On generalized concise words

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Words

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$w \in [F, F]$ a commutator word

$w \notin [F, F]$ a non-commutator word

$[x_1, [x_2, x_3], [x_4, [x_5, x_6]]]$ a multilinear commutator word

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$\gamma_1 = x_1, \quad \gamma_n = [\gamma_{n-1}, x_n] = [x_1, \ldots, x_{n-1}, x_n]$
the lower central words

$\delta_0 = x_1, \quad \delta_n = [\delta_{n-1}(x_1, \ldots, x_{2^{n-1}}), \delta_{n-1}(x_{2^{n-1}+1}, \ldots, x_{2^n})]$
the derived words

$[x, {}_n y] = [x, \underbrace{y, \ldots, y}_{n}]$ the n-th Engel word

The verbal subgroup

$w = w(x_1, \ldots, x_n)$ a word, G a group, $g_1, \ldots, g_n \in G$

$w(g_1, \ldots, g_n)$ a w-value in G

$G_w = \{w(g_1, \ldots, g_n) \,|\, g_i \in G\}$ the set of all w-values in G

$w(G) = \langle G_w \rangle$ the verbal subgroup of G corresponding to w

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- if G_w is finite then $w(G)$ is centre-by-finite
- if G_w is finite and $w(G)$ is periodic then $w(G)$ is finite

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(S.V. Ivanov, 1989)
Let $d > 10^{10}$ be an odd integer and $p > 5000$ a prime. There exists a 2-generator torsion-free group I such that $Z(I)$ is cyclic and $I/Z(I)$ is an infinite group of exponent $p^2 d$. Set

$$v(x,y) = [[x^{pd}, y^{pd}]^d, y^{pd}]^d.$$

Then $I_v = \{1, \epsilon\}$ and $Z(I) = \langle \epsilon \rangle = v(I)$ is infinite. Hence, v is not concise.

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- the derived word δ_n
  (R.F. Turner-Smith, 1966)
- multilinear commutator words
  (J.R.C. Wilson, 1974)
- the n-th Engel word, for $n \leq 4$
  (G.A. Fernández-Alcober, M. Morigi, G. Traustason, 2012)

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- $[x^r, y^s]$, for all integers r and s
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- $[u_1, u_2, u_3]$, where u_1, u_2 and u_3 are non-commutator words in disjoint sets of variables
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  (J. Azevedo, P. Shumyatsky, 2022)

- $w = w(u_1, \dots, u_r)$, where $w = w(x_1, \dots, x_r)$ is a multilinear commutator word and $u_1, \dots, u_r$ are non-commutator words in disjoint sets of variables

  (G.A. Fernández-Alcober, M. Pintonello, 2024)

Generalizing conciseness

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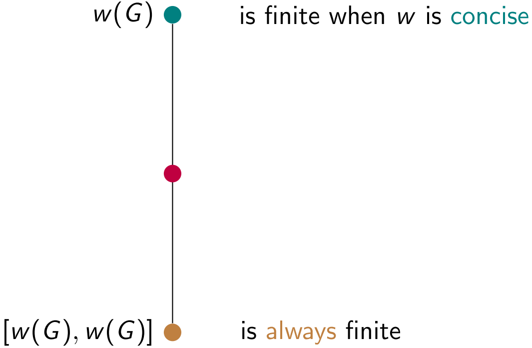

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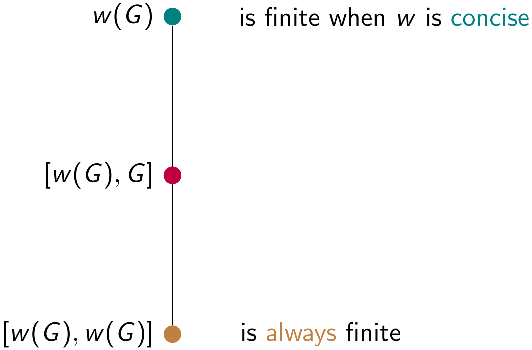

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$[w(G), G]$ need not be finite for an arbitrary w

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The n-th Engel word $[x,_n y]$ is semiconcise for all n.

(CD, P. Shumyatsky, A. Tortora, 2020)
If $x, y, z_1, \ldots, z_m$ are pairwise different variables, the word

$$v = [x,_n y, z_1, \ldots, z_m]$$

is semiconcise for all positive integers n and m.

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Open question
Is there any semiconcise word which is not concise?

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(CD, P. Shumyatsky, A. Tortora, 2020)
There exist words that are not semiconcise.

Generalizing conciseness: next step

Let w be any word, and G be a group such that G_w is finite. Then:

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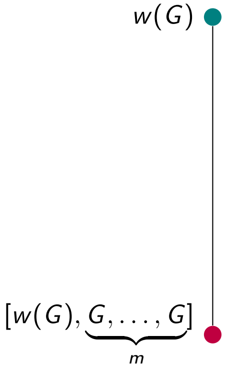

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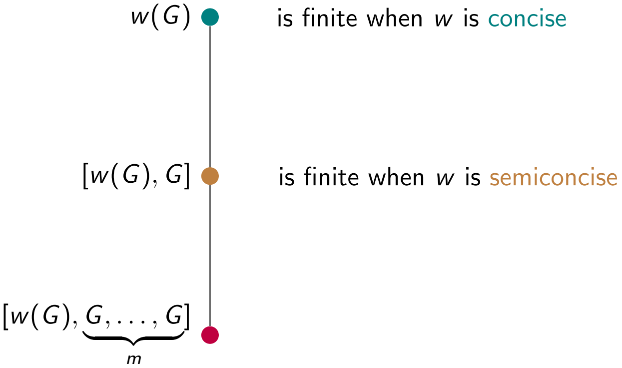

