GROUPS WHOSE PROPER SUBGROUPS SATISFY CERTAIN PROPERTIES: PART 2

Martyn R. Dixon[1]

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AGTA Workshop-Reinhold Baer Prize, Napoli, October 7-8, 2024

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- In memory of Francesco de Giovanni and Brian Hartley

Groups whose proper subgroups satisfy some additional property

Let $\mathfrak{X}$ be a class of groups:

- Let G be a group and suppose that every proper subgroup of G is in the class $\mathfrak{X}$. If $G \notin \mathfrak{X}$, then G is called a minimal non-$\mathfrak{X}$-group. Also call G an opponent of $\mathfrak{X}$.

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- Let G be a group and suppose that every proper subgroup of G is in the class $\mathfrak{X}$. If $G \notin \mathfrak{X}$, then G is called a minimal non-$\mathfrak{X}$-group. Also call G an opponent of $\mathfrak{X}$.
- $\mathfrak{X}$ is accessible if every locally graded opponent of $\mathfrak{X}$ is finite.
- A group G is locally graded if every nontrivial finitely generated subgroup has a nontrivial finite image. Examples: locally (soluble-by-finite) groups, residually finite groups, radical groups $\dots$.

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- Tarski monsters are infinite opponents of the class of abelian groups, the class of finite *p*-groups etc.
- Of course there are finite opponents such as S_3 to the class of abelian groups.
- For these reasons we often wish to exclude Tarski monsters and finite groups.

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- Related question (which typically applies when concepts like normality, subnormality, permutability are under consideration).What is the structure of a (locally graded) group all of whose subgroups satisfy such property?

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- (Asar 2000) The class of nilpotent-by-Chernikov groups is accessible
- (Casolo 2001, Smith 2001) Let G be a torsion-free group. If all subgroups of G are subnormal, then G is nilpotent.

All subgroups subnormal or nilpotent

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3. *Let G be a locally finite group with all non-nilpotent subgroups subnormal. Then G contains a normal subgroup K of finite index in which every subgroup is subnormal.*
4. *Let G be a torsion-free locally (soluble-by-finite) group with all non-nilpotent subgroups subnormal. Then G is nilpotent.*

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3. *Let G be a locally finite group with all non-nilpotent subgroups subnormal. Then G contains a normal subgroup K of finite index in which every subgroup is subnormal.*
4. *Let G be a torsion-free locally (soluble-by-finite) group with all non-nilpotent subgroups subnormal. Then G is nilpotent.*
5. *If G is a locally graded torsion-free group and the non-nilpotent subgroups are subnormal of defect at most d, then G is nilpotent.*

Questions

- If G is an infinite locally graded group with every non-nilpotent subgroup subnormal is G soluble?
- If G is torsion-free locally graded and every non-nilpotent subgroup is subnormal is G nilpotent?
- If G is a locally graded group with all non-nilpotent subgroups subnormal does G contain a normal subgroup K of finite index in which every subgroup is subnormal?

All subgroups soluble or subnormal

Theorem (Ersoy, Tortora, Tota 2014)

Let G be a locally (soluble-by-finite) group in which every subgroup is subnormal or soluble of derived length at most d. Then

1. *G is soluble or*
2. *G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.*

Let G be a locally graded group with all subgroups subnormal of defect at most n or soluble of derived length at most d.

1. *G is soluble of derived length bounded by a function of n, d or*
2. *G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.*

All subgroups permutable

- G is quasihamiltonian (q. h) if every subgroup of G is permutable.

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- (Iwasawa 1943) Let G be a q.h. group. Then G is metabelian, locally nilpotent.

Permutable or soluble

- (De Falco, de Giovanni, Musella, Schmidt 2003) G locally graded, all subgroups abelian or permutable. Then G is soluble of derived length at most 4; there is a finite normal subgroup N such that G/N is qh.

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  1. If G is soluble then G has derived length at most $d + 3$;
  2. If G is not soluble, then G'' is finite and perfect. Also all proper subgroups of G'' are soluble of derived length at most d.

Stonehewer's results on permutable subgroups

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- If $G = H\langle g \rangle$ with H permutable in G and $\langle g \rangle$ infinite cyclic such that $\langle g \rangle \cap H = 1$, then g normalizes H.
- If H is a core-free permutable subgroup of a group G, then H is a subdirect product of nilpotent groups and hence residually finite.

Permutable or nilpotent-Atlihan, MD, Evans

Theorem

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General Facts and Proofs

Let G be locally graded and suppose that every non-nilpotent subgroup of G is permutable.

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- X is a Fitting group.

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- X is a Fitting group.
  $Y =$ product of proper normals of X. If $Y \neq X$, then Y is nilpotent and X/Y is simple so has no proper non-permutables by Stonehewer. Asar then implies X/Y soluble.
- Every proper subgroup of X is soluble.

The non-periodic case

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- If X is not periodic, let T be its (proper) torsion subgroup. T is nilpotent and X/T is torsion-free. But then X/T is soluble (a bit of work needed) so X is soluble as is G, contradiction.

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- Thus X is periodic. Let H be a proper non-nilpotent subgroup of X. There is an element $g \in G$ such that $\langle g\rangle \cap H = 1$. Then H is normal in X so nilpotent. Contradiction. Thus H is nilpotent and Asar's result completes the proof.

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- Let P be non-nilpotent in X so P is permutable in G with core P_X in X

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- P_X is soluble with derived length d

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- Let P be non-nilpotent in X so P is permutable in G with core P_X in X
- P_X is soluble with derived length d
- P/P_X is a core-free permutable subgroup of X/P_X. Thus P/P_X is residually finite (Stonehewer).

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- Thus X is locally (soluble of derived length at most $d + 2$) so X is soluble of derived length at most $d + 2$. Final contradiction.

Other generalizations of normality

- A subgroup H of a group G is pronormal in G if for all $g \in G$ there exists $u \in \langle H, H^g \rangle$ such that $H^u = H^g$. Normal subgroups are pronormal.

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- A group G is a T-group if every subnormal subgroup of G is normal in G.
- A group G is a $\bar{T}$-group if every subgroup is a T-group.
- Let $\mathfrak{P}$ denote the class of groups G such that every subgroup of G is pronormal in G.

Accessibility etc

- (Kuzennyi-Subbotin, 1987, Robinson-Russo-Vincenzi 2007) Let G be a $\mathfrak{P}$-group. Then G is abelian or of Kuzennyi-Subbotin type. In particular G is metabelian.

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- Is the class of $\bar{T}$-groups accessible? Think yes at the moment!

Some inaccessible group classes

- Easy ones: The class of nilpotent groups, the class of Fitting groups, the class of Baer groups-use the locally dihedral 2-group, D. Note that in D every subgroup is finite nilpotent or it contains the lone copy of C_{p^∞} and hence is abelian or the whole of D.

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- HM-groups are infinite opponents of the class of hypercentral groups. The class of abelian-by-finite groups is also inaccessible-usie HM-groups.
- (Belyaev, 1978) If G is locally finite and a minimal non-$\mathfrak{AF}$-group, then G is metabelian. Periodic opponents of $\mathfrak{AF}$ are locally finite.

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- Also work of Asar, Arikan, Trabelisi, Badis

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- Is Chernikov's class the class of all locally graded groups?

Thanks

Grazie mille!