On Degenerations of Algebras over an Arbitrary Field [*]

Nataliya M. Ivanova — Christakis A. Pallikaros

Abstract

For each $n \geqslant 2$ we classify all n-dimensional algebras over an arbitrary infinite field which have the property that the n-dimensional abelian Lie algebra is their only proper degeneration.

1 Introduction and statement of results

The concept of degeneration probably first arises in the second half of the twentieth century when a lot of attention was paid to the study of various limit processes linking physical theories. One of the most famous examples of such a limit process is the relation between classical and quantum mechanics. Indeed, classical mechanics can be studied as a limit case of quantum mechanics where the quantum mechanical commutator $[x, p] = i\hbar$ (corresponding to the Heisenberg uncertainty principle) maps to the abelian case (that is the classical mechanics limit) as $\hbar \to 0$. The pioneering works in this

---

direction were a paper by Segal [26] and a series of papers of Inönü and Wigner [13],[14] devoted to the limit process $c \to \infty$ in special relativity theory showing how the symmetry group of relativistic mechanics (the Poincaré group) degenerates to the symmetry group of classical mechanics (Galilean group). The target algebras of such limit processes (which are nothing else but the points in the closure in metric topology of the orbit of the initial algebra under the "change of basis" action of the general linear group) are called contractions (or degenerations in the more general context of an arbitrary field and Zariski closure). These (and many other) examples of degenerations have a wide range of applications in physics based on the claim that if two physical theories are related by a limiting process, then the associated invariance groups (and invariance algebras) should also be related by some limiting process. Degenerations of algebras also have applications in other branches of mathematics. Thus, for example, they can be used as a tool for finding rigid algebras which are important for the investigation of varieties of algebras and their irreducible components. Degenerations are widely used for studying different properties of flat and curved spaces and in the theory of quantum groups. As the notion of degeneration is closely related to the notion of deformation (see [6]), degenerations can also be used for investigating deformations of Lie algebras (see [20] and [4]).

Despite their theoretical and practical interest, results about degenerations, especially in fields other than $\mathbb{C}$ or $\mathbb{R}$, are still fragmentary. In [10], it is shown that the closures in the Zariski topology and in the standard topology of the orbit of a point of an affine variety over $\mathbb{C}$ under the action of an algebraic group coincide. In the same work, again over $\mathbb{C}$, for G a reductive algebraic group with Borel subgroup B and X an algebraic set on which G acts, it is shown that for $x \in X$ the closure of the B-orbit of x has non-empty intersection with every orbit in the closure of its G-orbit. Various necessary conditions for the existence of degenerations of real and complex algebras have been obtained, see for example [10],[21],[23],[24] and [25]. A criterion for a Lie algebra to be a degeneration of another Lie algebra over an algebraically closed field is given in [11]. In [22] the question of whether or not a given orbit (of a point under the action of an algebraic group) lies in the Zariski closure of another orbit is being considered and a method of solving this problem is presented. However, in practice it is extremely difficult to apply the results of [11] and [22]. For some classes of algebras (like real and complex 3- and 4-dimensional Lie algebras, low-dimensional nilpo-

tent Lie algebras, some subclasses of Malcev algebras) the problem of determining all degenerations within the given class has been considered, see for example [2],[21],[16],[17] and references therein.

Our motivation comes from works of Gorbatsevich (see [7],[8],[9]) and in particular the notion of the level of complexity of a finite dimensional algebra. One of the main purposes of the present paper is to make a contribution in the general direction of studying degenerations of algebras over an arbitrary field, as the great majority of the works in this area so far have been over an algebraically closed field (mainly $\mathbb{C}$) or $\mathbb{R}$.

At this point we introduce some notation. Let V be an n-dimensional vector space over an arbitrary field $\mathbb{F}$ and let $G = GL(n, \mathbb{F})$. Also denote by $\mathcal{A}_n(\mathbb{F})$ the set of all algebra structures on V and by $\mathcal{K}_n^a(\mathbb{F})$ the subset of $\mathcal{A}_n(\mathbb{F})$ consisting of the algebra structures satisfying the identity $[x, x] = 0$ (we denote by $[x_1, x_2]$ the product of the elements x_1, x_2 of an algebra). Next we introduce some particular algebras in $\mathcal{A}_n(\mathbb{F})$. Let $\mathfrak{a}_n$ be the n-dimensional abelian Lie algebra. Also let $\mathfrak{h}_n \in \mathcal{A}_n(\mathbb{F})$ be isomorphic to the Lie algebra direct sum

$$\mathfrak{h}_3 \oplus \mathfrak{a}_{n-3}$$

(where $\mathfrak{h}_3$ is the Heisenberg algebra) and $\mathfrak{r}_n \in \mathcal{A}_n(\mathbb{F})$ be defined as the algebra structure for which the only nonzero products between the basis elements of a fixed basis $(v_1^*, \ldots, v_n^*)$ of V are

$$[v_i^*, v_n^*] = v_i^* = -[v_n^*, v_i^*]$$

for $1 \leqslant i \leqslant n-1$. In particular $\mathfrak{r}_n, \mathfrak{h}_n$ are both Lie algebras and hence they both belong to $\mathcal{K}_n^a(\mathbb{F})$. Note that $\mathfrak{r}_n$ is the only algebra structure in $\mathcal{K}_n^a(\mathbb{F})$ other than $\mathfrak{a}_n$ which satisfies the condition that the product of any two of its elements is a linear combination of the same two elements.

We consider the natural "change of basis" action of G on $\mathbb{F}^{n^3}$, regarded as a space of structure vectors (we say that

$$\lambda \in \mathbb{F}^{n^3}$$

is a structure vector for $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ if there exists a basis of V relative to which λ is the vector of structure constants for $\mathfrak{g}$). The algebra structure $\mathfrak{g}$ degenerates to $\mathfrak{g}_1$ ($\mathfrak{g}, \mathfrak{g}_1 \in \mathcal{A}_n(\mathbb{F})$) if there exist λ, μ structure vectors for $\mathfrak{g}, \mathfrak{g}_1$ respectively ($\lambda, \mu \in \mathbb{F}^{n^3}$) such that μ belongs to

the Zariski-closure of the orbit of λ resulting from the above action.

One can easily observe that if $\mathbb{F}$ is infinite then every $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ degenerates to $\mathfrak{a}_n$. Our first result deals with the problem of determining the isomorphism classes of algebras in $\mathcal{K}_n^a(\mathbb{F})$ that have $\mathfrak{a}_n$ as their only proper degeneration in the case $\mathbb{F}$ is an arbitrary infinite field. Note that the special case $\mathbb{F} = \mathbb{C}$ is already settled in Theorem 1 of [7]; see also Theorem 5.2 of [19] for the case $\mathbb{F} = \mathbb{R}$.

Theorem 1.1   *Let $n \geqslant 3$ and let $\mathbb{F}$ be an arbitrary infinite field. Then, among all n-dimensional algebras satisfying the identity $[x, x] = 0$, the algebras $\mathfrak{r}_n$ and $\mathfrak{h}_n$ are the only ones (up to isomorphism) which have the n-dimensional abelian Lie algebra $\mathfrak{a}_n$ as their only proper degeneration.*

Before stating the next result we need to introduce some more notation. Let
$$\delta_n = (\delta_{ijk}) \quad \text{and} \quad \varepsilon_n(\alpha) = (\varepsilon_{ijk}(\alpha)),$$

for $\alpha \in \mathbb{F}$, be the elements of $\mathbb{F}^{n^3}$ which are defined by $\delta_{112} = 1_{\mathbb{F}}$ (all other δ_{ijk} being zero) and

$$\varepsilon_{111}(\alpha) = 1_{\mathbb{F}}, \ \varepsilon_{1ii}(\alpha) = \alpha, \ \varepsilon_{i1i}(\alpha) = (1_{\mathbb{F}} - \alpha)$$

for $2 \leqslant i \leqslant n$ (all other $\varepsilon_{ijk}(\alpha)$ being zero). Also, for $\alpha \in \mathbb{F}$, let $\mathfrak{d}_n$ and $\mathfrak{e}_n(\alpha)$ be the elements of $\mathcal{A}_n(\mathbb{F})$ whose structure vectors relative to our fixed basis $(v_i^*)_{i=1}^n$ are δ_n and $\varepsilon_n(\alpha)$ respectively.

Theorem 1.2   *Let $n \geqslant 3$ and let $\mathbb{F}$ be an arbitrary infinite field. Then $\mathfrak{d}_n$ together with the family $\{\mathfrak{e}_n(\alpha) : \alpha \in \mathbb{F}\}$ give a complete list of non-isomorphic elements of $\mathcal{A}_n(\mathbb{F}) \setminus \mathcal{K}_n^a(\mathbb{F})$ which have $\mathfrak{a}_n$ as their only proper degeneration.*

We remark that the special case $\mathbb{F} = \mathbb{C}$ in Theorem 1.2 is already settled in Theorem 2.3 of [18].

The paper is organized as follows. In Section 2 we give some necessary background. In Section 3 we introduce the notion of degeneration and discuss some necessary conditions for an algebra to degenerate to another. Moreover, in Section 3, a key result is proved (Lemma 3.9) which enables us to "translate" results of limiting processes (involving diagonal matrices) in the metric topology of $\mathbb{C}$ or $\mathbb{R}$ to results in the Zariski topology over an arbitrary infinite field. In Section 4 we provide a completely elementary self-contained proof

of Theorem 1.1 (where we also use ideas from [18],[19]) and we discuss how the conclusion of this theorem can be used to obtain information about the composition series of $\mathcal{K}_n^a(\mathbb{F})$ (regarded as an $\mathbb{F}G$-module) via the action of $G = GL(n, \mathbb{F})$ we are considering. Finally, in Section 5 we provide a proof of Theorem 1.2.

2 Preliminaries

Fix a positive integer n with $n \geqslant 2$, an arbitrary field $\mathbb{F}$ and let $G = GL(n, \mathbb{F})$.