Generalizing conciseness: next step

Let w be any word, and G be a group such that G_w is finite. Then:

$w(G)$ is finite when w is concise

$[w(G), G]$ is finite when w is semiconcise

$[w(G), \underbrace{G, \ldots, G}_{m}]$ is finite when w is $\frac{1}{m+1}$-concise

$\frac{1}{m}$-concise words

Let m be any positive integer. A word w is $\frac{1}{m}$-concise if

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- w is 1-concise iff it is concise

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- w is 1-concise iff it is concise
- w is $\frac{1}{2}$-concise iff it is semiconcise
- if w is $\frac{1}{m}$-concise then it is $\frac{1}{t}$-concise for all $t \geq m$

A useful tool

(CD, M. Gaeta, C. Monetta, 2024)
Let $w = w(x_1, \ldots, x_n)$ be a word, and set

$$v = [w, x_{n+1}],$$

with $x_{n+1} \notin \{x_1, \ldots, x_n\}$. If w is $\frac{1}{m}$-concise for some positive integer m, then v is $\frac{1}{m}$-concise.

A hierarchy for words

A word w is 0-concise if for any group G such that G_w is finite there exists a positive integer m, depending on G, such that

$$[w(G), \underbrace{G, \ldots, G}_{m-1}]$$

is finite.

A hierarchy for words

A word w is 0-concise if for any group G such that G_w is finite there exists a positive integer m, depending on G, such that

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is finite.

Let W_m denote che set of all $\frac{1}{m}$-concise words, and W_∞ the set of all 0-concise words. Then

$$W_1 \subseteq W_2 \subseteq W_3 \subseteq \cdots \subseteq W_m \subseteq \cdots \subseteq \bigcup_{t \in \mathbb{N}} W_t \subseteq W_\infty \neq \{all\ words\}.$$

Existence of words which are not 0-concise

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Let $d > 10^{10}$ be an odd integer and $p > 5000$ a prime. There exists a 2-generator torsion-free group I such that $Z(I)$ is cyclic and $I/Z(I)$ is an infinite group of exponent $p^2 d$. Set

$$v(x, y) = [[x^{pd}, y^{pd}]^d, y^{pd}]^d.$$

(S. Brazil, A. Krasilnikov, P. Shumyatsky, 2006)
Let $B = I \ wr \ C$ where C has order 2. Set

$$w(x, y) = v(x^2, y^2).$$

Then $|B_w| = 4$.

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$$w(x,y) = v(x^2, y^2).$$

Then $|B_w| = 4$.

(CD, M. Gaeta, C. Monetta, 2024)
For all positive integers m

$$[w(B), \underbrace{B, \dots, B}_{m-1}]$$

is infinite. So the word $w(x,y)$ is not 0-concise.

FC-embedded subgroups

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If G is any FC-group then $G/Z(G)$ is locally finite. Hence $[G, G]$ is locally finite.

A subgroup H of a group G is FC-embedded in G if the set of conjugates $a^H = \{a^h \mid h \in H\}$ is finite for all $a \in G$.

A verbal generalization of FC-groups

Let w be a word. A group G is an $FC(w)$-group if the set of conjugates $a^{G_w} = \{a^g \mid g \in G_w\}$ is finite for all $a \in G$.

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(CD, P. Shumyatsky, A. Tortora, 2017)
Let w be a word. Then a group G is an $FC(w)$-group if and only if it is an $FC(w^{-1})$-group.

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(CD, M. Gaeta, C. Monetta, 2024)
Let $w = w(x_1, \dots, x_n)$ be a word, and set

$$v = [w, x_{n+1}, \dots, x_{n+m}],$$

where $x_{n+1}, \dots, x_{n+m} \notin \{x_1, \dots, x_n\}$. If G is an $FC(w)$-group then it is also an $FC(v)$-group.

Subgroups of $FC(w)$-groups

(CD, P. Shumyatsky, A. Tortora, 2017)
Let w be any word, and let G be an $FC(w)$-group. Then $[w(G), w(G)]$ is FC-embedded in G.

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(S. Franciosi, F. de Giovanni, P. Shumyatsky, 2002)
Let w be a concise word, and let G be an $FC(w)$-group. Then $w(G)$ is FC-embedded in G.

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Let w be a concise word, and let G be an $FC(w)$-group. Then $w(G)$ is FC-embedded in G.

(CD, M. Gaeta, C. Monetta, 2024) Let w be a $\frac{1}{m}$-concise word, and let G be an $FC(w)$-group. Then $[w(G), \underbrace{G, \ldots, G}_{m-1}]$ is FC-embedded in G.

Existence of bounds

Let w be any word. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $|[w(G), w(G)]| \leq f(r)$, for any group G with $|G_w| \leq r$.

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(G.A. Fernández-Alcober, M. Morigi, 2010)
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(CD, M. Gaeta, C. Monetta, 2024)
There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$|[w(G), \underbrace{G, \dots, G}_{m-1}]| \leq f(m, r),$$

for any $\frac{1}{m}$-concise word w and for any group G with $|G_w| \leq r$.

BFC-embedded subgroups

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A subgroup H of a group G is BFC-embedded in G if there exists a positive integer r such that $|a^H| \leq r$ for all $a \in G$.

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$$v = [w, x_{n+1}, \ldots, x_{n+m}]$$

where $x_{n+1}, \ldots, x_{n+m} \notin \{x_1, \ldots, x_n\}$. If G is a $BFC(w)$-group with $|a^{G_w}| \leq r$ for all $a \in G$, then G is also a $BFC(v)$-group and a^{G_v} has $\{n, r, m\}$-bounded order for all $a \in G$.

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