSL}_2(q)$ and $\mathrm{PSL}_2(r)$ have isomorphic Sylow 2-subgroups.

Example 5

$\mathrm{E}_6(2)$, ${}^2\mathrm{E}_6(7)$, $\mathrm{F}_4(2)$, HN and $\mathrm{PSL}_4(3)$ all have isomorphic Sylow 3-subgroups (order 3^6).

More examples

Its is very well known among finite groups theorists that $\mathrm{SL}_5(2)$, and the sporadic simple groups $\mathrm{Mat}(24)$ and He have isomorphic Sylow 2-subgroups.

Less well known is

Example 6

1. The sporadic simple groups Suz and Ly have isomorphic Sylow 3-subgroups [9310 groups of order 3^7].
2. The sporadic groups Ly, HN, BM have Sylow 5-subgroups isomorphic to that of $\mathrm{G}_2(5)$ [684 groups of order 5^6].
3. The monster group $\mathbb{M}$ and $\mathrm{G}_2(7)$ has isomorphic Sylow 7-subgroups.[860 groups of order 7^6.]

Broadly there are two type of p-groups in finite simple groups

We have two types of p-groups which occur:

- The Sylow p-subgroups of Lie type groups defined in characteristic p; and
- The Sylow r-subgroups of Lie type groups defined in characteristic p where $r \neq p$ is a prime.

Polynomial p-groups

Let $\mathbb{K}$ be a field of characteristic p with group of units $\mathbb{K}^*$. Set

$$A = \mathbb{K}[x, y].$$

Define $D = D(\mathbb{K}) = \mathbb{K}^* \times \mathrm{GL}_2(\mathbb{K})$ and

$$f(x, y) \cdot (\lambda, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \lambda f(ax + by, cx + dy).$$

This makes A into a $\mathbb{K}D$-module. This action preserves the submodules $V_n(\mathbb{K})$ of homogeneous polynomials of degree $n \geq 0$. Notice that $\dim_{\mathbb{K}} V_n(\mathbb{K}) = n + 1$.

Set

$$U(\mathbb{K}) = \{(1, \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)) \in D\}$$

Define

$$P_n(\mathbb{K}) = V_n(\mathbb{K}) \rtimes D$$

and

$$B_n(\mathbb{K}) = V_n(\mathbb{K}) \rtimes \{(\lambda, \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)) \in D \mid \lambda, a, c, d \in \mathbb{K}\}$$

$$S_n(\mathbb{K}) = V_n(\mathbb{K}) \rtimes U(\mathbb{K}).$$

Polynomial p-groups

Now assume that $|\mathbb{K}| = q = p^a$, then $S_n(\mathbb{K}) = S_n(q)$ has order q^{n+2}. We are only interested in the case when $n \leq p$.
In this case $V_n(q)$ is an irreducible $\mathbb{K}D$-module.
The focuss of our attention are the p-groups $S_n(q)$ and we call these groups polynomial p-groups.

Lemma 7

1. *$S_1(q)$ is isomorphic to a Sylow p-subgroup of $\mathrm{SL}_3(q)$.*
2. *$S_2(q)$ is isomorphic to a Sylow p-subgroup of $\mathrm{Sp}_4(q)$.*

For typical $q > p$, and $n \geq 3$, $S_n(q)$ is not a Sylow p-subgroup of a finite simple group.

Properties of polynomial p-groups

Let $\gamma_i(S_n(q))$, and $Z_i(S_n(q))$ denote the terms of the upper and lower central series of $S_n(q)$.

Lemma 8

1. *the terms of the upper and lower central series coincide.*
2. $|\gamma_i(S_n(q))/\gamma_{i+1}(S_n(q))| = q$ *for all* $1 \leq i \leq n-1$ *and* $S_n(q)/\gamma_1(S_n(q)) = q^2$.
3. $Z_i(S_n(q))U(q) \cong S_{i-1}(q)$ *for all* $1 \leq i \leq n$.
4. $S_n(q)/Z_i(S_n(q)) \cong S_{n-i}(q)$.
5. *The automorphism groups of* $S_0(q)$ *and* $S_1(q)$ *contain a subgroup isomorphic to* $\mathrm{SL}_2(q)$ *and for* $j > 2$, *the automorphism groups are soluble.*

What is a fusion system of a group?

Suppose that p is a prime, X is a group and S is a p-subgroup of X. For $P, Q \leq S$, define

$$\mathrm{Hom}_X(P, Q) = \{c_g \mid g \in X, P^g \leq Q\}$$

where c_g is the conjugation map induced by g:

$$c_g : x \mapsto g^{-1}xg.$$

If $P = Q$, then we write

$$\mathrm{Aut}_X(P) = \mathrm{Hom}_X(P, P) \cong N_X(P)/C_X(P).$$

Since we know S, the information about $N_G(P)$ which is lost when we make a fusion system is $C_X(P)$.

