On a family of groups generated by automata

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UNIVERSITÀ CUSANO

Words on q letters

$X = \{0, 1, \ldots, q-1\}$ q-ary alphabet

$X^n = \{x_1 x_2 \ldots x_n : x_i \in X\}$ q-ary words of length n

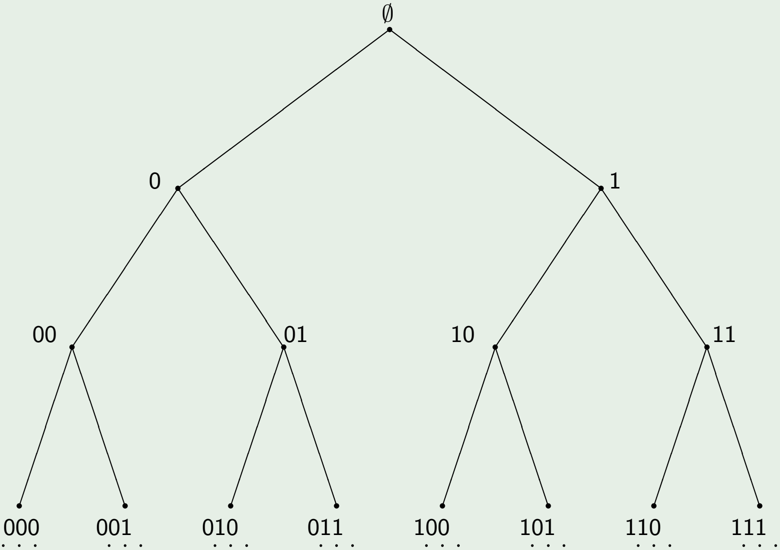
$X^* = \bigcup_{n \geq 0} X^n$, with $X^0 = \{\emptyset\}$

$X^\infty = \{x_1 x_2 \ldots \ldots : x_i \in X\}$ infinite q-ary words

Remark:
X^* can be identified with the vertex set of the *infinite rooted regular tree* T_q of degree q, where:

- empty word $\emptyset \longleftrightarrow$ root of T_q
- $X^n \longleftrightarrow$ vertices of the n-th level L_n of T_q
- $X^\infty \longleftrightarrow \partial T_q =$ boundary of T_q.

The rooted binary tree T_2 ($q = 2$)

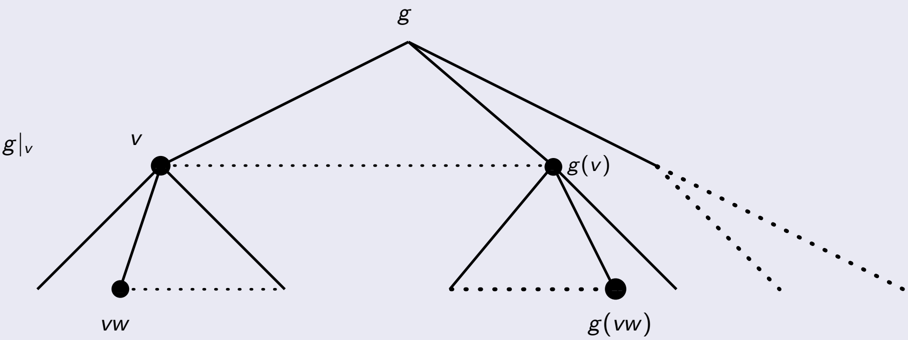


Restrictions of an element of $Aut(T_q)$

$Aut(T_q)$ = group of all automorphisms of T_q. Let $g \in Aut(T_q)$ and $v \in X^*$.

The *restriction* $g|_v : X^* \to X^*$ of g at v is defined by

$$g(vw) = g(v)g|_v(w), \qquad \text{for each } w \in X^*.$$



Self-similar representation of an element of $Aut(T_q)$

$$Aut(T_q) \cong Sym(q) \wr Aut(T_q) \cong \underbrace{(Aut(T_q) \times \cdots \times Aut(T_q))}_{q \text{ times}} \rtimes Sym(q),$$

where $Sym(q)$ is the symmetric group on q elements.

$\Rightarrow$ *self-similar representation* of $g \in Aut(T_q)$:

$$g = (g|_0, g|_1, \ldots, g|_{q-1})\pi_g,$$

where:
π_g = permutation induced by g on the first level L_1 of T_q
$g|_i$ = restriction of g at the vertex i of L_1, for $i = 0, 1, \ldots, q-1$.

Self-similar actions

The action of a group $G \leq Aut(T_q)$ is *self-similar* if $\forall g \in G$ and $x \in \{0, 1, \dots, q-1\}$ there exist $h \in G$ and $y \in \{0, 1, \dots, q-1\}$ such that

$$g(xw) = yh(w) \qquad \forall w \in \{0, 1, \dots, q-1\}^*. \tag{1}$$

By iterating, we get that $\forall g \in G$ and $v \in \{0, 1, \dots, q-1\}^*$ there exist $h \in G$ and $u \in \{0, 1, \dots, q-1\}^*$, with $|u| = |v|$, such that

$$g(vw) = uh(w) \qquad \forall w \in \{0, 1, \dots, q-1\}^*.$$

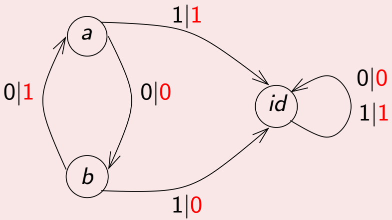
Eq. (1) can be interpreted as the work of a machine, which being in a **state** g and receiving as **input** letter x, goes into **state** h and returns the **output** letter y

$\Rightarrow$ *Automaton groups*

An automaton $\mathcal{A}$ over $X = \{0, 1, \ldots, q-1\}$ is a quadruple (S, X, λ, μ), where:

- S is a set (*states* of $\mathcal{A}$)
- X is an *alphabet*
- $\lambda : S \times X \to S$ *transition function*
- $\mu : S \times X \to X$ *output function*.