It will be convenient to regard the n^3 triples (i, j, k) for

$$1 \leqslant i, j, k \leqslant n$$

as an ordered n^3-tuple of triples by placing, for $1 \leqslant m \leqslant n^3$, at the mth position of this n^3-tuple the triple (i_1, j_1, k_1) where i_1, j_1, k_1 are the unique integers with $1 \leqslant i_1, j_1, k_1 \leqslant n$ satisfying

$$m - 1 = (i_1 - 1)n^2 + (j_1 - 1)n + (k_1 - 1).$$

This amounts to ordering the triples lexicographically.

Definition 2.1 (i) An n-*dimensional* $\mathbb{F}$-*algebra* (not necessarily associative) is a pair $(A, [,])$ where A is a vector space over $\mathbb{F}$ with $\dim_{\mathbb{F}} A = n$ and

$$[,] : A \times A \to A : \quad (x, y) \mapsto [x, y] \quad (x, y \in A)$$

is an $\mathbb{F}$-bilinear map (we call $[x, y]$ the product of x and y).

(ii) An *ideal* of an $\mathbb{F}$-algebra $(A, [,])$ is an $\mathbb{F}$-subspace J of A such that $[a, b]$ and $[b, a]$ belong to J for all $b \in J$ and $a \in A$.

Let $(A, [,])$ be an n-dimensional $\mathbb{F}$-algebra and suppose that

$$(b_1, \ldots, b_n) \in A^n$$

is an ordered $\mathbb{F}$-basis for the vector space A. Then, by bilinearity of the bracket, we see that the multiplication in $(A, [,])$ is completely determined by the products $[b_i, b_j]$, $1 \leqslant i, j \leqslant n$. The structure constants

of this algebra with respect to the basis $(b_1, \ldots, b_n)$ are the scalars $\alpha_{ijk} \in \mathbb{F}$ $(1 \leqslant i, j, k \leqslant n)$ given by

$$[b_i, b_j] = \sum_{k=1}^{n} \alpha_{ijk} b_k.$$

We will regard this set of structure constants α_{ijk} as an ordered n^3-tuple

$$(\alpha_{ijk})_{1 \leqslant i,j,k \leqslant n} \in \mathbb{F}^{n^3}$$

via the ordering on the triples (i, j, k) for $1 \leqslant i, j, k \leqslant n$ we have fixed above.

For example, when $n = 2$, we have

$$(\alpha_{ijk}) = (\alpha_{111}, \alpha_{112}, \alpha_{121}, \alpha_{122}, \alpha_{211}, \alpha_{212}, \alpha_{221}, \alpha_{222}) \in \mathbb{F}^8.$$

We call the n^3-tuple $\boldsymbol{\alpha} = (\alpha_{ijk}) \in \mathbb{F}^{n^3}$ the structure vector of $(A, [,])$ relative to the $\mathbb{F}$-basis $(b_1, \ldots, b_n)$ of A. More generally, we call the element

$$\boldsymbol{\beta} = (\beta_{ijk})_{1 \leqslant i,j,k \leqslant n} \in \mathbb{F}^{n^3}$$

a structure vector for $(A, [,])$ if there exists an ordered $\mathbb{F}$-basis

$$(b_1', \ldots, b_n')$$

for A relative to which the structure vector for $(A, [,])$ is $\boldsymbol{\beta} = (\beta_{ijk})$, that is

$$[b_i', b_j'] = \sum_{k=1}^{n} \beta_{ijk} b_k'$$

for $1 \leqslant i, j \leqslant n$.

Recall that the n-dimensional $\mathbb{F}$-algebras

$$(A_1, [,]_1) \quad \text{and} \quad (A_2, [,]_2)$$

are called $\mathbb{F}$-isomorphic if there exists a bijective $\mathbb{F}$-linear map

$$\Psi : A_1 \to A_2$$

such that

$$\Psi([x, y]_1) = [\Psi(x), \Psi(y)]_2$$

for all $x, y \in A_1$.

Remark 2.2 (i) Suppose that

$$\Psi : A_1 \to A_2$$

defines an isomorphism between the n-dimensional $\mathbb{F}$-algebras

$$(A_1, [,]_1) \quad \text{and} \quad (A_2, [,]_2).$$

Also let $(c_1, \ldots, c_n)$ be an $\mathbb{F}$-basis of A_1. It is easy to observe that the structure vectors of

$$(A_1, [,]_1) \quad \text{and} \quad (A_2, [,]_2)$$

relative to the $\mathbb{F}$-bases $(c_1, \ldots, c_n)$ and $(\Psi(c_1), \ldots, \Psi(c_n))$ of A_1 and A_2 respectively coincide.

(ii) Conversely, if the two n-dimensional $\mathbb{F}$-algebras $(A_1, [,]_1)$ and $(A_2, [,]_2)$ have a common structure vector, then they are necessarily $\mathbb{F}$-isomorphic. □

For the rest of the paper we fix V to be an n-dimensional $\mathbb{F}$-vector space. We also fix $(v_1^*, \ldots, v_n^*)$ to be an ordered $\mathbb{F}$-basis of V. We will be referring to this basis of V on several occasions in the sequel.

Definition 2.3 We call $\mathfrak{g}$ an *algebra structure on V* if $\mathfrak{g}$ is an $\mathbb{F}$-algebra having V as its underlying vector space (and hence has multiplication defined via a suitable $\mathbb{F}$-bilinear map $[,]_\mathfrak{g} : V \times V \to V$). We denote by $\mathcal{A}_n(\mathbb{F})$ the set of all algebra structures on V.

It is then clear from the preceding discussion that any n-dimensional $\mathbb{F}$-algebra is $\mathbb{F}$-isomorphic to an element of $\mathcal{A}_n(\mathbb{F})$. For $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}_n(\mathbb{F})$ we write

$$\mathfrak{g}_1 \simeq \mathfrak{g}_2$$

to denote the fact that $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_2$ as $\mathbb{F}$-algebras.

Also observe that if $\mathfrak{g} = (V, [,]_\mathfrak{g})$ is an algebra structure on V, we have

$$[0_V, x]_\mathfrak{g} = 0_V = [x, 0_V]_\mathfrak{g}$$

for all $x \in V$. Hence, the zero vector 0_V of V is the zero element of any $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$.

Remark 2.4   We can regard $\mathcal{A}_n(\mathbb{F})$ as an $\mathbb{F}$-vector space. For

$$\mathfrak{g}_1 = (V, [,]_1), \ \mathfrak{g}_2 = (V, [,]_2) \in \mathcal{A}_n(\mathbb{F})$$

and $\alpha \in \mathbb{F}$ we define $\mathfrak{g}_1 + \mathfrak{g}_2 = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$ where

$$[x, y] = [x, y]_1 + [x, y]_2$$

and $\alpha\mathfrak{g}_1 = (V, [,]_\alpha)$ where $[x, y]_\alpha = \alpha[x, y]_1$ for all $x, y \in V$. It is then easy to check that the maps $[,], [,]_\alpha : V \times V \to V$ are both $\mathbb{F}$-bilinear and that they turn $\mathcal{A}_n(\mathbb{F})$ into an $\mathbb{F}$-space.

   Regarding $\mathbb{F}^{n^3}$ as an $\mathbb{F}$-vector space, as usual, via the natural (componentwise) addition and scalar multiplication, we can then obtain an isomorphism of $\mathbb{F}$-vector spaces

$$\Theta : \mathcal{A}_n(\mathbb{F}) \to \mathbb{F}^{n^3}$$

such that the image of an algebra structure $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ is the structure vector of $\mathfrak{g}$ relative to the basis $(v_1^*, \ldots, v_n^*)$ of V we have fixed.   □

Definition 2.5   With the help of the bijection $\Theta : \mathcal{A}_n(\mathbb{F}) \to \mathbb{F}^{n^3}$, given in Remark 2.4 we can define a map

$$\Omega : \mathbb{F}^{n^3} \times G \to \mathbb{F}^{n^3} : \quad (\lambda, g) \mapsto \lambda g \quad (\lambda \in \mathbb{F}^{n^3}, \ g = (g_{ij}) \in G)$$

where $\lambda g \in \mathbb{F}^{n^3}$ is the structure vector of $\Theta^{-1}(\lambda) \in \mathcal{A}_n(\mathbb{F})$ relative to the basis $(v_1, \ldots, v_n)$ of V given by

$$v_j = \sum_{i=1}^{n} g_{ij} v_i^*$$

(we call $g \in G$ the transition matrix from the basis $(v_i^*)_{i=1}^n$ to the basis $(v_i)_{i=1}^n$ of V).

Remark 2.6   With the setup and notation of Definition 2.5, we see that $\Theta^{-1}(\lambda)$ is the algebra structure $\mathfrak{g}_1$ on V having λ as its structure vector relative to the basis

$$(v_1^*, \ldots, v_n^*)$$

of V we have fixed. If now $\mathfrak{g}_2 \in \mathcal{A}_n(\mathbb{F})$ also has structure vector λ

relative to some basis $(u_i)_{i=1}^n$ of V, then the structure vector of $\mathfrak{g}_2$ relative to the basis $(u_i')_{i=1}^n$ of V given by

$$u_j' = \sum_{i=1}^n g_{ij} u_i$$

is again λg. This means that the resulting vector $\lambda g \in \mathbb{F}^{n^3}$ obtained from $\lambda \in \mathbb{F}^{n^3}$ via the "change of basis" process in Definition 2.5 which is determined by the element $g = (g_{ij}) \in G$ does not depend on the choice of the pair $(\mathfrak{g} \in \mathcal{A}_n(\mathbb{F}),$ ordered basis for $V)$ that determines λ (note that in general there are more than one such pairs). $\square$

It is then easy to observe that the map Ω defines a linear right action of G on $\mathbb{F}^{n^3}$. We can thus regard $\mathbb{F}^{n^3}$ as a right $\mathbb{F}G$-module via this action.

The orbit of $\lambda \in \mathbb{F}^{n^3}$ with respect to the action given in Definition 2.5 will be denoted by $O(\lambda)$.

Note that the orbits resulting from this action correspond precisely to the isomorphism classes of n-dimensional $\mathbb{F}$-algebras (this is immediate from Remark 2.2).

We now recall briefly some basic facts on algebraic sets.

