Rack Automorphisms and Reflections to Solutions of the Yang-Baxter Equation

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AGTA Workshop - Reinhold Baer Prize 2024
Department of Mathematics and Applications “Renato Caccioppoli” - Naples

This talk is based on some results obtained in:

A. Albano, M. Mazzotta, and P. Stefanelli. Reflections to set-theoretic solutions of the Yang-Baxter equation. *arXiv:2405.19105* (2024).

This work was supported by the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA - INdAM)

Andrea Albano was supported by a scholarship financed by the Ministerial Decree no. 118/2023, based on the NRRP - funded by the European Union - NextGenerationEU - Mission 4.

The Yang-Baxter Equation

V. G. Drinfeld, On some unsolved problems in quantum group theory, in: *Quantum groups (Leningrad, 1990)*, 1–8, *Lecture Notes in Math.*, **1510**, Springer, Berlin.

If X is a set, a map $r : X \times X \to X \times X$ satisfying the *braid relation*

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r),$$

is called a *set-theoretic solution* to the *YBE*.

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If we project r onto its components and write

$$r(a, b) = (\lambda_a(b), \rho_b(a))\,,$$

where $\lambda_a, \rho_b : X \to X$ are maps, for all $a, b \in X$, then r is named:

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- *left non-degenerate* if λ_a is bijective, for all $a \in X$;

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where $\lambda_a, \rho_b : X \to X$ are maps, for all $a, b \in X$, then r is named:

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- *right non-degenerate* if ρ_b is bijective, for all $b \in X$;
- *non-degenerate* if it is both left and right non-degenerate.

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A solution (X, r) is *involutive* if $r^2 = \mathrm{id}_{X\times X}$.

Shelves, racks and what's more

A *(left) shelf* $(X, \triangleright)$ is a set X equipped with a binary operation $\triangleright$ such that:

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)\,,$$

for all $x, y, z \in X$.

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- *rack*, if the map $L_x : X \ni y \mapsto x \triangleright y \in X$ is bijective, for all $x \in X$;
- *quandle*, if $(X, \triangleright)$ is a rack such that $x \triangleright x = x$, for all $x \in X$.

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D. Joyce, A classifying invariant of knots, the knot quandle, *J. Pure Appl. Algebra* **23** (1982), no. 1, 37–65.

S. V. Matveev, Distributive groupoids in knot theory, *Mat. Sb. (N.S.)* **119(161)** (1982), no. 1, 78–88, 160.

Some examples

- **Cyclic Shelf:** Let X be a non-empty set, let $f : X \to X$ be a map and define
$$x \triangleright y = f(y)\,.$$
Then, $(X, \triangleright)$ is a shelf and a
  i) *rack* if and only if f is bijective;
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- **Dihedral Quandle:** Let $n \in \mathbb{N}_0$ and define on $X = \mathbb{Z}/n\mathbb{Z}$
$$a \triangleright b = 2a - b\,.$$
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Then, $(X, \triangleright)$ is a quandle.

- **Conjugation Quandle:** Let G be a group and define
$$x \triangleright y = x^{-1}yx\,.$$
Then, $\mathrm{Conj}(G) = (G, \triangleright)$ is a quandle.

Self-distributivity in the YBE

Let $(X, \triangleright)$ be a shelf and define $r_\triangleright : X \times X \to X \times X$ by

$$r_\triangleright(a, b) = (b, b \triangleright a)\,,$$

Then, $r_\triangleright$ is a *left non-degenerate solution*.

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Conversely let (X, r) be a *left non-degenerate solution* and define $\triangleright_r$ on X by setting

$$a \triangleright_r b = \lambda_a \rho_{\lambda_b^{-1}(a)}(b)\,.$$

for all $a, b \in X$.

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$$a \triangleright_r b = \lambda_a \rho_{\lambda_b^{-1}(a)}(b)\,.$$

for all $a, b \in X$. Then, $(X, \triangleright_r)$ is a shelf called the *derived shelf* of (X, r).

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If (X, r) is left non-degenerate then

$$(X, r) \text{ is bijective} \iff (X, \triangleright_r) \text{ is a rack},$$
$$(X, r) \text{ is involutive} \iff (X, \triangleright_r) \text{ is trivial}, \text{ i.e. } a \triangleright_r b = b.$$

Theorem

Let $(X, \triangleright)$ be a shelf and consider a map $\lambda : X \to \mathrm{Sym}_X$. Then, the map $r : X \times X \to X \times X$ defined for all $a, b \in X$ by

$$r(a,b) = \left(\lambda_a(b), \lambda^{-1}_{\lambda_a(b)}(\lambda_a(b) \triangleright_r a)\right),$$

is a solution if and only if the following holds for all $a, b \in X$:

$$\lambda_a \in \mathrm{Aut}(X, \triangleright_r), \text{ for all } a \in X;$$
$$\lambda_a \lambda_b = \lambda_{\lambda_a(b)} \lambda_{\lambda^{-1}_{\lambda_a(b)}(\lambda_a(b) \triangleright a)}, \text{ for all } a, b \in X.$$

Moreover, any *left non-degenerate solution* can be constructed in this way.