Example ($q = 2$)



$X = \{0, 1\}$ $\quad \mu(a, 0) = 0 \quad \lambda(a, 0) = b \quad \mu(b, 1) = 0 \quad \lambda(b, 1) = id$

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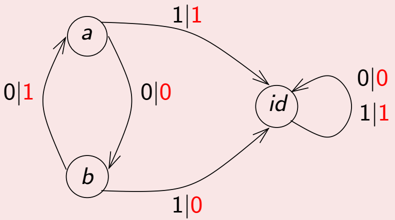

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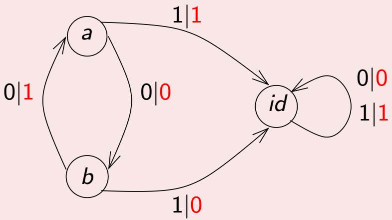

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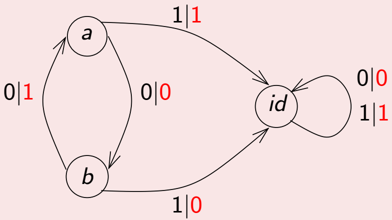

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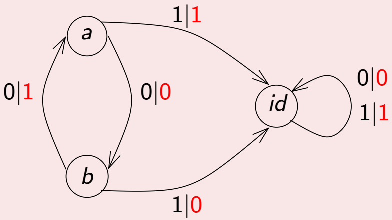

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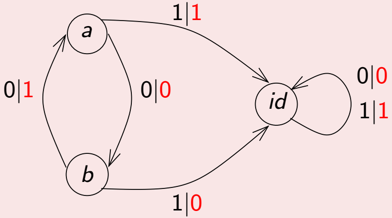

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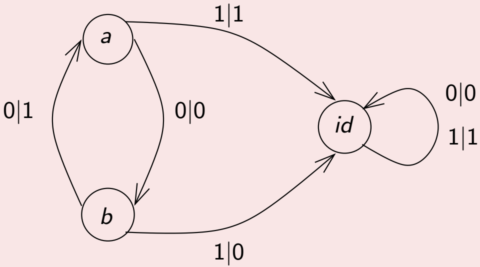
The Basilica group

The *Basilica group* is the group of automorphisms of T_2 generated by the automaton



Self-similar representation of the generators:

$$a = (b, id) \qquad b = (a, id)\varepsilon,$$

with $\varepsilon = (0\ 1) =$ nontrivial permutation of $Sym(2)$.

[R. Grigorchuk, A. Żuk, On a torsion-free weakly branch group defined by a three state automaton, *Int. J. Algebra Comput.* **12** (2002), 223–246]

Branch properties

Let $G \leq Aut(T_q)$ be an automaton group transitive on each level of T_q.

- $Stab_G(v) = \{g \in G : g(v) = v\}$ *vertex stabilizer*;
- $Stab_G(L_n) = \bigcap_{v \in X^n} Stab_G(v)$ *level stabilizer.*

Let $\phi : Stab_G(L_1) \to G^q$ such that:

$$g = (g|_0, g|_1, \ldots, g|_{q-1})id \longmapsto (g|_0, g|_1, \ldots, g|_{q-1}).$$

Then:

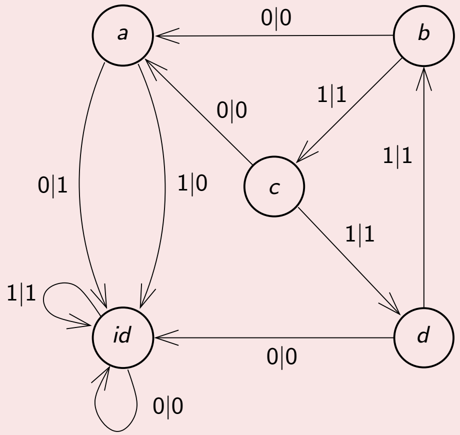
◇ G is *regular weakly branch* over K if there exists a normal subgroup $K \neq \{1\}$ in G, with $K \leq Stab_G(L_1)$, such that

$$\phi(K) > \underbrace{K \times K \times \cdots \times K}_{q \text{ times}}.$$

◇ G is *regular branch* over K if it is regular weakly branch over K and

$$[G : K] < \infty.$$

The automaton generating the Grigorchuk group $\mathcal{G}$



$a = (id, id)(01)$

$b = (a, c)$

$c = (a, d)$

$d = (id, b)$

[R. Grigorchuk. On Burnside's problem on periodic groups, *Funktsional. Anal. i Prilozhen.* **14** (1980), no. 1, 53–54.]

Some remarkable properties of the Grigorchuk group $\mathcal{G}$

- $\mathcal{G}$ is a finitely generated, infinite, torsion group (*Burnside Problem*)
- $\mathcal{G}$ is not finitely presented, but it admits the recursive presentation of L-type:

$$\mathcal{G} = \langle a, b, c, d | a^2, b^2, c^2, d^2, bcd, \sigma^i(ad)^4, \sigma^i(adacac)^4, i \geq 0 \rangle$$

where σ is the substitution on $\{a, b, c, d\}^*$ defined by

$$\sigma(a) = aca, \ \sigma(b) = d, \ \sigma(c) = b, \ \sigma(d) = c.$$

(Lysenok 1985)

- $\mathcal{G}$ is just-infinite
- $\mathcal{G}$ has solvable word problem and solvable conjugacy problem
- $\mathcal{G}$ was the first example of a group of intermediate growth (*Milnor Problem*)

$$e^{n^\alpha} \leq \gamma(n) \leq e^{n^\beta}, \qquad 0 < \alpha, \beta < 1.$$

Grigorchuk (1984) : $\alpha = 0.5$; $\beta = \log_{32} 31 \approx 0.991$

Leonov (2001) : $\alpha \approx 0.504$; *Bartholdi* (2001) : $\beta \approx 0.7675$

Some remarkable properties of the Grigorchuk group $\mathcal{G}$

- $\mathcal{G}$ is *residually finite*:
  the approximating sequence of finite quotient groups is $\{\mathcal{G}_n = \mathcal{G}/Stab_{\mathcal{G}}(L_n)\}$, with

$$\mathcal{G}_1 \cong C_2; \quad \mathcal{G}_2 \cong C_2 \wr C_2; \quad \mathcal{G}_3 \cong C_2 \wr C_2 \wr C_2; \qquad |\mathcal{G}_n| = 2^{5\cdot 2^{n-3}+2}, n \geq 3.$$

- $\mathcal{G}$ is *regular branch* over its normal subgroup $K = \langle (ab)^2 \rangle^{\mathcal{G}}$:

$$[\mathcal{G} : K] = 16 \qquad \phi(K) \geq K \times K$$

  Put $K_n = \underbrace{K \times \cdots \times K}_{2^n \text{ times}}$, where each factor acts on a subtree rooted at L_n

  $\Rightarrow \{K_n\}_{n\geq 1}$ is a descending sequence of normal subgroups of finite index in $\mathcal{G}$ with trivial intersection

  $\Rightarrow$ *branching subgroup structure*

- $\mathcal{G}$ is *amenable* but not elementary amenable (*Day Problem*).