Let $\mathbb{F}[\mathbf{X}]$ be the ring

$$\mathbb{F}[X_{ijk} : 1 \leqslant i, j, k \leqslant n]$$

of polynomials in the indeterminates X_{ijk} $(1 \leqslant i, j, k \leqslant n)$ over $\mathbb{F}$. For each

$$\mu = (\mu_{ijk}) \in \mathbb{F}^{n^3}$$

we can define the evaluation map

$$\mathbf{ev}_\mu : \mathbb{F}[\mathbf{X}] \to \mathbb{F}$$

to be the unique ring homomorphism $\mathbb{F}[\mathbf{X}] \to \mathbb{F}$ such that

$$X_{ijk} \mapsto \mu_{ijk}$$

for $1 \leqslant i, j, k \leqslant n$ and which is the identity on $\mathbb{F}$. A subset W of $\mathbb{F}^{n^3}$ is algebraic (and thus closed in the Zariski topology on $\mathbb{F}^{n^3}$) if there

exists a subset $S \subseteq \mathbb{F}[\mathbf{X}]$ such that

$$W = \{\boldsymbol{\mu} = (\mu_{ijk}) \in \mathbb{F}^{n^3} : \quad \mathbf{ev}_{\boldsymbol{\mu}}(f) = 0_{\mathbb{F}} \quad \text{for all} \quad f \in S\}.$$

The Zariski closure of a subset Y of $\mathbb{F}^{n^3}$ will be denoted by $\overline{Y}$. Finally, for $U \subseteq \mathbb{F}^{n^3}$, the vanishing ideal $\mathbf{I}(U)$ of U is defined by

$$\mathbf{I}(U) = \{f \in \mathbb{F}[\mathbf{X}] : \mathbf{ev}_{\boldsymbol{\mu}}(f) = 0_{\mathbb{F}} \quad \text{for all} \quad \boldsymbol{\mu} \in U\}.$$

Next we introduce some subsets of $\mathcal{A}_n(\mathbb{F})$ which are defined via polynomial identities.

(i) $\mathcal{B}_n^a(\mathbb{F}) = \{\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F}) : [[x_1, x_2], x_3] = 0_V$ for all $x_1, x_2, x_3 \in V\}$ (this is the set of algebra structures on V which are nilpotent of class at most 2).

(ii) $\mathcal{K}_n^a(\mathbb{F}) = \{\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F}) : [x, x] = 0_V$ for all $x \in V\}$ (note that $[x, x] = 0_V$ for all $x \in V$ implies that $[x, y] = -[y, x]$ for all $x, y \in V$; in particular, for char $\mathbb{F} \neq 2$, we have that $\mathcal{K}_n^a(\mathbb{F})$ is precisely the set of skew-symmetric algebra structures on V).

(iii) $\mathcal{L}_n^a(\mathbb{F}) = \{\mathfrak{g} = (V, [,]) \in \mathcal{K}_n^a(\mathbb{F}) : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0_V$ for all $x, y, z \in V\}$ (this is the set of algebra structures in $\mathcal{K}_n^a(\mathbb{F})$ satisfying the Jacobi identity and hence it is precisely the set of Lie algebra structures on V).

Denote by $\mathcal{B}_n(\mathbb{F})$, $\mathcal{K}_n(\mathbb{F})$, $\mathcal{L}_n(\mathbb{F})$ the subsets of $\mathbb{F}^{n^3}$ which are the images of $\mathcal{B}_n^a(\mathbb{F})$, $\mathcal{K}_n^a(\mathbb{F})$, $\mathcal{L}_n^a(\mathbb{F})$ respectively via the map Θ given in Remark 2.4.

Remark 2.7   It is immediate from the way they are defined that

$$\mathcal{B}_n(\mathbb{F}), \quad \mathcal{K}_n(\mathbb{F}) \quad \text{and} \quad \mathcal{L}_n(\mathbb{F})$$

are all unions of orbits. Below we will also see that these subsets of $\mathbb{F}^{n^3}$ are algebraic (Zariski-closed) as they can be described via polynomial equations. For the rest of the discussion in this remark

$$(e_1, \ldots, e_n)$$

denotes an arbitrary $\mathbb{F}$-basis of V and $\mathfrak{g} = (V, [,])$ an element of $\mathcal{A}_n(\mathbb{F})$.

(i) To establish that $\mathcal{B}_n(\mathbb{F})$ is algebraic, first observe that $\mathfrak{g} \in \mathcal{B}_n^a(\mathbb{F})$ if, and only if, $[[e_i, e_j], e_k] = 0_V$ for $1 \leqslant i, j, k \leqslant n$. Consequently, we have

$$\mathcal{B}_n(\mathbb{F}) = \left\{ \boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^{n^3} : \sum_l \lambda_{ijl}\lambda_{lkm} = 0_{\mathbb{F}} \text{ for } 1 \leqslant i, j, k, m \leqslant n \right\}$$

showing that $\mathcal{B}_n(\mathbb{F})$ is a Zariski-closed subset of $\mathbb{F}^{n^3}$.

(ii) Also observe that $\mathfrak{g} \in \mathcal{K}_n^a(\mathbb{F})$ if, and only if, we have $[e_i, e_i] = 0_V$ and

$$[e_i, e_j] + [e_j, e_i] = 0_V$$

for $1 \leqslant i, j \leqslant n$. It follows that

$$\mathcal{K}_n(\mathbb{F}) = \{ \boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^{n^3} : \lambda_{iik} = 0_{\mathbb{F}}$$

$$\text{and } \lambda_{ijk} + \lambda_{jik} = 0_{\mathbb{F}} \text{ for all } 1 \leqslant i, j, k \leqslant n \}.$$

Clearly $\mathcal{K}_n(\mathbb{F})$ is an $\mathbb{F}$-subspace of $\mathbb{F}^{n^3}$ with

$$\dim_{\mathbb{F}} \mathcal{K}_n(\mathbb{F}) = \tfrac{1}{2} n^2(n-1)$$

and hence it is an algebraic set. Alternatively, it is easy to see that $\boldsymbol{\mu} \in \mathcal{K}_n(\mathbb{F})$ if, and only if, $\mathbf{ev}_{\boldsymbol{\mu}}(f) = 0_{\mathbb{F}}$ for all

$$f \in \{X_{iik} : 1 \leqslant i, k \leqslant n\} \cup \{X_{ijk} + X_{jik} : 1 \leqslant i, j, k \leqslant n\} \subseteq \mathbb{F}[\mathbf{X}].$$

(iii) Similar argument shows that

$$\mathcal{L}_n(\mathbb{F}) = \left\{ \boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathcal{K}_n(\mathbb{F}) : \right.$$

$$\left. \sum_k (\lambda_{ijk}\lambda_{klm} + \lambda_{jlk}\lambda_{kim} + \lambda_{lik}\lambda_{kjm}) = 0_{\mathbb{F}} \text{ for } 1 \leqslant i, j, l, m \leqslant n \right\}$$

(see also page 4 of [15]). □

Since $\mathcal{K}_n(\mathbb{F})$ is an $\mathbb{F}$-subspace of $\mathbb{F}^{n^3}$ which is also a union of orbits, it can be regarded as an $\mathbb{F}G$-module via the (linear) action of G on $\mathbb{F}^{n^3}$ we are considering.

For the rest of the paper, by an $\mathbb{F}G$-submodule of $\mathbb{F}^{n^3}$ we will mean

any subspace of $\mathbb{F}^{n^3}$ which is also an $\mathbb{F}G$-module via the action of G on $\mathbb{F}^{n^3}$ given in Definition 2.5.

Example 2.8 Let $\mathfrak{a}_n$ be the algebra structure on V with multiplication defined by $[x, y] = 0_V$ for all $x, y \in V$. Then the structure vector of $\mathfrak{a}_n$ relative to any basis of V is the zero vector

$$\mathbf{0} = (0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}})$$

of $\mathbb{F}^{n^3}$. The orbit of $\mathbf{0}$ under the action of G we have described thus consists of precisely one point and hence it is Zariski-closed. Clearly $\mathfrak{a}_n \in \mathcal{L}_n^{\mathfrak{a}}(\mathbb{F})$, in other words it is a Lie algebra structure on V. The algebra $\mathfrak{a}_n$ is known as the n-dimensional abelian Lie algebra over $\mathbb{F}$. Observe that for char $\mathbb{F} \neq 2$ the only algebra structure

$$\mathfrak{g} = (V, [,]) \in \mathcal{K}_n(\mathbb{F})$$

satisfying the commutativity relation

$$[x, y] = [y, x]$$

for all $x, y \in V$ is the abelian Lie algebra $\mathfrak{a}_n$. Note, however, that when char $\mathbb{F} = 2$, we have that every skew-symmetric algebra satisfies the commutativity relation.   □

3 Degenerations

Recall that for $\lambda \in \mathbb{F}^{n^3}$ and $g \in G$ we denote by λg the image of

$$(\lambda, g) \in \mathbb{F}^{n^3} \times G$$

under the map Ω in Definition 2.5.

For each $g \in G$ define a function

$$\Phi_g : \mathbb{F}^{n^3} \to \mathbb{F}^{n^3} : \quad \mu \mapsto \mu g (\mu \in \mathbb{F}^{n^3}).$$

Then Φ_g is a regular map for each $g \in G$ and hence continuous in the Zariski topology. To see this, fix $g \in G$. It follows from the change

of basis process that for each $\mu \in \mathbb{F}^{n^3}$ we get

$$\Phi_g(\mu) = (\mathbf{ev}_\mu(f_1), \ldots, \mathbf{ev}_\mu(f_{n^3}))$$

where, for $1 \leqslant i \leqslant n^3$, f_i is a polynomial in $\mathbb{F}[X]$ which only depends on the element $g \in G$.

The following result is elementary but for completeness we provide a proof for it.

Lemma 3.1 *Let $\lambda, \mu \in \mathbb{F}^{n^3}$ with $\mu \in \overline{O(\lambda)}$. Then*

$$O(\mu) \subseteq \overline{O(\lambda)}$$

(and hence $\overline{O(\mu)} \subseteq \overline{O(\lambda)}$).

