Pronormality in infinite groups

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In loving memory of professor Francesco de Giovanni

Pronormal subgroups

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Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups.

It is also clear that Sylow subgroups of finite groups, Hall subgroups of finite soluble groups are pronormal.

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- if H is ascendant subgroup of G, then H is normal in G.

In particular, H is normal in G if and only if it is pronormal and subnormal in G. So, a group in which every subgroup is pronormal is a group in which normality is transitive (a so called *T-group).*

Finite T-groups

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(Gaschütz, 1957) Let G be a finite soluble T-group and $L = [G', G] = \gamma_3(G)$. Then L is abelian and it is the last term of the lower central series, G/L is a Dedekind group and the orders of L and of G/L are coprime

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(Robinson, 1964) Let G be a non abelian soluble $\bar{T}$-group. Then G is periodic and $L = [G', G]$ has no elements of order 2

Groups with only pronormal subgroups

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On the other hand, the structure of a periodic locally graded group in which every subgroup is pronormal is well-known and in this case the solubility of G is garanteed.

Recall that a group is said to be locally graded if every finitely generated non-trivial subgroup contains a proper subgroup of finite index.

If G is a periodic group, $\pi(G)$ denote the set of all prime numbers p such that G has an element of order p.

Groups with only pronormal subgroups

(N. F. Kuzennyi, I. Ya. Subbotin, 1987) A periodic locally graded group G has only pronormal subgroups if and only if it admits a semidirect decomposition $G = B \ltimes A$ satisfying the following conditions:

(a) B is a Dedekind group and A is an abelian group all of whose subgroups are normal in G;

(b) $\pi(A) \cap \pi(B) = \emptyset$ and $2 \notin \pi(A)$;

(c) $G' = A \times B'$;

(d) every Sylow $\pi(B)$-subgroup of G is a complement of A.

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(F. de Giovanni, G. Vincenzi, 1995) Let G be a non-periodic soluble group. If every infinite subgroup of G is pronormal, then G is abelian.

Groups of infinite rank

A group G is said to have finite rank r if there exists an r such that every finitely generated subgroup of G can be genarated by at most r elements and r is the least positive integer with such a property. Otherwise, G is said to have infinite rank.

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Let G be a soluble non-periodic group in which every subgroup of infinite rank is pronormal. If the largest periodic normal subgroup T of G has infinite rank, then G is abelian.

Countable Subgroups

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Uncountable groups

(F. de Giovanni, M. Trombetti, 2020) Let G be an uncountable locally graded group of cardinality $\aleph$. If all proper subgroups of cardinality $\aleph$ have all subgroups pronormal and G has no simple homomorphic images af cardinality $\aleph$, then every subgroup of G is pronormal.

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In this context, a subgroup H of a group G is called *large* if it has the same cardinality as G and *small* otherwise; the uncountable cardinal $\aleph$ is supposed to be regular

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M. De Falco, F. de Giovanni, C.M.) Let G be a group of cardinality $\aleph$ whose Hirsch-Plotkin radical H is large. If all large subgroups of G are pronormal, then H is nilpotent.

It follows that G is soluble

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(M. De Falco, F. de Giovanni, C.M.) Let G be a non-periodic locally graded group of cardinality $\aleph$ in which all large subgroups are pronormal. If the Hirsch-Plotkin radical H of G is large, then G is nilpotent and all its large subgroups are normal.

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Denote by $\mathcal{C}_{\aleph}(G)$ the set of commutator subgroups of all large subgroups of G and put

$$I_{\aleph}(G) = \bigcap_{X \in \mathcal{C}_{\aleph}(G)} X.$$

The characteristic subgroup $I_{\aleph}(G)$ plays a relevant role in the study of groups whose large subgroups are normal.

Note that in a soluble group G, $I_{\aleph}(G)$ has cardinality less than $\aleph$.

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(a) $I_{\aleph}(G)$ is a p-group for some prime number p and $G/I_{\aleph}(G)$ is a Dedekind group with $\pi(G) = \pi\big(G/I_{\aleph}(G)\big)$;

(b) G contains a subgroup M such that $G' = M'$ and $\pi(M) \subseteq \{2, p\}$;

(c) if $\left|\mathcal{C}_{\aleph}(G)\right| = 1$, then the unique Sylow p'-subgroup $G_{p'}$ of G is abelian;

(d) if $\left|\mathcal{C}_{\aleph}(G)\right| = 2$, then either $I_{\aleph}(G) = G'_p$ or $I_{\aleph}(G) = N'$ for a suitable subgroup N of G of finite index.

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(M. De Falco, F. de Giovanni, C.M.) Let G be a periodic locally graded group of cardinality $\aleph$ whose subgroups cardinality $\aleph$ are pronormal. If the Hirsch-Plotkin radical H of G has cardinality $\aleph$, then all subgroups of $G/I_{\aleph}(H)$ are pronormal.

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Let G be a periodic group of cardinality $\aleph$.
We say that G is a $\mathcal{P}_{\aleph}$-*group* if it admits a semidirect decomposition $G = B \ltimes C$, where B is a large group whose large subgroups are normal (in B), C is a small abelian group whose subgroups are normal in G and the following conditions hold:

- $\pi(B) \cap \pi(C) = \emptyset$ and $2 \notin \pi(C)$;
- $C = \gamma_3(G)$ is the last term of the lower central series of G;
- every Sylow $\pi(B)$-subgroup of G is a complement of C;
- the subgroup $I_{\aleph}(B)$ is normal in G and $G/I_{\aleph}(B)$ is metabelian.

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(M. De Falco, F. de Giovanni, C.M.) Let G be a periodic locally graded group of cardinality $\aleph$ whose Hirsch-Plotkin radical H is large. Then all large subgroups of G are pronormal if and only if the following conditions hold:

(a) all large subgroups of H are normal in H;

(b) all subgroups of $G/I_{\aleph}(H)$ are pronormal;

(c) $X \cap I_{\aleph}(H)$ is normal in G for each large subgroup X of G;

(d) if X is a large subgroup of G and there exists a pronormal subgroup Y of G such that $Y \leqslant X$ and $|X : Y| < \aleph$, then X is pronormal in G;

(e) if U/V is a section of G with a large Hirsch-Plotkin radical and $\gamma_3(U/V)$ is small, then U/V is a $P_{\aleph}$-group.