Example: the Basilica Group as Iterated Monodromy Group

Consider the map $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ defined as

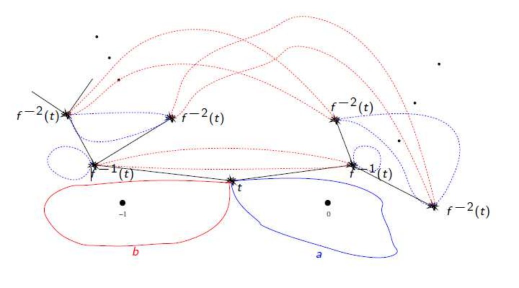
$$f(z) = z^2 - 1$$

Critical set: $C_f = \{0, \infty\}$ *Post-critical set:* $P_f = \bigcup f^n(C_f) = \{-1, 0, \infty\}$

Look at the restriction

$$f : \hat{\mathbb{C}} \setminus f^{-1}(P_f) \longrightarrow \hat{\mathbb{C}} \setminus P_f$$

If $t \in \hat{\mathbb{C}} \setminus P_f$, the set $\bigcup f^{-n}(t)$ can be identified with the tree T_2.



The action of $\pi_1(\mathbb{C} \setminus \{-1, 0\})$ on such a tree coincides with the action of the Basilica group on $T_2 \quad \Longrightarrow \quad IMG(z^2 - 1)$

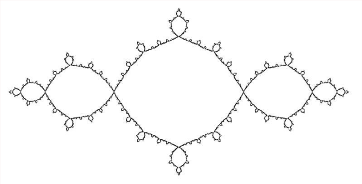
Let $f : \mathbb{C} \to \mathbb{C}$ be a rational map.

The *Julia set* $J(f)$ of f is the set of limit points of the full backwards orbit $\bigcup_n f^{-n}(z)$.

It often has a *fractal* structure!

Example: The Julia set of the map $f(z) = z^2 - 1$

The Julia set of $f(z) = z^2 - 1$ is the so-called *Basilica* fractal!



This explains the reason for the name of the Basilica group $IMG(z^2 - 1)$!

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Graph Automaton Groups *

Matteo Cavaleri — Daniele D'Angeli — Alfredo Donno
Emanuele Rodaro

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Graph automaton groups

Let $\Gamma = (V, E)$ be a *finite graph*, with $V = \{x_1, \ldots, x_q\}$.
Let E' be the set of edges, where an orientation of each edge has been chosen, so that an element of E' is an ordered pair of type (x_i, x_j).
Define an automaton $\mathcal{A}_\Gamma = (E' \cup \{id\}, V, \lambda, \mu)$ s.t.:

- $E' \cup \{id\}$ is the set of states;
- V is the alphabet;
- $\lambda : E' \times V \to E'$ is such that, for each $e = (x, y) \in E'$:

$$\lambda(e,z) = \begin{cases} e & \text{if } z = x \\ id & \text{if } z \neq x; \end{cases} \qquad \text{(transition function)}$$

- $\mu : E' \times V \to V$ is such that, for each $e = (x, y) \in E'$:

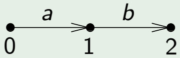
$$\mu(e,z) = \begin{cases} y & \text{if } z = x \\ x & \text{if } z = y \\ z & \text{if } z \neq x, y. \end{cases} \qquad \text{(output function)}$$

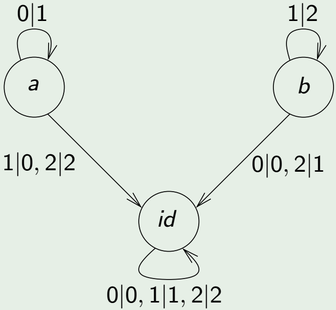
In words: the state $e = (x, y)$ has one transition to itself (given by $\lambda(e, x) = e$) and all other transitions to the sink id.
It acts nontrivially only on x and y, which are switched as $\mu(e, x) = y$ and $\mu(e, y) = x$.

$\Rightarrow$ The *graph automaton group* $\mathcal{G}_\Gamma$ is the automaton group generated by $\mathcal{A}_\Gamma$.

Example: the *Tangled odometer*

The path graph P_3 on 3 vertices and the associated automaton $\mathcal{A}_{P_3}$.





The group $\mathcal{G}_{P_3}$ is the so-called *Tangled odometer*.

Its generators have the self-similar representation:

$$a = (a, id, id)(0\ 1) \qquad b = (id, b, id)(1\ 2).$$

Some basic properties of graph automaton groups

- Any loop in Γ gives rise to the trivial element of $\mathcal{G}_\Gamma$.
- Any multiedge produces a set of equal generators (up to consider the inverse)

  $\Longrightarrow$ The graph $\Gamma = (V, E)$ can be supposed to be simple.

- The group $\mathcal{G}_\Gamma$ does not depend on the choice of the edge orientation, since changing the orientation of an edge corresponds to invert a generator.
- If $e, f \in E$ do not share any vertex, then

$$[e, f] = e^{-1}f^{-1}ef = id,$$

  since their action are nontrivial on disjoint subsets of V.

- A rearrangement of the vertex set of $\Gamma = (V, E)$ produces groups that are isomorphic.
- If $\widetilde{\Gamma} = (\widetilde{V}, \widetilde{E})$ is a graph isomorphic to a subgraph of $\Gamma = (V, E)$, then

$$\mathcal{G}_{\widetilde{\Gamma}} \leq \mathcal{G}_\Gamma$$

- If $\Gamma = (V, E)$ is the disjoint union of the graphs $\Gamma_1 = (V_1, E_1), \ldots, \Gamma_t = (V_t, E_t)$, then

$$\mathcal{G}_\Gamma = \mathcal{G}_{\Gamma_1} \times \cdots \times \mathcal{G}_{\Gamma_t}$$

  $\Longrightarrow$ The graph $\Gamma = (V, E)$ can be supposed to be connected.

Theorem [Cavaleri, D'Angeli, Donno, Rodaro, *Adv. Group Theory Appl.* (2021)]

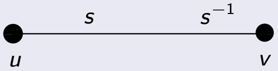
Let $\Gamma = (V, E)$ be a graph, with $|E| \geq 2$, and let $\mathcal{G}_\Gamma$ be the associated graph automaton group. Then:

1. $\mathcal{G}_\Gamma$ is weakly regular branch over its commutator subgroup;
2. $\mathcal{G}_\Gamma$ contains an element of finite order;
3. $\mathcal{G}_\Gamma$ has trivial center;
4. $\mathcal{G}_\Gamma$ is amenable;
5. If $|V| \geq 5$, then $\mathcal{G}_\Gamma$ is not solvable;
6. If the graph Γ contains a cycle $e_1, \ldots, e_t$, then $(e_1^{\varepsilon_1} \cdots e_t^{\varepsilon_t})^{t-1}$, with $\varepsilon_i \in \{\pm 1\}$, is a relation in $\mathcal{G}_\Gamma$ whenever $e_1^{\varepsilon_1}, \ldots, e_t^{\varepsilon_t}$ is an oriented cycle in Γ;
7. If e, f are two edges sharing a vertex in the graph Γ, then the semigroup generated by e and f is free
   $\Rightarrow$ $\mathcal{G}_\Gamma$ has exponential growth.