PROOF — Assume the hypothesis and let $\mu^* \in O(\mu)$. It follows that

$$\mu^* = \mu g^*$$

for some $g^* \in G$. Suppose now that U is an open subset of $\mathbb{F}^{n^3}$ containing μ^*. It is enough to show that $O(\lambda) \cap U \neq \varnothing$. Invoking the fact that the map

$$\Phi_{g^*} : \mathbb{F}^{n^3} \to \mathbb{F}^{n^3}$$

is continuous, we see that $\Phi_{g^*}^{-1}(U)$ is an open subset of $\mathbb{F}^{n^3}$. Now

$$\mu^* = \mu g^* = \Phi_{g^*}(\mu) \in U,$$

so $\mu \in \Phi_{g^*}^{-1}(U)$. But $\mu \in \overline{O(\lambda)}$, hence there exists

$$\lambda' \in O(\lambda)$$

such that $\lambda' \in \Phi_{g^*}^{-1}(U)$. This means that

$$\Phi_{g^*}(\lambda') = \lambda' g^* \in U.$$

Hence $\lambda' g^* \in O(\lambda) \cap U$ ensuring that $O(\lambda) \cap U$ is non-empty as required. $\qquad\square$

More can be said under the assumption that $\mathbb{F}$ is algebraically closed using the theory of algebraic groups (see [1],[5],[12]). In particular, the closed orbit lemma ensures that orbits are locally closed.

Definition 3.2   Let $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}_n(\mathbb{F})$. We say that $\mathfrak{g}_1$ *degenerates* to $\mathfrak{g}_2$ (respectively, $\mathfrak{g}_1$ *properly degenerates* to $\mathfrak{g}_2$) if there exist structure vectors λ_1 of $\mathfrak{g}_1$ and λ_2 of $\mathfrak{g}_2$ such that $\lambda_2 \in \overline{O(\lambda_1)}$ (respectively, $\lambda_2 \in \overline{O(\lambda_1)} \setminus O(\lambda_1)$).

In the following remark we include some observations which are immediate from this definition.

Remark 3.3   Let $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}_n(\mathbb{F})$.

(i) We have that $\mathfrak{g}_1$ degenerates to $\mathfrak{g}_2$ if, and only if, $\mu_2 \in \overline{O(\mu_1)}$ whenever μ_1 is a structure vector of $\mathfrak{g}_1$ and μ_2 is a structure vector of $\mathfrak{g}_2$ (similarly for proper degenerations).

(ii) Suppose that $\mathfrak{g}_1^*, \mathfrak{g}_2^* \in \mathcal{A}_n(\mathbb{F})$ are such that $\mathfrak{g}_1^* \simeq \mathfrak{g}_1$ and $\mathfrak{g}_2^* \simeq \mathfrak{g}_2$. It then follows that $\mathfrak{g}_1^*$ degenerates (resp., properly degenerates) to $\mathfrak{g}_2^*$ whenever $\mathfrak{g}_1$ degenerates (resp., properly degenerates) to $\mathfrak{g}_2$.   □

It is then clear from the above definition and remark that the notion of degeneration (since it only depends on the isomorphism classes of the algebras involved) can be extended to cover all $\mathbb{F}$-algebras and not just algebra structures on our vector space V.

Example 3.4   (i): *The case $\mathbb{F}$ is a finite field.*
If $\mathbb{F}$ is finite, it is then obvious that any orbit in $\mathbb{F}^{n^3}$ under the action of G we are considering consists of a finite set of points. But points are closed sets in the Zariski topology. Consequently, when $\mathbb{F}$ is finite, all orbits in $\mathbb{F}^{n^3}$ are closed in the Zariski topology. We conclude that there exist no proper degenerations over finite fields.

(ii): *Degenerations in $\mathcal{K}_2(\mathbb{F})$.*
Let
$$\lambda = (\lambda_{ijk})_{1 \leqslant i,j,k \leqslant 2} \in \mathbb{F}^{2^3}$$
where $\lambda_{121} = 1_{\mathbb{F}} = -\lambda_{211}$, $\lambda_{122} = 1_{\mathbb{F}} = -\lambda_{212}$ (all other λ_{ijk} equal to $0_{\mathbb{F}}$); then λ is a structure vector for $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$ relative to basis (e_1, e_2) of V where

$$[e_1, e_2] = e_1 + e_2 = -[e_2, e_1] \quad \text{and} \quad [e_1, e_1] = 0_V = [e_2, e_2].$$

Note that λ is a Lie algebra structure vector. Consider the change of basis
$$e_1' = \alpha e_1, \ e_2' = \beta e_2$$

where α, β are both nonzero elements of $\mathbb{F}$. The products between the elements of this new basis are given by: $[e_1', e_1'] = 0_V = [e_2', e_2']$,

$$[e_1', e_2'] = \alpha\beta[e_1, e_2] = \alpha\beta(e_1 + e_2) = \alpha\beta(\alpha^{-1}e_1' + \beta^{-1}e_2') = \beta e_1' + \alpha e_2'$$

and $[e_2', e_1'] = -\beta e_1' - \alpha e_2'$. If, instead, we set

$$e_1' = \beta(e_1 + e_2), \ e_2' = \beta e_2$$

with $\beta \neq 0_{\mathbb{F}}$ (resp., $e_1' = \alpha e_1, \ e_2' = \alpha(e_1 + e_2)$ with $\alpha \neq 0_{\mathbb{F}}$), we get that

$$(0_{\mathbb{F}}, 0_{\mathbb{F}}, \beta, 0_{\mathbb{F}}, -\beta, 0_{\mathbb{F}}, 0_{\mathbb{F}}, 0_{\mathbb{F}}) \in O(\boldsymbol{\lambda})$$

(resp., $(0_{\mathbb{F}}, 0_{\mathbb{F}}, 0_{\mathbb{F}}, \alpha, 0_{\mathbb{F}}, -\alpha, 0_{\mathbb{F}}, 0_{\mathbb{F}}) \in O(\boldsymbol{\lambda})$). Consequently,

$$\mathcal{K}_2(\mathbb{F}) = \{(0_{\mathbb{F}}, 0_{\mathbb{F}}, \beta, \alpha, -\beta, -\alpha, 0_{\mathbb{F}}, 0_{\mathbb{F}}) : \ \alpha, \beta \in \mathbb{F}\} \subseteq O(\boldsymbol{\lambda}) \cup \{\mathbf{0}\}$$

$$\subseteq \mathcal{L}_2(\mathbb{F}) \subseteq \mathcal{K}_2(\mathbb{F}).$$

We conclude that

$$\mathcal{K}_2(\mathbb{F}) = O(\boldsymbol{\lambda}) \cup \{\mathbf{0}\}$$

and hence it is a union of two orbits. But $\mathcal{K}_2(\mathbb{F})$ is an irreducible subset of $\mathbb{F}^{n^3}$ if $\mathbb{F}$ is infinite (this follows, for example, from Example 1.1.3 and Remark 1.3.2 of [5]) verifying that $\mathbf{0} \in \overline{O(\boldsymbol{\lambda})}$ under the assumption that $\mathbb{F}$ is infinite. Moreover, the above argument shows that $\mathcal{K}_2(\mathbb{F})$ is an irreducible $\mathbb{F}G$-submodule of $\mathbb{F}^{2^3}$ for any field $\mathbb{F}$ (including finite fields). $\qquad\qquad\square$

In view of the above example, for the rest of the paper we will assume that $\mathbb{F}$ is an arbitrary infinite field.

Our next aim is to prove Lemma 3.9 which will play some part in the proof of our main result.

Definition 3.5 Let

$$S \subseteq T = \{(i, j, k) \in \mathbb{Z}^3 : \ 1 \leqslant i, j, k \leqslant n\}.$$

We define $\deg_S f$ for a non-zero monomial

$$f \in \mathbb{F}[X_{i,j,k} : (i, j, k) \in T]$$

to be the degree of the monomial obtained from f by specializing to $1_{\mathbb{F}}$ each of the indeterminates $X_{i,j,k}$ with

$$(i, j, k) \in T \setminus S$$

and which occur in f. In particular, $\deg_T f = \deg f$.

Definition 3.6 Let

$$\widehat{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n$$

be given. Also let T be as in Definition 3.5.

(i) For each $r \in \mathbb{Z}$, define the subset $S(\widehat{q}, r)$ of T by

$$S(\widehat{q}, r) = \{(i, j, k) \in T : q_i + q_j - q_k = r\}.$$

(ii) Define the $\widehat{q}$-*auxiliary degree* of a nonzero monomial

$$f \in \mathbb{F}[X_{ijk} : (i, j, k) \in T]$$

by

$$\widehat{q}\_aux\_deg(f) = \begin{cases} \sum_{r \geqslant 0} r \deg_{S(\widehat{q},r)} f, & \text{if no indeterminate } X_{ijk} \text{ with} \\ & (i, j, k) \in \cup_{r < 0} S(\widehat{q}, r) \text{ occurs in } f, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) Given a structure vector $\lambda = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$, define a new structure vector

$$\lambda(\widehat{q}) = (\lambda_{ijk}(\widehat{q})) \in \mathbb{F}^{n^3}$$

by

$$\lambda_{ijk}(\widehat{q}) = \begin{cases} \lambda_{ijk}, & \text{if } (i, j, k) \in \cup_{r \leqslant 0} S(\widehat{q}, r), \\ 0_{\mathbb{F}}, & \text{otherwise.} \end{cases}$$

It is clear from the above definition that, given $\widehat{q} \in \mathbb{Z}^n$, only finitely many of the $S(\widehat{q}, r)$ are nonempty as r runs through $\mathbb{Z}$, and T is the disjoint union of these nonempty $S(\widehat{q}, r)$.

Moreover, nonzero constants (which are monomials of degree zero) have $\widehat{q}$-auxiliary degree zero for any choice of $\widehat{q}$. However, in general, it is possible for a nonzero monomial f (with $\deg f > 0$) to have $\widehat{q}$-auxiliary degree zero for a certain choice of $\widehat{q} \in \mathbb{Z}^n$, even though none

of the indeterminates X_{ijk} with

$$(i, j, k) \in \bigcup_{r < 0} S(\widehat{q}, r)$$

occurs in f (these are precisely the monomials which are composed solely by indeterminates X_{ijk} with $(i, j, k) \in S(\widehat{q}, 0)$, provided $S(\widehat{q}, 0) \neq \varnothing$).