What is a fusion system of a group?

A fusion system determined by X and S is the category $\mathcal{F} = \mathcal{F}_S(X)$ with objects all the subgroups of S and, for objects P and Q of $\mathcal{F}$, morphisms from P to Q

$$\mathrm{Mor}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_X(P, Q).$$

For Q a subgroup of S, set

$$\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Hom}_X(Q, Q) \cong N_X(Q)/C_X(Q).$$

If S is a Sylow p-subgroup of X, then the fusion system is saturated.
Abstractly saturated fusion systems on a p-group S capture the properties of the fusion systems just defined with S a Sylow p-subgroup of G.

The Alperin-Goldschmidt theorem

Theorem 9 (Alperin-Goldschmidt Theorem for Fusion Systems)

Suppose that $\mathcal{F}$ is a saturated fusion system on S. Then

$$\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(E) \mid E \text{ is } \mathcal{F}\text{-essential} \rangle.$$

This means that to describe a saturated fusion system we should list the $\mathcal{F}$-automorphism group of S, as well as $\mathrm{Aut}_{\mathcal{F}}(E)$ for a representative of each $\mathcal{F}$-class of essential subgroups.

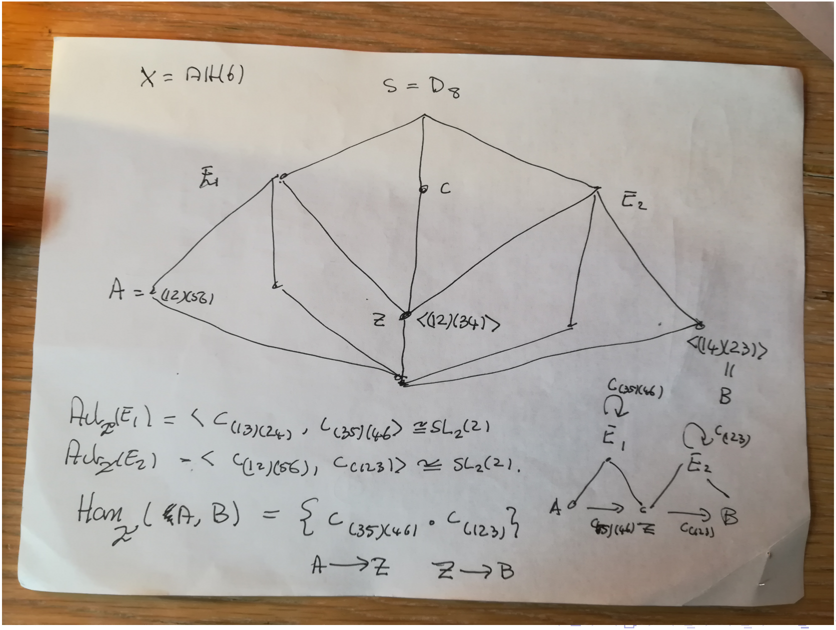
Essential subgroup:

- Fully-normalized (Normalizer in S as big as possible).
- S-centric (all $\mathcal{F}$-conjugates satisfy $C_S(E\phi) = Z(E\phi)$).
- $\mathrm{Aut}_{\mathcal{F}}(E)/\mathrm{Inn}(E)$ has a strongly p-embedded subgroup.

Exotic or realizable

If we have an abstract saturated fusion system $\mathcal{F}$ defined on S, we can ask if there is a finite group G and $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} = \mathcal{F}_S(G)$?
If there is such a group then we say that $\mathcal{F}$ is realized by G. If there's no such group, then $\mathcal{F}$ is exotic.

An Example



X = Alt(6)
S = D8
E1
C
E2
A = ⟨(12)(56)⟩
Z ⟨(12)(34)⟩
⟨(14)(23)⟩
B
Aut_Z(E1) = ⟨c_(13)(24), c_(35)(46)⟩ ≅ SL2(2)
Aut_Z(E2) = ⟨c_(12)(56), c_(123)⟩ ≅ SL2(2).
Hom_Z(A, B) = { c_(35)(46) ∘ c_(123) }
A → Z
Z → B
c_(35)(46)
c_(123)

Some free amalgamated products

Define $R_0(q) = N_{P_n(q)}(S_0(q))$ and $R_1(q) = N_{P_n(q)}(S_1(q))$. Then these groups have Sylow p-subgroups $S_0(q)U \cong S_1(q)$ and $S_1(q)U \cong S_2(q)$.
Then we have $R_0(q)/Z(R_0(q))$ is isomorphic to a Borel subgroup of $\mathrm{GL}_3(q)$ and $R_1(q)/Z(R_1(q))$ is isomorphic to a Borel subgroup of $\mathrm{GSp}_4(q)$.
Choose parabolic subgroups $L_1(q) \leq \mathrm{SL}_3(q)$ (fixing a point) and $L_2(q)$ is $\mathrm{Sp}_4(q)$ (fixing a point) then neglecting quotients we can form the free amalgamated products

$$X(n,q) = P_n(q) *_{R_0(q)} L_1(q)$$

$$Y(n,q) = P_n(q) *_{R_1(q)} L_2(q).$$

These are infinite groups. Still we may form the fusion systems as we do for finite groups.