Finite Schreier graphs

Given an automaton group $\mathcal{G}$ acting on the rooted tree T_q, the n-th *Schreier graph* Γ_n of $\mathcal{G}$ w.r.t. the symmetric generating set S is defined as:

1. $V(\Gamma_n) = \{0, 1, \ldots, q-1\}^n$
2. $u \sim v$ in Γ_n if $\exists\, s \in S$ such that $s(u) = v$. In this case, we have



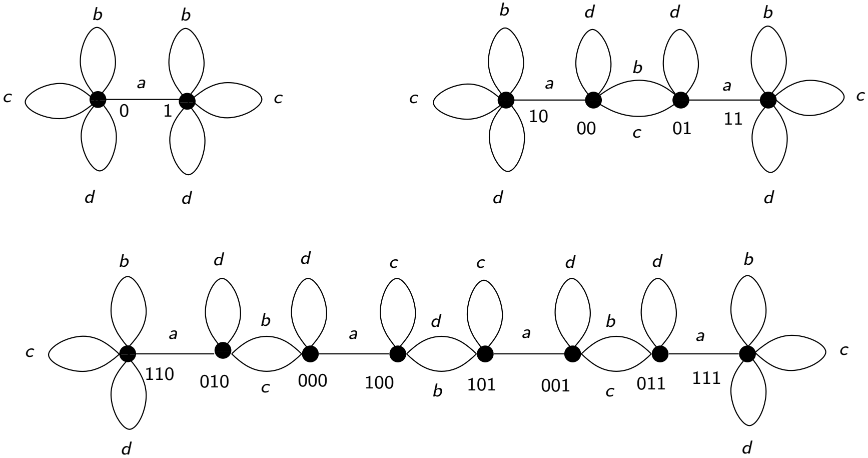
Remarks:

- Γ_n is regular of degree $|S|$ on q^n vertices
- Γ_n is connected if the action of $\mathcal{G}$ on L_n is transitive
- The map

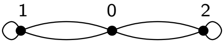
$$\begin{aligned} \pi_{n+1} : \Gamma_{n+1} &\longrightarrow \Gamma_n \\ x_1 \ldots x_n x_{n+1} &\longmapsto x_1 \ldots x_n \end{aligned} \qquad \forall\, x_1 \ldots x_{n+1} \in \{0, 1, \ldots, q-1\}^{n+1}$$

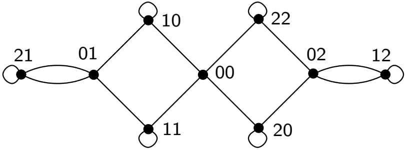
  induces a graph covering of order q.

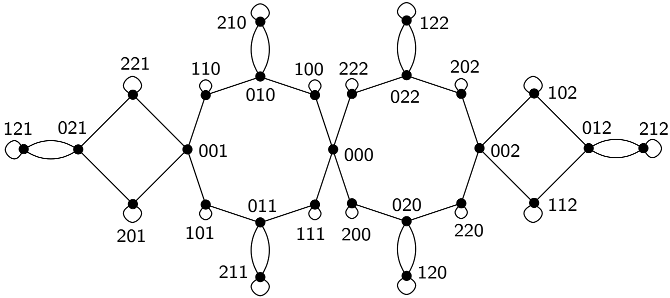
The Schreier graph Γ_n of the action of the Grigorchuk group on T_2, for $n = 1, 2, 3$.



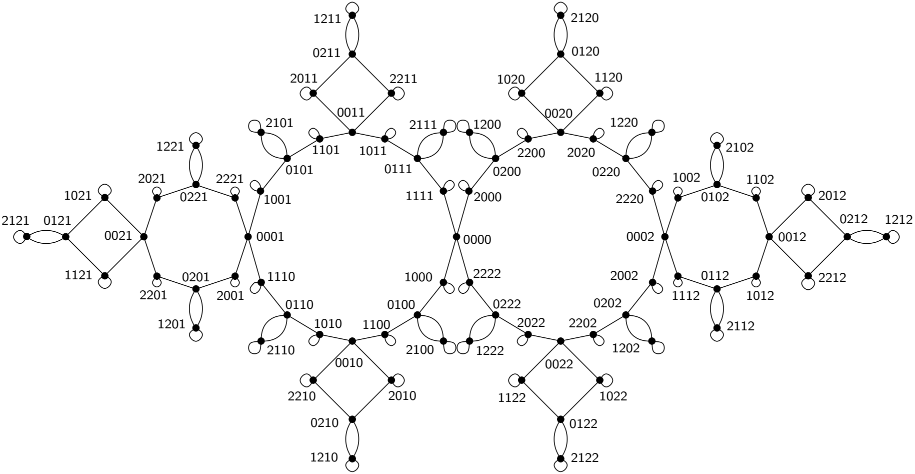
The Schreier graph Γ_n of the action of the graph automaton group $\mathcal{G}_{P_3}$ on T_3, for $n = 1, 2, 3$.



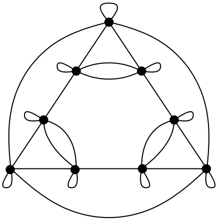


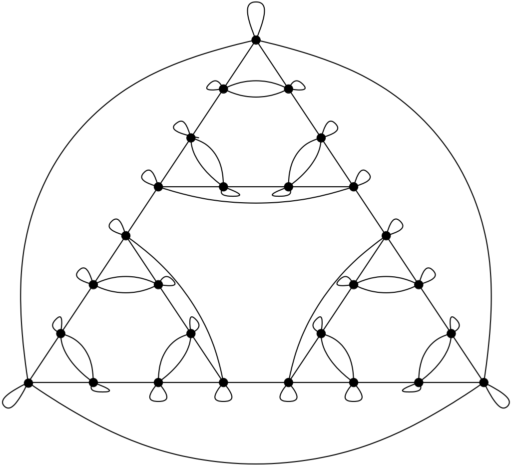


The Schreier graph Γ_4 of the action of the graph automaton group $\mathcal{G}_{P_3}$ on T_3.

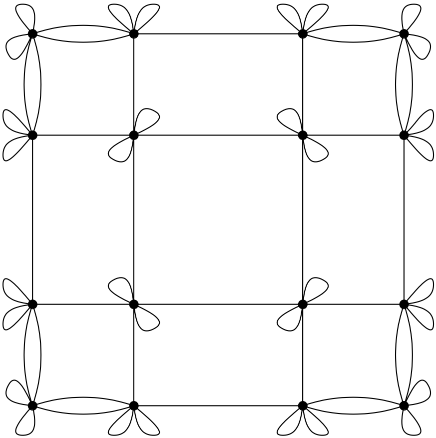


The Schreier graphs Γ_2 and Γ_3 of the graph automaton group $\mathcal{G}_\Gamma$ acting on T_3, when Γ is a cycle of length 3.