Example 3.7 (i) Let

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

with $q_i = 1$ for $1 \leqslant i \leqslant n$. Then $S(\widehat{q}, r) = \varnothing$ for $r \neq 1$ and $T = S(\widehat{q}, 1)$. Then, for a nonzero monomial

$$f \in \mathbb{F}[X_{ijk} : (i, j, k) \in T],$$

we have $\widehat{q}\_\text{aux}\_\deg f = \deg f$.

(ii) Fix $m \in \mathbb{Z}$ with $1 \leqslant m < n$ and let

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

with $q_i = 0$ for $1 \leqslant i \leqslant m$ and $q_i = 1$ for $m + 1 \leqslant i \leqslant n$. Then $S(\widehat{q}, r) = \varnothing$ for all $r \in \mathbb{Z} \setminus \{-1, 0, 1, 2\}$. We also have

$$S(\widehat{q}, -1) = \{(i, j, k) \in T : 1 \leqslant i, j \leqslant m \text{ and } m + 1 \leqslant k \leqslant n\}$$

and

$$S(\widehat{q}, 0) = \{(i, j, k) \in T : 1 \leqslant i, j, k \leqslant m\}$$

$$\cup \{(i, j, k) \in T : 1 \leqslant i \leqslant m \text{ and } m + 1 \leqslant j, k \leqslant n\}$$

$$\cup \{(i, j, k) \in T : 1 \leqslant j \leqslant m \text{ and } m + 1 \leqslant i, k \leqslant n\}.$$

(iii) Suppose $n \geqslant 3$ and let

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

with $q_i = 1$ for $1 \leqslant i \leqslant 2$ and $q_i = 2$ for $i \geqslant 3$. Here

$$S(\widehat{q}, 0) = \{(i, j, k) \in T : 1 \leqslant i, j \leqslant 2 \text{ and } 3 \leqslant k \leqslant n\}$$

and $S(\widehat{q}, r) = \varnothing$ for all $r \in \mathbb{Z} \setminus \{0, 1, 2, 3\}$. $\qquad\square$

Remark 3.8   Given $\widehat{q} \in \mathbb{Z}^n$ and a nonzero polynomial

$$h \in \mathbb{F}[X_{ijk} : (i, j, k) \in T],$$

it will be useful to consider the (unique) decomposition of h (relative to $\widehat{q}$) in the form $h = h_0 + h_1 + \ldots + h_m$ (with

$$h_l \in \mathbb{F}[X_{ijk} : (i, j, k) \in T]$$

for $0 \leqslant l \leqslant m$, $h_m \neq 0$), where, for each $l = 0, 1, \ldots, m$, the polynomial h_l is the sum of all nonzero monomials of $\widehat{q}$-auxiliary degree l occurring in h (if for some l, with $0 \leqslant l \leqslant m-1$, no monomial of $\widehat{q}$-auxiliary degree l occurs in h, then h_l is taken to be the zero polynomial). □

Lemma 3.9   *Let $\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$ and let $\lambda = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$ (where $\mathbb{F}$ is infinite). Suppose further that $\lambda_{ijk} = 0_{\mathbb{F}}$ whenever*

$$(i, j, k) \in \bigcup_{r < 0} S(\widehat{q}, r).$$

Then $\lambda(\widehat{q}) \in \overline{O(\lambda)}$ (in particular, the hypothesis of the lemma is satisfied regardless of the choice of λ by all $\widehat{q} \in \mathbb{Z}^n$ such that $\cup_{r<0} S(\widehat{q}, r) = \varnothing$).

Proof — Let

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

and suppose the hypothesis of the lemma is satisfied for

$$\lambda = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$$

(that is, $\lambda_{ijk} = 0_{\mathbb{F}}$ whenever $(i, j, k) \in \cup_{r<0} S(\widehat{q}, r)$). Temporarily fix $\tau \in \mathbb{F}^*$ and let

$$\lambda(\tau) = (\lambda_{ijk}(\tau)) \in \mathbb{F}^{n^3}$$

where

$$\lambda_{ijk}(\tau) = \tau^{q_i + q_j - q_k} \lambda_{ijk}$$

(as usual, we take $\tau^0 = 1_{\mathbb{F}}$ and for $m \in \mathbb{Z}$, $m < 0$, we take

$$\tau^m = (\tau^{-1})^{-m}$$

with τ^{-1} being the unique multiplicative inverse of τ in $\mathbb{F}^*$, where $\mathbb{F}^*$ denotes the set of invertible elements of $\mathbb{F}$).

Observe that if λ is the structure vector of the algebra $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ relative to the ordered $\mathbb{F}$-basis $(e_1, \ldots, e_n)$ of V, then $\lambda(\tau)$ is the structure vector of $\mathfrak{g}$ relative to a new basis $(e_1(\tau), \ldots, e_n(\tau))$ of V defined by

$$e_i(\tau) = \tau^{q_i} e_i$$

for $1 \leqslant i \leqslant n$. It follows that $\lambda(\tau) \in O(\lambda)$. It is also easy to see that for each $r \in \mathbb{Z}$, we have $\lambda_{ijk}(\tau) = \tau^r \lambda_{ijk}$ if, and only if, $(i, j, k) \in S(\widehat{q}, r)$. Consequently, our hypothesis that $\lambda_{ijk} = 0_{\mathbb{F}}$ whenever

$$(i, j, k) \in \bigcup_{r < 0} S(\widehat{q}, r)$$

ensures that

$$\lambda_{ijk}(\tau) = \lambda_{ijk}$$

whenever

$$(i, j, k) \in \bigcup_{r \leqslant 0} S(\widehat{q}, r).$$

Now let h be any nonzero polynomial in the vanishing ideal of $O(\lambda)$ and consider the decomposition $h = h_1 + \ldots + h_m$ with $h_m \neq 0$ described in Remark 3.8. By setting $\alpha_l = \mathbf{ev}_\lambda(h_l) \in \mathbb{F}$ for $0 \leqslant l \leqslant m$ we get, in view of the discussion above, that

$$\mathbf{ev}_{\lambda(\tau)}(h_l) = \tau^l \, \mathbf{ev}_\lambda(h_l) = \tau^l \alpha_l$$

for $0 \leqslant l \leqslant m$. Invoking the fact that $\lambda(\tau) \in O(\lambda)$ we get that

$$0_{\mathbb{F}} = \mathbf{ev}_{\lambda(\tau)}(h) = \mathbf{ev}_{\lambda(\tau)}(h_0) + \mathbf{ev}_{\lambda(\tau)}(h_1) + \ldots + \mathbf{ev}_{\lambda(\tau)}(h_m)$$

$$= \alpha_0 + \alpha_1 \tau + \ldots + \alpha_m \tau^m.$$

This last equality is true independent of the original choice of $\tau \in \mathbb{F}^*$ so the fact that $\mathbb{F}$ is an infinite field ensures that

$$\alpha_0 = \alpha_1 = \ldots = \alpha_m = 0_{\mathbb{F}}.$$

Next observe that it is immediate from the definition of $\lambda(\widehat{q})$ and the hypothesis in the statement of the lemma we have assumed, that

$$\mathbf{ev}_{\lambda(\widehat{q})}(h_l) = 0_{\mathbb{F}}$$

for $1 \leqslant l \leqslant m$ and that $\mathbf{ev}_{\lambda(\widehat{q})}(h_0) = \mathbf{ev}_\lambda(h_0)$. To see this last point there are two cases to consider:

(i) a nonzero monomial f occurring in h_0 contains an indeterminate X_{ijk} with $(i,j,k) \in \cup_{r<0}S(\widehat{q},r)$ and

(ii) a nonzero monomial f occurring in h_0 does not contain an indeterminate X_{ijk} with $(i,j,k) \in \cup_{r<0}S(\widehat{q},r)$ in which case it also does not contain X_{ijk} with $(i,j,k) \in \cup_{r>0}S(\widehat{q},r)$. Then clearly

$$\mathbf{ev}_{\lambda(\widehat{q})}\,f = \mathbf{ev}_\lambda\,f$$

in both cases.

We conclude that

$$\mathbf{ev}_{\lambda(\widehat{q})}(h) = \mathbf{ev}_{\lambda(\widehat{q})}(h_0) = \mathbf{ev}_\lambda(h_0) = \alpha_0 = 0_{\mathbb{F}}$$

and this is enough to complete the proof that $\lambda(\widehat{q}) \in \overline{O(\lambda)}$.   □

Example 3.10 (i) Let

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

where $q_i = 1$, $1 \leqslant i \leqslant n$. We have seen in Example 3.7 (i) that

$$\bigcup_{r<0} S(\widehat{q},r) = \varnothing.$$

Moreover, $\lambda(\widehat{q}) = \mathbf{0}$ (the structure vector of the abelian Lie algebra $\mathfrak{a}_n$) for any λ in $\mathbb{F}^{n^3}$. It is now immediate from Lemma 3.9 that any (n-dimensional) algebra $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ degenerates to $\mathfrak{a}_n$ (a well-known result). It also follows from the proof of Lemma 3.9 (with the choice of $\widehat{q}$ we have made here) that the vanishing ideal of $O(\lambda)$ is generated by homogeneous polynomials for any $\lambda \in \mathbb{F}^{n^3}$.

(ii) Suppose

$$\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$$

has an m-dimensional ideal $\mathfrak{g}_1$ for some m with $1 \leqslant m < n$. Pick a basis $(e_1,\ldots,e_n)$ for V by completing a basis $(e_1,\ldots,e_m)$ of the underlying space for $\mathfrak{g}_1$ and let λ be the structure vector of $\mathfrak{g}$ relative to $(e_1,\ldots,e_n)$. Also let

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

with $q_i = 0$ for $1 \leqslant i \leqslant m$ and $q_i = 1$ for $m+1 \leqslant i \leqslant n$. Comparing with Example 3.7 (ii) we see that with this choice of λ and $\widehat{q}$ the hypothesis of Lemma 3.9 is satisfied. Also observe that $\lambda_{rst} = 0_{\mathbb{F}}$ whenever

$$(r, s, t) \in \{(i, j, k) \in T : 1 \leqslant i \leqslant m \text{ and } m+1 \leqslant j, k \leqslant n\}$$

$$\cup \{(i, j, k) \in T : 1 \leqslant j \leqslant m \text{ and } m+1 \leqslant i, k \leqslant n\}$$

since $\mathfrak{g}_1$ is an ideal. We conclude that $\lambda(\widehat{q}) \in \overline{O(\lambda)}$. But $\lambda(\widehat{q})$ is a structure vector for $\mathfrak{g}_2 \in \mathcal{A}_n(\mathbb{F})$, where $\mathfrak{g}_2$ is isomorphic to $\mathfrak{g}_1 \oplus \mathfrak{a}_{n-m}$ (algebra direct sum). Later on we will see that this result is not true in general if we only assume that $\mathfrak{g}_1$ is a subalgebra of $\mathfrak{g}$. $\qquad\square$

3.1 Some necessary conditions

Various authors (see for example [10],[21],[23],[24],[25]) have considered necessary conditions for degenerations within special classes of algebras. In this subsection we consider some of these conditions in the more general context of algebras over an arbitrary field.