A. Doikou, B. Rybołowicz and P. Stefanelli, Quandles as pre-Lie skew braces, set-theoretic Hopf algebras & universal $\mathcal{R}$-matrices, *J. Phys. A: Math. Theor.* **57** (2024) 405203.

The Reflection Equation

V. Caudrelier and Q. C. Zhang, Yang-Baxter and reflection maps from vector solitons with a boundary, *Nonlinearity* **27** (2014), no. 6, 1081-1103.

Let (X, r) be a solution. A map $\kappa : X \to X$ is a *solution to the set-theoretic reflection equation*, or simply a *reflection*, for (X, r) if the following holds:

$$r\ (\mathrm{id}_X \times \kappa)\ r\ (\mathrm{id}_X \times \kappa) = (\mathrm{id}_X \times \kappa)\ r\ (\mathrm{id}_X \times \kappa)\ r$$

I. Cherednik, Factorizing particles on a half line, and root systems, *Teoret. Mat. Fiz.* **61** (1984), no. 1, 35-44.

E. K. Sklyanin, Boundary conditions for integrable quantum systems, *J. Phys. A* **21** (1988), no. 10, 2375-2389.

Notation:

$\mathcal{K}(X, r)$ is the set of all reflections for (X, r).

$\mathcal{K}_{\mathrm{bij}}(X, r)$ is the set of all bijective reflections for (X, r).

State of the Art

- [Caudrelier, Crampé, and Zhang (2013)] deepens the interplay between solutions to the YBE and reflections.
- [De Commer (2019)] introduced a notion of braided action of a group with braiding and showed that it provides reflections.
- [Smoktunowicz, Vendramin, and Weston (2020)] proposed a systematic approach employing general methods coming from the theory of *braces*, to produce families of reflections.
- [Doikou and Smoktunowicz (2021)] investigated connections between set-theoretic Yang–Baxter and reflection equations and quantum integrable systems.
- [Lebed and Vendramin (2022)] focused on reflections for involutive non-degenerate solutions.

λ-centralizing and ρ-invariant maps

Let X be a set and consider a map $\sigma : X \to X^X, \sigma(a) = \sigma_a$.
A map $\kappa : X \to X$ is called

- σ-centralizing if $\sigma_a\kappa = \kappa\sigma_a$, for all $a \in X$;
- σ-invariant if $\sigma_{\kappa(a)} = \sigma_a$, for all $a \in X$.

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Let (X, r) be a solution and if we write $r(a, b) = (\lambda_a(b), \rho_b(a))$, then:

- [SVW (2020)] if r is involutive left non-degenerate then

$$\kappa : X \to X \text{ is } \lambda\text{-centralizing} \implies \kappa \in \mathcal{K}(X, r);$$

- [LV (2022)] if r is involutive right non-degenerate then

$$\kappa : X \to X \text{ is } \rho\text{-invariant} \implies \kappa \in \mathcal{K}(X, r);$$

The non-involutive case

Example

There exists a *skew-brace* $(B, +, \circ)$ where $(B, +) \cong D_8$ and $(B, \circ) \cong C_8$ whose associated solution (B, r) has

- **288** reflections;
- **256** *only* λ-centralizing reflections;
- **16** *only* ρ-invariant reflections;
- **16** of *both types*.

Recall that a *skew-brace* $(B, +, \circ)$ is the datum of two groups $(B, +)$ and $(B, \circ)$ such that

$$a \circ (b + c) = a \circ b - a + a \circ c\,,$$

for all $a, b, c \in B$. The map $r : B \times B \to B \times B$ defined by

$$r(a, b) = (\lambda_a(b), (\lambda_a(b))^- \circ a \circ b)\,,$$

is a *bijective non-degenerate solution*.

Recovering lost reflections

Theorem [AMS (2024x)]

Let (X, r) be a solution and let $\kappa \in \mathcal{K}(X, r)$.
If $\varphi, \psi : X \to X$ are λ, ρ-centralizing and λ, ρ-invariant maps, then

$$\varphi\,\kappa\,\psi \in \mathcal{K}(X, r)\,.$$

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This generalizes an analogous result contained in [LV (2022)] valid for *involutive* solutions.