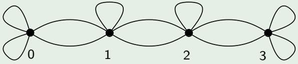


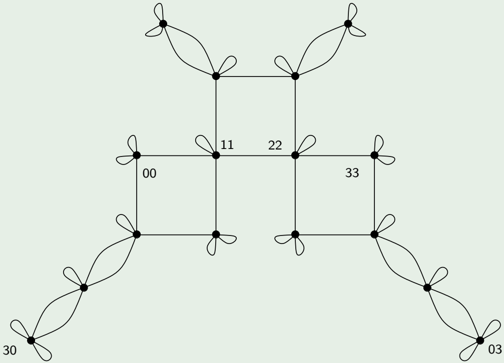
The Schreier graph Γ_2 of the graph automaton group $\mathcal{G}_\Gamma$ acting on T_4, when Γ is a cycle of length 4.



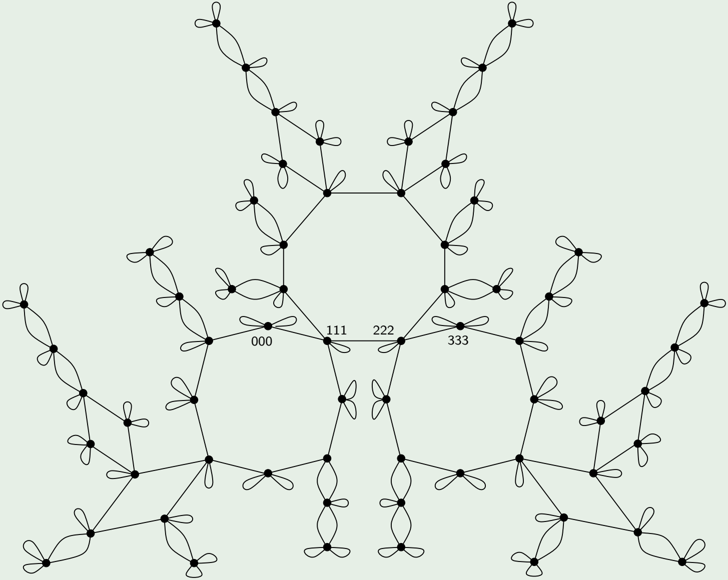
The self-similarity of the action reflects on the structure of the Schreier graphs!

The Schreier graphs Γ_1 and Γ_2 of $\mathcal{G}_{P_4}$



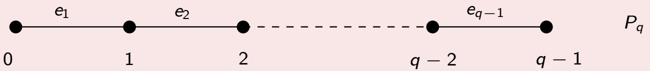


The Schreier graph Γ_3 of $\mathcal{G}_{P_4}$



How to construct Γ_n from Γ_{n-1} for the group $\mathcal{G}_{P_q}$

Consider the path graph P_q on q vertices



1. Take q copies of Γ_{n-1} and append to the end of the vertices of the i-th copy the letter i, for $i = 0, \dots, q-1$.
2. For each $i = 1, \dots, q-2$, remove the edges

$$\{i^n, (i-1)^{n-1} i\} \quad \text{and} \quad \{i^n, (i+1)^{n-1} i\}$$

together with the edges

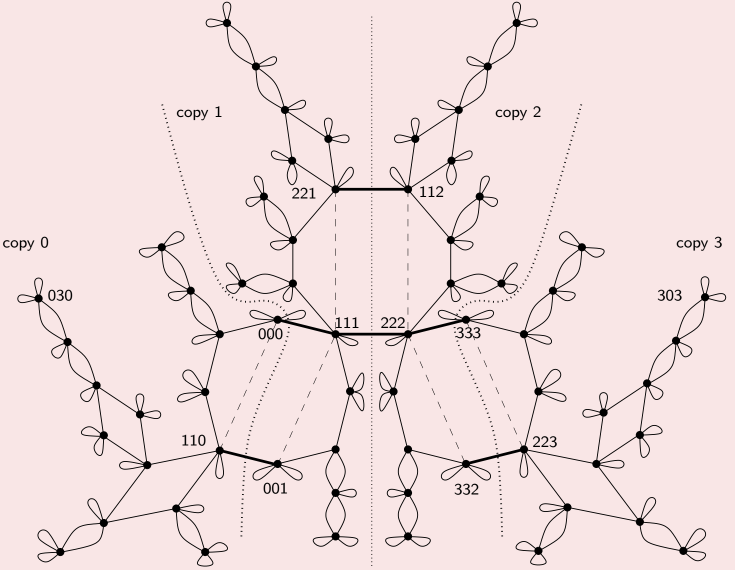
$$\{0^n, 1^{n-1}0\} \quad \text{and} \quad \{(q-1)^n, (q-2)^{n-1}(q-1)\}.$$

3. For $i = 0, \dots, q-2$, join the i-th and $(i+1)$-th copies by adding the edges

$$\{i^n, (i+1)^n\} \quad \text{and} \quad \{(i+1)^{n-1} i, i^{n-1}(i+1)\}.$$

The last operation gives rise to new cycles of doubled length with respect to the level $n-1$.

The construction of Γ_3 from Γ_2 for the group $\mathcal{G}_{P_4}$ $(q = 4)$



The copies are separated by dotted lines; the deleted edges are represented by dashed lines; the new edges producing cycles of length 8 are in bold lines.

Using this recursive construction of the Schreier graphs $\{\Gamma_n\}_{n\geq 1}$ of the graph automaton group $\mathcal{G}_{P_q}$, for $q \geq 3$, it is possible:

- to determine the number and the length of the cycles in Γ_n;
- to determine the *diameter* of Γ_n, via the recursion

$$\begin{cases} diam(\Gamma_n) = diam(\Gamma_{n-1}) + 2(q-3) + 2^n \\ diam(\Gamma_1) = q-1, \end{cases}$$

  obtaining $diam(\Gamma_n) = 2^{n+1} + (q-1)(2n-1) - 4n$;
- to establish that the automorphism group of Γ_n is isomorphic to the group $\mathbb{Z}_2^{\phi_q(n)+1}$, where

$$\phi_q(n) = \frac{2(q^{n-1} - 2q^{n-2} + 1)}{q-1}$$

  is the number of cycles of length greater or equal to 4 in Γ_n generated by the action of the generators e_1 and e_{q-1}.

The Wiener index of Schreier graphs of a tree automaton group

Let $\Gamma = (V, E)$ be a connected graph.

$d_\Gamma(u, v) =$ *geodesic distance* of the vertices $u, v \in V$ in Γ
$=$ length of a *shortest path* in Γ connecting u and v.