Definition 3.11 Let $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$. Define the *left annihilator* of $\mathfrak{g}$ by
$$\text{ann}_L \, \mathfrak{g} = \{c \in V : [c, a] = 0_V \text{ for all } a \in V\}.$$

Similarly we can define the *right annihilator* $\text{ann}_R \, \mathfrak{g}$. The two-sided annihilator of $\mathfrak{g}$ is defined by $\text{ann} \, \mathfrak{g} = \text{ann}_L \, \mathfrak{g} \cap \text{ann}_R \, \mathfrak{g}$.

Note that $\text{ann}_L \, \mathfrak{g}$, $\text{ann}_R \, \mathfrak{g}$ and $\text{ann} \, \mathfrak{g}$ are all $\mathbb{F}$-subspaces of V. In fact, $\text{ann} \, \mathfrak{g}$ is an ideal of $\mathfrak{g}$. Also observe that if $\mathfrak{g} \in \mathcal{K}_n^a(\mathbb{F})$ then

$$\text{ann}_L \, \mathfrak{g} = \text{ann} \, \mathfrak{g} = \text{ann}_R \, \mathfrak{g}.$$

In particular, $\text{ann}_L \, \mathfrak{g} = Z(\mathfrak{g})$, the center of $\mathfrak{g}$, when $\mathfrak{g} \in \mathcal{L}_n^a(\mathbb{F})$.

Remark 3.12 If
$$\mathfrak{g} = (V, [,]_{\mathfrak{g}}) \in \mathcal{K}_n^a(\mathbb{F})$$

then $\dim_{\mathbb{F}}(\text{ann} \, \mathfrak{g}) \neq n-1$.

Indeed, if
$$\dim_{\mathbb{F}} \text{ann}(\mathfrak{g}) = n-1$$

we can then complete a basis $(e_i)_{i=1}^{n-1}$ of $\text{ann} \, \mathfrak{g}$ to a basis $(e_i)_{i=1}^{n}$ of V. But then
$$e_n \in \text{ann} \, \mathfrak{g}$$

since

$$[e_n, e_i]_\mathfrak{g} = 0_V = [e_i, e_n]_\mathfrak{g}$$

for all $1 \leqslant i \leqslant n$, which is a contradiction. In particular, whenever $\mathfrak{g} \in \mathcal{L}_n^\mathfrak{a}(\mathbb{F})$ with $\mathfrak{g} \neq \mathfrak{a}_n$ then $\dim_\mathbb{F} Z(\mathfrak{g}) \leqslant n - 2$. $\qquad\square$

Proposition 3.13 (see Theorem 19.11 at page 153 of [3])   *Let*

$$r, s, t \in \mathbb{Z}$$

with $r \geqslant 1$, $s \geqslant 1$ and $t \geqslant 0$. Also let $\widetilde{\mathfrak{a}} \in \mathbb{F}^{r \times s}$. Then $\operatorname{rank} \widetilde{\mathfrak{a}} \leqslant t$ if, and only if, all k-rowed minor determinants of $\widetilde{\mathfrak{a}}$ are zero for $k > t$.

Definition 3.14   Let

$$\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^{n^3}.$$

We define the $n \times n^2$ matrices

$$\widetilde{\mathfrak{a}}(\boldsymbol{\lambda}) = (\widetilde{\alpha}_{lm})_{1 \leqslant l \leqslant n, 1 \leqslant m \leqslant n^2}$$

and

$$\widetilde{\mathfrak{b}}(\boldsymbol{\lambda}) = (\widetilde{\beta}_{lm})_{1 \leqslant l \leqslant n, 1 \leqslant m \leqslant n^2}$$

in $\mathbb{F}^{n \times n^2}$ as follows: The coefficient $\widetilde{\alpha}_{lm}$ (resp., $\widetilde{\beta}_{lm}$) is the entry in position

$$(l-1)n^2 + m \quad (\text{resp.}, (m-1)n + l)$$

of the ordered n^3-tuple (λ_{ijk}) (relative to the ordering we have fixed at the beginning).

For example, for $n = 3$ we have that

$$\widetilde{\mathfrak{a}}(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_{111} & \lambda_{112} & \lambda_{113} & \lambda_{121} & \lambda_{122} & \lambda_{123} & \lambda_{131} & \lambda_{132} & \lambda_{133} \\ \lambda_{211} & \lambda_{212} & \lambda_{213} & \lambda_{221} & \lambda_{222} & \lambda_{223} & \lambda_{231} & \lambda_{232} & \lambda_{233} \\ \lambda_{311} & \lambda_{312} & \lambda_{313} & \lambda_{321} & \lambda_{322} & \lambda_{323} & \lambda_{331} & \lambda_{332} & \lambda_{333} \end{bmatrix}.$$

Note that $\boldsymbol{\lambda} = (\lambda_{ijk})$ can be recovered from $\widetilde{\mathfrak{a}}(\boldsymbol{\lambda})$ by placing its rows one next to the other (starting from the first row and ending with the last one). Also, for $n = 3$,

$$\widetilde{\mathfrak{b}}(\boldsymbol{\lambda}) = \begin{bmatrix} \lambda_{111} & \lambda_{121} & \lambda_{131} & \lambda_{211} & \lambda_{221} & \lambda_{231} & \lambda_{311} & \lambda_{321} & \lambda_{331} \\ \lambda_{112} & \lambda_{122} & \lambda_{132} & \lambda_{212} & \lambda_{222} & \lambda_{232} & \lambda_{312} & \lambda_{322} & \lambda_{332} \\ \lambda_{113} & \lambda_{123} & \lambda_{133} & \lambda_{213} & \lambda_{223} & \lambda_{233} & \lambda_{313} & \lambda_{323} & \lambda_{333} \end{bmatrix}.$$

Note that the "transpose" of $\boldsymbol{\lambda}$ can be recovered from $\widetilde{\mathfrak{b}}(\boldsymbol{\lambda})$ by placing the columns of $\widetilde{\mathfrak{b}}(\boldsymbol{\lambda})$ one below the other (starting from the first

column and ending with the last one).

Remark 3.15 It is then immediate from Proposition 3.13 that the sets

$$\{\lambda \in \mathbb{F}^{n^3} : \text{rank } \widetilde{a}(\lambda) \leqslant t\}$$

and

$$\{\lambda \in \mathbb{F}^{n^3} : \text{rank } \widetilde{b}(\lambda) \leqslant t\}$$

are both closed subsets of $\mathbb{F}^{n^3}$ in the Zariski topology for any non-negative integer t.

For

$$\mathfrak{g} = (V, [,]_{\mathfrak{g}}) \in \mathcal{A}_n(\mathbb{F})$$

we denote by $\mathfrak{g}^2$ the $\mathbb{F}$-subspace of $\mathfrak{g}$ spanned by all products of the form $[x, y]_{\mathfrak{g}}$ with $x, y \in V$.

Lemma 3.16 *Let* $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$ *and let* λ *be any structure vector of* $\mathfrak{g}$. *Then,*

 (i) $\dim_{\mathbb{F}}(\text{ann}_L \mathfrak{g}) = n - \text{rank } \widetilde{a}(\lambda)$

 (ii) $\dim_{\mathbb{F}} \mathfrak{g}^2 = \text{rank } \widetilde{b}(\lambda)$

PROOF — Assume the hypothesis. Then there exists an $\mathbb{F}$-basis $(e_i)_{i=1}^n$ of V relative to which λ is the structure vector of $\mathfrak{g}$.

 (i) Let

$$\Psi : \mathbb{F}^n \to \mathbb{F}^{n^2} : \widetilde{x} \mapsto \widetilde{x}\widetilde{a}(\lambda), \quad (\widetilde{x} \in \mathbb{F}^n)$$

(here we identify $\mathbb{F}^r$ with $\mathbb{F}^{1 \times r}$). It is then easy to see that

$$\alpha_1 e_1 + \ldots + \alpha_n e_n \in \text{ann}_L \mathfrak{g} \quad (\alpha_i \in \mathbb{F})$$

if, and only if,

$$(\alpha_1, \ldots, \alpha_n) \in \ker \Psi.$$

Moreover,

$$\dim_{\mathbb{F}} \text{im } \Psi = \text{rank } \widetilde{a}(\lambda)$$

(since im Ψ is the $\mathbb{F}$-span of the rows of $\widetilde{a}(\lambda)$). We conclude that

$$\dim_{\mathbb{F}}(\text{ann}_L \mathfrak{g}) = \dim_{\mathbb{F}} \ker \Psi = n - \text{rank } \widetilde{a}(\lambda).$$

 (ii) This is immediate once we observe that $\mathfrak{g}^2$ is the $\mathbb{F}$-span of all the products of the form $[e_i, e_j]$ for $1 \leqslant i, j \leqslant n$. □

Lemma 3.17   *Let* $\mathfrak{g}, \mathfrak{g}_1 \in \mathcal{A}_n(\mathbb{F})$ *and suppose that* $\mathfrak{g}$ *degenerates to* $\mathfrak{g}_1$. *Then,*

$$\dim_{\mathbb{F}}(\operatorname{ann}_L \mathfrak{g}_1) \geqslant \dim_{\mathbb{F}}(\operatorname{ann}_L \mathfrak{g})$$

and $\dim_{\mathbb{F}} \mathfrak{g}_1^2 \leqslant \dim_{\mathbb{F}} \mathfrak{g}^2$.