Notice that *all* reflections share this form.

Reflections for left non-degenerate solutions

Theorem [AMS (2024x)]

Let (X, r) be left non-degenerate and $\kappa : X \to X$ be a λ-centralizing map. For all $a, b \in X$ write $L_a(b) = \lambda_a \rho_{\lambda_b^{-1}(a)}(b)$.

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$$\kappa \in \mathcal{K}(X, r) \iff \begin{cases} \forall\, a, b \in X \;\; \kappa L_{L_a(b)}(a) = L_{\kappa L_a(b)} \kappa(a) \\ \forall\, a \in X \;\; \kappa L_a = \kappa L_{\kappa(a)} \end{cases}$$

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Corollary [AMS (2024x)]

Let (X, r) be a bijective non-degenerate solution. If $\kappa : X \to X$ is a λ-centralizing map, then

$$\kappa \in \mathcal{K}(X, r) \iff \begin{cases} \kappa \in \mathrm{End}(X, \triangleright_r) \\ \forall\, x \in X \quad \kappa L_x = \kappa L_{\kappa(x)} \end{cases}$$

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If (X, r) is *involutive* left non-degenerate then $(X, \triangleright_r)$ is *trivial*.

Reflections for right non-degenerate solutions

If (X, r) is a right non-degenerate solution, one can consider the right rack $(X, \triangleleft_r)$ where $a \triangleleft_r b := \rho_a \lambda_{\rho_b^{-1}(a)}(b)$, for all $a, b \in X$.

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Theorem [AMS (2024x)]

Let (X, r) be right non-degenerate and $\kappa : X \to X$ be a ρ-invariant map.

$$\kappa \in \mathcal{K}(X, r) \iff \forall\, a \in X \quad \kappa R_a = R_a \kappa\,,$$

where $R_a(b) = b \triangleleft_r a$, for all $a, b \in X$.

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If (X, r) is a right non-degenerate solution and $\kappa : X \to X$ is ρ-invariant, then

$$\kappa \in \mathcal{K}(X, r) \iff \kappa \in \mathcal{K}(X, \triangleleft_r)\,.$$

Reflections for solutions of derived type

Let $(X, \triangleright)$ be a rack. The *left multiplication group* of $(X, \triangleright)$ is the normal subgroup of $\operatorname{Aut}(X, \triangleright)$ defined by

$$\operatorname{LMlt}(X, \triangleright) := \langle L_x \mid x \in X \rangle \, .$$

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Proposition [AMS (2024x)]

If $(X, \triangleright)$ is a rack, then

$$\mathcal{K}(X, r_\triangleright) \supseteq C_{\mathrm{End}(X,\triangleright)}\left(\mathrm{LMlt}\left(X, \triangleright\right)\right) \, ,$$
$$\mathcal{K}_{\mathrm{bij}}(X, r_\triangleright) = C_{\mathrm{Aut}(X,\triangleright)}\left(\mathrm{LMlt}\left(X, r_\triangleright\right)\right) \, .$$

Studying rack automorphisms

Problem:

Let $\mathcal{F}$ be a class of racks. For $X \in \mathcal{F}$ determine $\mathrm{Aut}(X)$ and $\mathrm{LMlt}(X)$.

M. Elhamdadi, J. Macquarrie and R. R. López, Automorphism groups of quandles, *J. Algebra Appl.* **11** (2012), no. 1, 1250008, 9 pp.

The authors solve the problem for *dihedral quandles*.

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The authors solve the problem for *dihedral quandles*.

Let G be a group and for all $a, b \in G$ define

$$a \triangleright b = ba^{-1}b\,.$$

The pair $\mathrm{Core}(G) = (G, \triangleright)$ is a quandle named the *core quandle* of G.

If G is an abelian group, then $\mathrm{Core}(G)$ takes the name of *Takasaki quandle* of G.

V. G. Bardakov, T. R. Nasybullov and M. Singh, Automorphism groups of quandles and related groups, *Monatsh. Math.* **189** (2019), no. 1, 1–21.

The authors describe $\mathrm{Aut}(X)$ for the class of *Takasaki quandles* of a *finite abelian group*, generalizing previous results on the matter.

For conjugation quandles it still an open problem. Among all, an embedding is known [BNS (2019)]

$$\mathrm{Aut}(G) \times \mathrm{Sym}_{Z(G)} \hookrightarrow \mathrm{Aut}(X)\,,$$

where $X = \mathrm{Conj}(G)$ of a group G. There are similar results for core quandles.

Thank you for the attention!

References

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