Wiener index $W(\Gamma)$ of Γ:

$$W(\Gamma) = \frac{1}{2} \sum_{u,v \in V} d_\Gamma(u, v).$$

[H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* **69** (1947), 17–20.]

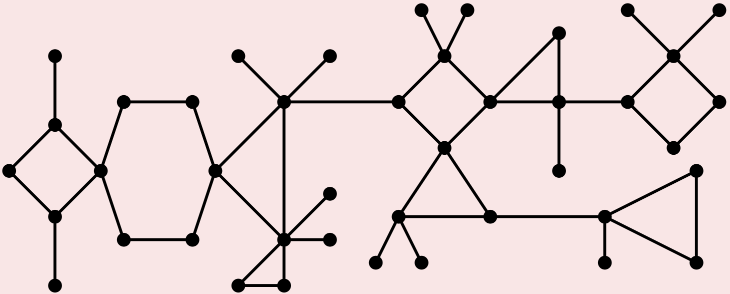
Applications

- Chemical graph theory
- Centrality measures in graphs
- Social networks and Game theory

The Wiener index of Schreier graphs of a tree automaton group

A connected graph $\Gamma = (V, E)$ is a *cactus* if it satisfies one of the following equivalent conditions:

- any two cycles have no edge in common;
- any two cycles have at most one vertex in common.



Theorem [D'Angeli, Hammer, Rodaro (2024)]

Let $T = (V_T, E_T)$ be a tree, with $|V_T| = k$, and let $\mathcal{G}_T$ be the associated graph automaton group.
For each $n \geq 1$, let Γ_n be the Schreier graph of $\mathcal{G}_T$. Then Γ_n is a cactus.
Moreover:

$$\begin{aligned} W(\Gamma_n) &= \frac{(k-1)^2(2k^2-2k-1)}{k^3(2k-1)} \cdot 2^n k^{2n} - \frac{2(k-1)^2}{k^2} \cdot nk^{2n} \\ &+ \frac{2(k+1)(k-1)}{k^3} \cdot k^{2n} - \frac{2(k+1)(k-1)}{k(2k-1)} \cdot k^n \\ &+ \left(\frac{2}{k^2} \cdot nk^{2n} - \frac{(k^2+2)}{k^3(k-1)} \cdot k^{2n} + \frac{(k+2)}{k^2(k-1)} \cdot k^n \right) W(T). \end{aligned}$$

Remark

Extremal bounds are obtained for $T = S_{k-1}$ and $T = P_k$, since it is known that:

$$(k-1)^2 = W(S_{k-1}) \leq W(T) \leq W(P_k) = \binom{k+1}{3} = \frac{1}{6}k(k^2-1)$$

The spectrum of a graph $\Gamma = (V, E)$

The *adjacency matrix* A_Γ of Γ is the matrix defined by $(A_\Gamma)_{i,j} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$
and its (real) spectrum is called the *adjacency spectrum* (or *spectrum*) of Γ.

Spectra of Schreier graphs of automaton groups have been largely studied in the last decades, in connection with problems in Algebra, Operator algebras, Random walks, Combinatorics, Fractal geometry.

In some cases, a recursive approach allowed to explicitly compute the spectrum of the adjacency matrix (e.g., Grigorchuk group, Hanoi Towers group, Lamplighter group).

This approach failed for the Basilica group, and an explicit description of the spectrum of its Schreier graphs does not exist, even if some efforts have been made with this aim:

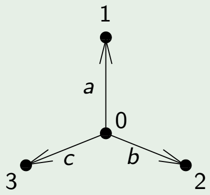
- Grigorchuk, Żuk (2002): two-dimensional dynamical system describing the spectrum.
- Rogers, Teplyaev (2010): Dirichlet forms and associated Laplace operators on the Basilica Julia set.
- Cavaleri, D'Angeli, Donno (2022): study of the characteristic polynomial using the Coefficient Theorem for signed graphs and the spectral theory of cover graphs.

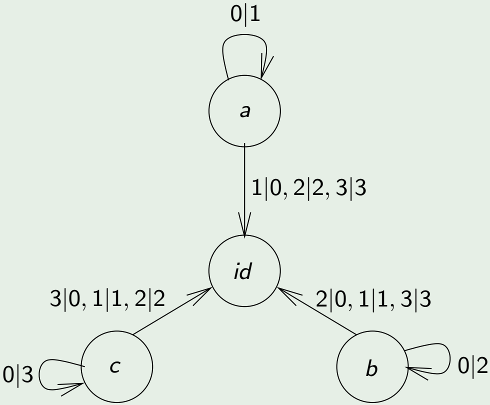
[Cavaleri, D'Angeli, Donno, Rodaro, *Groups, Geometry, and Dynamics* (2024)]:

Explicit computation of the spectrum of the Schreier graphs of the graph automaton group $\mathcal{G}_{S_p}$, where S_p is the *star graph* on $p+1$ vertices.

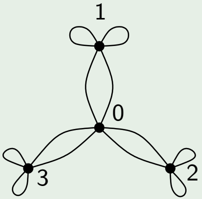
The case $p = 3$

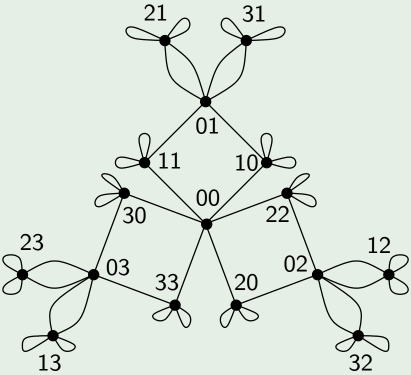
Consider the oriented star S_3 on the vertices $\{0, 1, 2, 3\}$ and the associated automaton.