PROOF — Assume the hypothesis and let λ and ν be structure vectors of $\mathfrak{g}$ and $\mathfrak{g}_1$ respectively. Also let

$$S_1(\lambda) = \{\mu \in \mathbb{F}^{n^3} : \ \operatorname{rank} \widetilde{\mathfrak{a}}(\mu) \leqslant \operatorname{rank} \widetilde{\mathfrak{a}}(\lambda)\}$$

and

$$S_2(\lambda) = \{\mu \in \mathbb{F}^{n^3} : \ \operatorname{rank} \widetilde{\mathfrak{b}}(\mu) \leqslant \operatorname{rank} \widetilde{\mathfrak{b}}(\lambda)\}.$$

In the argument below $S(\lambda)$ can be any one of $S_1(\lambda)$, $S_2(\lambda)$. It is clear from Lemma 3.16 that $S(\lambda)$ is a union of orbits, in particular

$$O(\lambda) \subseteq S(\lambda).$$

Moreover, the set $S(\lambda)$ is Zariski-closed (see Remark 3.15). It follows that

$$\overline{O(\lambda)} \subseteq S(\lambda)$$

and hence $\nu \in S(\lambda)$ since we have assumed that $\nu \in \overline{O(\lambda)}$. Hence

$$\operatorname{rank} \widetilde{\mathfrak{a}}(\nu) \leqslant \operatorname{rank} \widetilde{\mathfrak{a}}(\lambda)$$

and $\operatorname{rank} \widetilde{\mathfrak{b}}(\nu) \leqslant \operatorname{rank} \widetilde{\mathfrak{b}}(\lambda)$. Invoking Lemma 3.16 again we get the desired result.   □

4 Orbit closures in $\mathcal{K}_n(\mathbb{F})$ consisting of precisely two orbits

We continue with our assumption that $\mathbb{F}$ is an arbitrary infinite field. For this section we also assume that $n \geqslant 3$ (compare with Example 3.4).

Our next aim is to provide a proof of Theorem 1.1. This is achieved via a sequence of lemmas. We also discuss how this theorem can be used in order to obtain information about the composition series of the $\mathbb{F}G$-module $\mathcal{K}_n(\mathbb{F})$.

First we introduce the algebra structures $\mathfrak{r}_n$ and $\mathfrak{h}_n \in \mathcal{L}_n^a(\mathbb{F})$.

Definition 4.1   Let

$$\rho = (\rho_{ijk}) \quad \text{and} \quad \eta = (\eta_{ijk}) \in \mathbb{F}^{n^3}$$

be such that the only nonzero components of ρ are

$$\rho_{ini} = 1_{\mathbb{F}} = -\rho_{nii}$$

for $1 \leqslant i \leqslant n-1$ and the only nonzero components of η are

$$\eta_{123} = 1_{\mathbb{F}} = -\eta_{213}.$$

It is easy to observe that $\rho, \eta \in \mathcal{L}_n(\mathbb{F})$. Now define $\mathfrak{r}_n, \mathfrak{h}_n \in \mathcal{L}_n^a(\mathbb{F})$ by

$$\mathfrak{r}_n = \Theta^{-1}(\rho) \quad \text{and} \quad \mathfrak{h}_n = \Theta^{-1}(\eta)$$

with Θ as in Remark 2.4.

We have chosen the simpler notation ρ, η instead of the more precise notation ρ_n, η_n as n will not vary within our arguments (we work within a fixed n with $n \geqslant 3$).

Remark 4.2   Keeping the notation of the previous definition, we see that ρ is the structure vector of the algebra

$$\mathfrak{r}_n = (V, [,]_{\mathfrak{r}_n}) \in \mathcal{L}_n^a(\mathbb{F})$$

relative to the basis $(v_i^*)_{i=1}^n$ of V we have fixed, where the only nonzero products between elements of this basis are

$$[v_i^*, v_n^*]_{\mathfrak{r}_n} = v_i^* = -[v_n^*, v_i^*]_{\mathfrak{r}_n}$$

with $1 \leqslant i \leqslant n-1$.

Moreover, η is the structure vector, again relative to the basis $(v_i^*)_{i=1}^n$ of V, for the algebra

$$\mathfrak{h}_n = (V, [,]_{\mathfrak{h}_n}) \in \mathcal{L}_n^a(\mathbb{F}),$$

where the nonzero products between the elements of this basis are

$$[v_1^*, v_2^*]_{\mathfrak{h}_n} = v_3^* = -[v_2^*, v_1^*]_{\mathfrak{h}_n}.$$

Note that $\mathfrak{h}_3$ is isomorphic to the Heisenberg algebra and that $\mathfrak{h}_n$ is isomorphic to the Lie algebra direct sum $\mathfrak{h}_3 \oplus \mathfrak{a}_{n-3}$. $\qquad\square$

We denote by $\mathbb{F}\text{-sp}(x_1, \ldots, x_n)$ the set of $\mathbb{F}$-linear combinations of the elements $x_1, \ldots, x_n$ of V.

Definition 4.3   Let $\mathfrak{g} = (V, [,]_\mathfrak{g}) \in \mathcal{A}_n(\mathbb{F})$.

(i) We say that $\mathfrak{g}$ satisfies condition $(*)$ if $[x, y]_\mathfrak{g} \in \mathbb{F}\text{-sp}(x, y)$ for all $x, y \in V$.

(ii) We say that $\mathfrak{g}$ satisfies condition $(**)$ if $[x, x]_\mathfrak{g} \in \mathbb{F}\text{-sp}(x)$ for all $x \in V$.

It is then immediate that every $\mathfrak{g} \in \mathcal{K}_n^\mathfrak{a}(\mathbb{F})$ satisfies condition $(**)$. Moreover, $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$ satisfies condition $(**)$ whenever $\mathfrak{g}$ satisfies condition $(*)$ but the converse is not true in general.

It is also clear from the above definition that if $\mathfrak{g}, \mathfrak{g}_1 \in \mathcal{A}_n(\mathbb{F})$ are isomorphic and in addition $\mathfrak{g}$ satisfies condition $(*)$, then $\mathfrak{g}_1$ also satisfies condition $(*)$. Hence, the subset

$$\{\lambda \in \mathbb{F}^{n^3} : \lambda \text{ is a structure vector for some } \mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$$

$$\text{that satisfies condition } (*)\}$$

of $\mathbb{F}^{n^3}$ is a union of orbits.

Finally we remark that $\mathfrak{h}_n$ does not satisfy condition $(*)$ whereas $\mathfrak{a}_n$, the n-dimensional abelian Lie algebra, satisfies condition $(*)$. Later on we will show that $\mathfrak{r}_n$ also satisfies condition $(*)$ and even more, that any algebra $\mathfrak{g} \in \mathcal{K}_n^\mathfrak{a}(\mathbb{F})$ which satisfies condition $(*)$ is in fact isomorphic to either $\mathfrak{a}_n$ or $\mathfrak{r}_n$.

We first focus attention on algebras not satisfying condition $(*)$. In the proof of the following lemma we use ideas from the proofs of Theorem 5.2 in [19] and Proposition 2.2 in [18].

Lemma 4.4   *Let $\mathfrak{g} = (V, [,]_\mathfrak{g}) \in \mathcal{A}_n(\mathbb{F})$ and suppose that $\mathfrak{g}$ satisfies condition $(**)$. Suppose further that $\mathfrak{g}$ does not satisfy condition $(*)$. Then $\mathfrak{g}$ degenerates to $\mathfrak{h}_n$. In particular, the hypothesis of the lemma is satisfied by any $\mathfrak{g} \in \mathcal{K}_n^\mathfrak{a}(\mathbb{F})$ which does not satisfy condition $(*)$.*

PROOF — Assume the hypothesis. Then there exist elements $x, y \in V$ such that

$$[x, y]_{\mathfrak{g}} \notin \mathbb{F}\text{-sp}(x, y).$$

Observe that the set

$$\{x, y, [x, y]_{\mathfrak{g}}\}$$

is $\mathbb{F}$-linearly independent. To see this we also need to invoke the fact that $\mathfrak{g}$ satisfies condition $(**)$ so, in particular, y cannot be a scalar multiple of x otherwise we would have that

$$[x, y]_{\mathfrak{g}} \in \mathbb{F}\text{-sp}(x) \subseteq \mathbb{F}\text{-sp}(x, y).$$

We can thus complete this set to a basis $e_1 = x$, $e_2 = y$, $e_3 = [x, y]_{\mathfrak{g}}$, $e_4, \ldots, e_n$ of V (recall we assume $n \geqslant 3$). Let

$$\lambda = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$$

be the structure vector of $\mathfrak{g}$ relative to $(e_i)_{i=1}^n$. Then

$$\lambda_{123} = 1_{\mathbb{F}} \quad \text{and} \quad \lambda_{12k} = 0_{\mathbb{F}}$$

for all $k > 3$. Moreover,

$$\lambda_{213} = -1_{\mathbb{F}} \quad \text{and} \quad \lambda_{21k} = 0_{\mathbb{F}}$$

for all $k > 3$ (this follows from the fact that

$$[e_1, e_2]_{\mathfrak{g}} + [e_2, e_1]_{\mathfrak{g}} \in \mathbb{F}\text{-sp}(e_1, e_2)$$

in view of the hypothesis that $\mathfrak{g}$ satisfies condition $(**)$). Finally note that

$$\lambda_{11k} = \lambda_{22k} = 0_{\mathbb{F}}$$

for all $k \geqslant 3$.

We want to use Lemma 3.9 in order to complete the proof, so let

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

where $q_i = 1$ for $1 \leqslant i \leqslant 2$ and $q_i = 2$ for $i \geqslant 3$. Comparing with Example 3.7 (iii), we see that $\lambda(\widehat{q}) = \eta$ and, moreover, the hypothesis of Lemma 3.9 is satisfied. We conclude that $\eta \in \overline{O(\lambda)}$. $\square$

Next we introduce some more subsets of $\mathbb{F}^{n^3}$. Let

$$P = \{\boldsymbol{\lambda} = (\lambda_{ijk})_{1 \leqslant i,j,k \leqslant n} \in \mathcal{K}_n(\mathbb{F}) : \lambda_{ijk} = 0_{\mathbb{F}} \text{ whenever } k \notin \{i,j\}$$

$$\text{and } \lambda_{iji} = \lambda_{kjk} \text{ whenever } j \notin \{i,k\}\}.$$

We also denote by $\rho(\mathbb{F}G)$ and $\eta(\mathbb{F}G)$ the $\mathbb{F}G$-submodules of $\mathbb{F}^{n^3}$ generated by ρ and η respectively.