Some fusion systems

Theorem 10 (Clelland-CWP, 2010)

$\mathcal{F}_{S_n(q)}(X(n,q))$ and $\mathcal{F}_{S_n(q)}(Y(n,q))$ are saturated and exotic. Furthermore, they have no normal p-subgroups.

Draw a picture.Mention weak BN-pairs.

Pruning

The essential subgroups of $\mathcal{F}_{S_n(q)}(X(n,q))$ of are $V_n(q)$ and $S_0(q)$. So they are both elementary abelian. For $\mathcal{F}_{S_n(q)}(Y(n,q))$, $V_n(q)$ is abelian and $S_1(q)$ is a generalization of an extraspecial group.

Lemma 11 (CWP-Semeraro, 2021)

Suppose that $\mathcal{F}$ is a saturated fusion system on S and P is an $\mathcal{F}$-essential subgroup of S. Let $\mathcal{C}$ be a set of $\mathcal{F}$-conjugacy class representatives of $\mathcal{F}$-essential subgroups with $P \in \mathcal{C}$. Assume P has the minimality property:

$$\text{if } Q < P, \text{ then } Q \text{ is not } S\text{-centric.}$$

If $H_{\mathcal{F}}(P) \leq K \leq \mathrm{Aut}_{\mathcal{F}}(P)$, then

$$\mathcal{G} = \langle \mathrm{Aut}_{\mathcal{F}}(S), K, \mathrm{Aut}_{\mathcal{F}}(E) \mid E \in \mathcal{C} \setminus \{P\} \rangle$$

is saturated. Furthermore, P is $\mathcal{G}$-essential if and only if $K > H_{\mathcal{F}}(P)$ and in this case $\mathrm{Aut}_{\mathcal{G}}(P) = K$.

Uniqueness

Theorem 12 (Grazian, CWP, Semeraro, van Beek, 2025)

Suppose that $\mathcal{F}$ is a saturated fusion system on $S_n(q)$, then $\mathcal{F}$ comes from the above construction or n and q are very small.

Discuss proof.
Step 1: what are the essential subgroups. Step 2: what are their automorphism groups.
Show that they fit together in a unique way and that their $\mathcal{F}$-automorphism groups $\mathrm{Aut}_{\mathcal{F}}(E)$ are uniquely determined.

An invitation to fusion systems

Research directions centred on fusion systems:

- Determining fusion systems on a particular family of groups: Max. class (CWP-Grazian), sectional rank 3, Grazian.

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- Determine all the fusion systems on important p-groups (van Beek, Oliver, Raul Moragues Moncho, CWP-Semeraro,Baccanelli-Franchi-Mainardis).

Computation: Fusion systems on groups of order 3^6

Theorem 13 (CWP, Semeraro 2021)

Suppose that $\mathcal{F}$ is a saturated fusion system on a 3-group of order 3^6 with $O_3(\mathcal{F}) = 1$ and $\mathcal{F} = O^3(\mathcal{F})$. Then there are a total of 70 fusion systems up to isomorphism.

group #	*rank*	*# s.f.s*	*type*
95	2	7	$\mathrm{PSL}_3^{\pm}(q)$, $\nu_3(q \pm 1) = 3$
97	2	2	$B(3, 6; 0, 1, 0)$
98	2	2	$B(3, 6; 0, 2, 0)$
99	2	1	$B(3, 6; 1, 0, 0)$
100	2	3	$B(3, 6; 1, 0, 2)$
149	4	2	$\mathrm{G}_2(3)$
307	4	10	$\mathrm{PSL}_4(3)$
321	4	13	$\mathrm{PSU}_4(3)$
453	4	21	$\mathrm{PSL}_3(3) \times \mathrm{PSL}_3(3)$, $3_+^{1+2} \times 3_+^{1+2}$
469	4	5	$\mathrm{PSL}_3(9)$, $(3^2)_+^{1+2}$
479	5	4	$\mathrm{Alt}(15)$, $(3 \wr 3) \times 3^2$

Thanks!