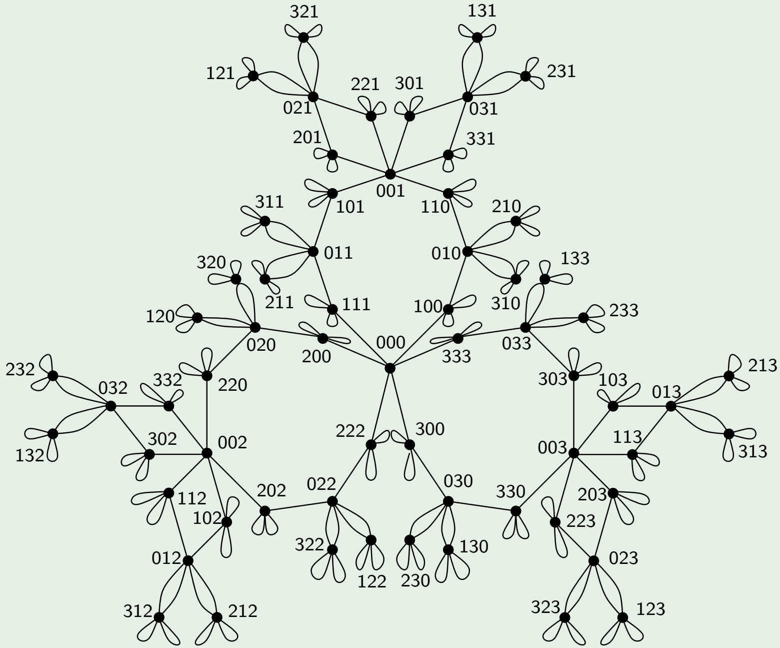


The Schreier graphs Γ_1 and Γ_2 of $\mathcal{G}_{S_3}$





The Schreier graph Γ_3 of $\mathcal{G}_{S_3}$



The adjacency matrix of the Schreier graphs of $\mathcal{G}_{S_3}$

Using the self-similar representation of the generators of $\mathcal{G}_{S_3}$

$$a = (a, id, id, id)(0\ 1) \qquad b = (b, id, id, id)(0\ 2) \qquad c = (c, id, id, id)(0\ 3)$$

we write by recursion the permutation matrices of size 4^{n+1}

$$a_{n+1} = \left(\begin{array}{c|c|c|c} 0 & a_n & 0 & 0 \\ \hline I_n & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ \hline 0 & 0 & 0 & I_n \end{array}\right) \; b_{n+1} = \left(\begin{array}{c|c|c|c} 0 & 0 & b_n & 0 \\ \hline 0 & I_n & 0 & 0 \\ \hline I_n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_n \end{array}\right) \; c_{n+1} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & c_n \\ \hline 0 & I_n & 0 & 0 \\ \hline 0 & 0 & I_n & 0 \\ \hline I_n & 0 & 0 & 0 \end{array}\right)$$

together with

$$a_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; b_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; c_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and similarly for a_{n+1}^{-1}, b_{n+1}^{-1}, c_{n+1}^{-1}, a_1^{-1}, b_1^{-1}, c_1^{-1}.

The adjacency matrix of the Schreier graphs of $\mathcal{G}_{S_3}$

Then the adjacency matrix of the graph Γ_{n+1} is obtained as

$$\begin{aligned} A_{n+1} &= a_{n+1} + a_{n+1}^{-1} + b_{n+1} + b_{n+1}^{-1} + c_{n+1} + c_{n+1}^{-1} \\ &= \left(\begin{array}{c|c|c|c} 0 & a_n + I_n & b_n + I_n & c_n + I_n \\ \hline a_n^{-1} + I_n & 4I_n & 0 & 0 \\ \hline b_n^{-1} + I_n & 0 & 4I_n & 0 \\ \hline c_n^{-1} + I_n & 0 & 0 & 4I_n \end{array}\right), \qquad n \geq 1 \end{aligned}$$

with

$$A_1 = a_1 + a_1^{-1} + b_1 + b_1^{-1} + c_1 + c_1^{-1} = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 4 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix}.$$

Schur Complement technique

Let $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be a block matrix, where A has size $k \times k$, B has size $k \times (n-k)$, C has size $(n-k) \times k$, and D has size $(n-k) \times (n-k)$. If D is nonsingular, one has

$$\det M = \det D \cdot \det(A - BD^{-1}C),$$

where the matrix $A - BD^{-1}C$ is called the *Schur complement* of D.

Theorem

Let $P_n(\lambda)$ be the characteristic polynomial of the adjacency matrix A_n of the Schreier graph Γ_n of $\mathcal{G}_{S_3}$, for each $n \geq 1$. Then

$$P_{n+1}(\lambda) = (\lambda - 4)^{2\cdot 4^n} P_n(f(\lambda)),$$

with $f(\lambda) = \lambda^2 - 4\lambda - 6$ and $P_1(\lambda) = (\lambda-6)(\lambda+2)(\lambda-4)^2$.

Proof

We use the Schur complement technique for the computation of

$$P_{n+1}(\lambda) = \det(A_{n+1} - \lambda I_{n+1}) = \det \left(\begin{array}{c|ccc} -\lambda I_n & a_n + I_n & b_n + I_n & c_n + I_n \\ \hline a_n^{-1} + I_n & (4-\lambda)I_n & 0 & 0 \\ b_n^{-1} + I_n & 0 & (4-\lambda)I_n & 0 \\ c_n^{-1} + I_n & 0 & 0 & (4-\lambda)I_n \end{array}\right)$$

and we obtain

$$\begin{aligned} \det(A_{n+1} - \lambda I_{n+1}) &= (4-\lambda)^{3\cdot 4^n} \cdot \det\left(\frac{\lambda^2 - 4\lambda - 6}{4-\lambda} I_n - \frac{1}{4-\lambda} A_n\right) \\ &= (4-\lambda)^{2\cdot 4^n} \cdot \det(A_n - (\lambda^2 - 4\lambda - 6) I_n). \end{aligned}$$

Theorem

For each $n \geq 1$, the following factorization of the characteristic polynomial $P_n(\lambda)$ holds:

$$P_n(\lambda) = (\lambda - 6) \cdot \prod_{i=0}^{n-1} \left(f^{\circ i}(\lambda) + 2 \right) \cdot \prod_{i=0}^{n-1} \left(f^{\circ i}(\lambda) - 4 \right)^{2 \cdot 4^{n-i-1}},$$

where $f^{\circ i}(\lambda) = \underbrace{f(f(\dots f(\lambda)))}_{i \text{ times}}$. In particular, the spectrum of the graph Γ_n is

$$\Sigma(\Gamma_n) = \{6\} \coprod \left(\bigcup_{i=0}^{n-1} f^{-i}(-2) \right) \coprod \left(\bigcup_{i=0}^{n-1} (f^{-i}(4))^{2 \cdot 4^{n-i-1}} \right).$$

A more explicit description of the eigenvalues

$$f^{-i}(-2) = \left\{ 2 \pm \sqrt{12 \pm \sqrt{12 \pm \sqrt{\dots \pm 2\sqrt{2}}}} \right\}, \; i \geq 1$$

$$f^{-i}(4) = \left\{ 2 \pm \sqrt{12 \pm \sqrt{12 \pm \sqrt{\dots \pm \sqrt{14}}}} \right\}, \; i \geq 1$$

where the double sign $\pm$ occurs i times.

In the general case of the star S_p on $p+1$ vertices:

Theorem

Let $P_n(\lambda)$ be the characteristic polynomial of the adjacency matrix A_n of the Schreier graph Γ_n of the group $\mathcal{G}_{S_p}$, for each $n \geq 1$. Then

$$P_{n+1}(\lambda) = (\lambda - 2(p-1))^{(p-1)\cdot(p+1)^n} P_n(f_p(\lambda)),$$

with $f_p(\lambda) = \lambda^2 - 2(p-1)\lambda - 2p$ and $P_1(\lambda) = (\lambda - 2p)(\lambda+2)(\lambda - 2(p-1))^{p-1}$.

Theorem

For each $n \geq 1$, the following factorization of the characteristic polynomial $P_n(\lambda)$ holds:

$$P_n(\lambda) = (\lambda - 2p) \cdot \prod_{i=0}^{n-1} \left(f_p^{\circ i}(\lambda) + 2 \right) \cdot \prod_{i=0}^{n-1} \left(f_p^{\circ i}(\lambda) - 2(p-1) \right)^{(p-1)\cdot(p+1)^{n-i-1}},$$

where $f_p^{\circ i}(\lambda) = \underbrace{f_p(f_p(\ldots f_p(\lambda)))}_{i \text{ times}}$. In particular:

$$\Sigma(\Gamma_n) = \{2p\} \coprod \left(\bigcup_{i=0}^{n-1} f_p^{-i}(-2) \right) \coprod \left(\bigcup_{i=0}^{n-1} (f_p^{-i}(2(p-1)))^{(p-1)\cdot(p+1)^{n-i-1}} \right).$$

Automaton groups

Graph automaton groups

Schreier graphs

Spectral computations

Infinite Schreier graphs

What about the action of an automaton group $\mathcal{G}$ on $\{0, 1, \ldots, q-1\}^\infty$ or, equivalently, on the boundary ∂T_q?

Boundary action

Fix $\omega = x_1 x_2 x_3 \ldots\ldots \in \{0, 1, \ldots, q-1\}^\infty$.

The Schreier graph Γ_ω of $\mathcal{G}$ w.r.t. the generating set S is defined by:

1. $V(\Gamma_\omega) = \{\eta \in \{0, 1, \ldots, q-1\}^\infty : \exists\, g \in \mathcal{G} \mid g(\omega) = \eta\}$
2. $\eta \sim \xi$ in Γ_ω if $\exists\, s \in S$ such that $s(\xi) = \eta$.

Remark:

The action of $\mathcal{G}$ is not transitive on ∂T_q and it factorizes into uncountably many orbits

$\Rightarrow$ there exist uncountably many orbital Schreier graphs Γ_ω, with $\omega \in \partial T_q$, which are nonisomorphic a priori.

Question:
What is the relationship between finite and infinite Schreier graphs?

Gromov-Hausdorff convergence

The sequence of rooted graphs (G_n, v_n) converges to the rooted graph (G, v) if

$$\forall\ R > 0\ \exists\ N \text{ s.t. } \forall\ n > N \text{ one has } B_{G_n}(v_n, R) \cong B_G(v, R),$$

where $B_G(v, R)$ denotes the ball of radius R centered at v in G.

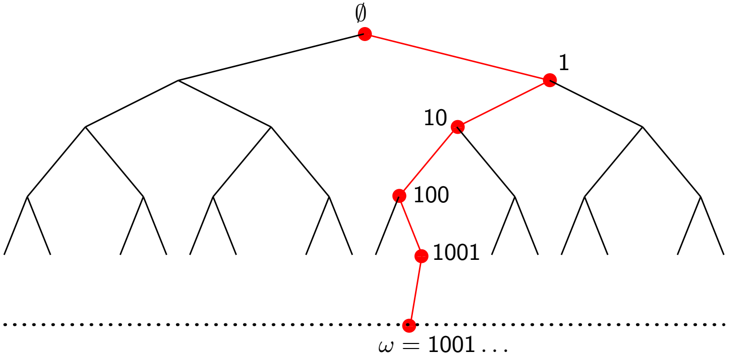
Theorem: Convergence of rooted Schreier graphs

Let $\omega = x_1 x_2 \ldots\ldots \in \partial T_q = \{0, 1, \ldots, q-1\}^\infty$. Put $\omega_n = x_1 \ldots x_n$. Then:

$$\lim_{n\to\infty} (\Gamma_n, \omega_n) = (\Gamma_\omega, \omega)$$

in the Gromov-Hausdorff sense.

$\Rightarrow$ Finite Schreier graphs are *approximations* of infinite Schreier graphs.



The sequence of rooted graphs

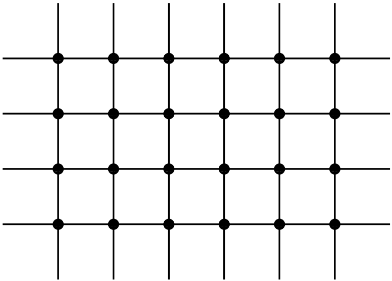
$$(\Gamma_1, 1), (\Gamma_2, 10), (\Gamma_3, 100), (\Gamma_4, 1001), \ldots\ldots$$

converges to the infinite rooted Schreier graph (Γ_ω, ω).

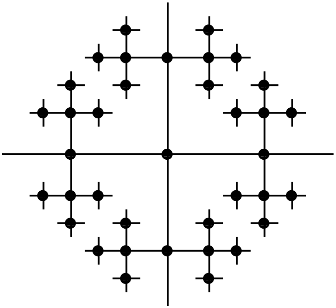
What is the shape of infinite Schreier graphs?

An infinite graph Γ is said to be a *k-ended graph* if *k* is the supremum of the number of connected infinite components of Γ, when a finite subgraph is removed from Γ.

2 ends



1 end



infinitely many ends

The case of the graph automaton group $\mathcal{G}_{S_p}$

Let $\mathcal{G}_{S_p}$ be the graph automaton group associated with the star graph S_p on $p+1$ vertices, acting on the rooted tree T_{p+1}, so that $X = \{0, 1, \ldots, p\}$.

Put $E_k = \{\omega \in X^\infty : \Gamma_\omega \text{ is } k\text{-ended}\}$.

- For each $\omega \in X^\infty$, the elements in the orbit of ω under $\mathcal{G}_{S_p}$ can be described.
- The graph Γ_ω is either $2p$-ended, or 2-ended, or 1-ended.
  The 1-ended case is generic w.r.t. the uniform measure on X^∞.
- The sets E_{2p}, E_2, E_1 can be explicitly characterized.

Finally, we studied isomorphism classes of infinite Schreier (unrooted) graphs.

Theorem (Cavaleri, D'Angeli, Donno, Rodaro)

- There is one isomorphism class of $2p$-ended graphs, consisting of the only graph Γ_{0^∞}.
- There are uncountably many isomorphism classes of 2-ended graphs, each consisting of $2p$ graphs.
- There are uncountably many isomorphism classes of 1-ended graphs, each consisting of uncountably many graphs.

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Thank you for your attention!