It is easy to see that P is an $\mathbb{F}$-subspace of $\mathcal{K}_n(\mathbb{F})$ (and also an algebraic subset of $\mathbb{F}^{n^3}$). To compute $\dim_{\mathbb{F}} P$, first observe that at most $2n(n-1)$ of the "positions" $(i,j,k) \in T$ (with T as in Definition 3.5) can afford coefficients λ_{ijk} which can possibly be nonzero. This is because for each fixed j only the coefficients λ_{iji} and λ_{ijj} (with $i \neq j$) can possibly be nonzero. Now the condition $\lambda_{iji} = \lambda_{kjk}$ whenever $j \notin \{i,k\}$ forces

$$\lambda_{i1i} = \alpha_1 \ (i \neq 1), \ \lambda_{i2i} = \alpha_2 \ (i \neq 2), \ldots, \lambda_{ini} = \alpha_n \ (i \neq n)$$

for some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. Invoking the fact that

$$\lambda_{ijj} = -\lambda_{jij} = -\alpha_i$$

for $j \neq i$, we conclude that $\dim_{\mathbb{F}} P = n$.

Lemma 4.5 $P = O(\rho) \cup \{\mathbf{0}\} = \overline{O(\rho)} = \rho(\mathbb{F}G)$.

PROOF — We keep the notation about ρ and $\mathfrak{r}_n$ we have fixed in Remark 4.2. In particular, we have that

$$[v_i^*, v_n^*]_{\mathfrak{r}_n} = v_i^* = -[v_n^*, v_i^*]_{\mathfrak{r}_n} \quad (1 \leqslant i \leqslant n-1)$$

are the only nonzero products between the elements of our basis $(v_i^*)_{i=1}^n$. Consider now the new basis $(u_i)_{i=1}^n$ of V where

$$u_i = \sum_{j=1}^n \beta_{ji} v_j^*$$

(with β_{ij} for $1 \leqslant i,j \leqslant n$ being the (i,j)-component of an invertible $n \times n$ matrix over $\mathbb{F}$). We then have, for $i \neq j$, that

$$[u_i, u_j]_{\mathfrak{r}_n} = \beta_{nj} u_i - \beta_{ni} u_j$$

(and $[u_i, u_i]_{\mathfrak{r}_n} = 0_{\mathbb{F}}$). By setting

$$\alpha_i = \beta_{ni}$$

for $i = 1, \ldots, n$ (note that at least one of the α_i is nonzero since the matrix (β_{ij}) is invertible) and comparing with the discussion immediately before the lemma we get that

$$P = O(\boldsymbol{\rho}) \cup \{\mathbf{0}\}.$$

Invoking now the fact that P is an $\mathbb{F}$-subspace of $\mathcal{K}_n(\mathbb{F})$, we get that P is in fact an $\mathbb{F}G$-submodule of $\mathbb{F}^{n^3}$, from which the equality

$$P = \boldsymbol{\rho}(\mathbb{F}G)$$

follows easily. Finally, combining the fact that P is Zariski-closed with the fact that $\mathbf{0} \in \overline{O(\boldsymbol{\rho})}$ we get that $P = \overline{O(\boldsymbol{\rho})}$. □

We remark in passing that if $\mathfrak{s}$ is the subalgebra of $\mathfrak{r}_n$ having $\mathbb{F}$-sp(v_1^*, v_n^*) as its underlying vector space, then a consequence of Lemma 4.5 is that there is no degeneration from $\mathfrak{r}_n$ to $\mathfrak{s} \oplus \mathfrak{a}_{n-2}$ (Lie algebra direct sum). However, this is not a counterexample to the result in Example 3.10 (ii) as $\mathfrak{s}$ in fact is not an ideal of $\mathfrak{r}_n$.

Corollary 4.6 (i) $\mathfrak{r}_n$ *satisfies condition* $(*)$.

(ii) *Let* $\boldsymbol{\lambda} \in \mathbb{F}^{n^3}$. *Then* $\boldsymbol{\lambda}$ *belongs to* P *if, and only if,* $\boldsymbol{\lambda}$ *is a structure vector for some algebra* $\mathfrak{g} \in \mathcal{K}_n^a(\mathbb{F})$ *which satisfies condition* $(*)$.

PROOF — (i) Let $x, y \in V$. Keeping the notation we have fixed in Remark 4.2 we see that if $y \in \mathbb{F}$-sp(x) then $[x, y]_{\mathfrak{r}_n} = 0_V$ ($\in \mathbb{F}$-sp(x, y)) since $\mathfrak{r}_n \in \mathcal{K}_n^a(\mathbb{F})$. If $y \notin \mathbb{F}$-sp(x) then we can complete $\{x, y\}$ to a basis of V. But we have seen in the proof of the previous lemma that for any choice of basis $(u_i')_{i=1}^n$ of V we have

$$[u_i', u_j']_{\mathfrak{r}_n} \in \mathbb{F}\text{-sp}(u_i', u_j')$$

for all i, j. It follows that $[x, y]_{\mathfrak{r}_n} \in \mathbb{F}$-sp$(x, y)$.

(ii) For the "if" part. Suppose $\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$ is the structure vector, relative to the basis $(e_i)_{i=1}^n$ of V, of $\mathfrak{g} = (V, [,]) \in \mathcal{K}_n^a(\mathbb{F})$, where $\mathfrak{g}$ satisfies condition $(*)$. It follows from the properties we have assumed for $\mathfrak{g}$ that

$$\lambda_{iik} = 0_{\mathbb{F}} \quad \text{and} \quad \lambda_{ijk} = -\lambda_{jik}$$

for all i, j, k and that $\lambda_{ijk} = 0_{\mathbb{F}}$ whenever $k \notin \{i, j\}$. Now let $j \notin \{k, i\}$ with $k \neq i$. We then have,

$$[e_i, e_j] = \lambda_{iji} e_i + \lambda_{ijj} e_j$$

and

$$[e_k, e_j] = \lambda_{kjk} e_k + \lambda_{kjj} e_j.$$

This gives

$$[(e_i + e_k), e_j] = [e_i, e_j] + [e_k, e_j] = \lambda_{iji} e_i + \lambda_{kjk} e_k + (\lambda_{ijj} + \lambda_{kjj}) e_j.$$

But from hypothesis $[(e_i + e_k), e_j] \in \mathbb{F}\text{-sp}(e_i + e_k, e_j)$. This forces

$$\lambda_{iji} = \lambda_{kjk}$$

since e_i, e_j, e_k are three distinct elements of the above basis. We conclude that $\lambda \in P$.

The "only if" part is immediate from item (i) of this corollary and Lemma 4.5 since both $\mathfrak{r}_n$ and $\mathfrak{a}_n$ belong to $\mathcal{L}_n^a(\mathbb{F}) \subseteq \mathcal{K}_n^a(\mathbb{F})$.   □

Now $\mathfrak{r}_n$, $\mathfrak{h}_n$ are not isomorphic, so $P \cap O(\eta) = \varnothing$ in view of Lemma 4.5. Moreover, combining Lemmas 4.4, 4.5 and Corollary 4.6 with the fact that

$$\mathbf{0} \in \overline{O(\lambda)}$$

for any $\lambda \in \mathbb{F}^{n^3}$ we get the following corollary.

Corollary 4.7   *Let* $\lambda \in \mathcal{K}_n(\mathbb{F}) \setminus P$.
   (i) $\eta \in \overline{O(\lambda)}$.
   (ii) *If, in addition,* $\lambda \notin O(\eta)$, *then* $\overline{O(\lambda)}$ *contains at least 3 distinct orbits.*

In the following remark we collect some facts about the algebra $\mathfrak{h}_n$ we will use later.

Remark 4.8   (i): $\dim_{\mathbb{F}} Z(\mathfrak{h}_n) = n - 2$.
Indeed, comparing with the discussion in Remark 4.2 we easily see that

$$\{v_3^*, \dots, v_n^*\} \subseteq Z(\mathfrak{h}_n),$$

so $\mathbb{F}\text{-sp}(v_3^*, \dots, v_n^*) \subseteq Z(\mathfrak{h}_n)$. In order to conclude that

$$\dim_{\mathbb{F}} Z(\mathfrak{h}_n) = n - 2,$$

it is enough to show that $Z(\mathfrak{h}_n) \subseteq \mathbb{F}\text{-sp}(v_3^*, \ldots, v_n^*)$. But if

$$z = \alpha_1 v_1^* + \ldots + \alpha_n v_n^* \in Z(\mathfrak{h}_n) \quad (\alpha_i \in \mathbb{F}),$$

then

$$\alpha_1 v_3^* = [z, v_2^*]_{\mathfrak{h}_n} = 0_V = [v_1^*, z]_{\mathfrak{h}_n} = \alpha_2 v_3^*$$

which forces $\alpha_1 = 0_{\mathbb{F}} = \alpha_2$.

(ii): $\mathfrak{h}_n \in \mathcal{B}_n^{\mathfrak{a}}(\mathbb{F})$.
To see this, note that for $x_1, x_2 \in V$,

$$[x_1, x_2]_{\mathfrak{h}_n} = \alpha v_3^*$$

for some $\alpha \in \mathbb{F}$ which in turn shows that $[x_1, x_2]_{\mathfrak{h}_n} \in Z(\mathfrak{h}_n)$. Alternatively, we can observe that

$$\sum_l \eta_{ijl} \eta_{lkm} = 0_{\mathbb{F}}$$

for $1 \leqslant i, j, k, m \leqslant n$ which in turn shows that $\eta \in \mathcal{B}_n(\mathbb{F})$.

Lemma 4.9 *Let*

$$\mathfrak{g} = (V, [,]) \in \mathcal{K}_n^{\mathfrak{a}}(\mathbb{F}) \cap \mathcal{B}_n^{\mathfrak{a}}(\mathbb{F})$$

with $\dim_{\mathbb{F}}(\operatorname{ann} \mathfrak{g}) = n - 2$. *Then* $\mathfrak{g} \simeq \mathfrak{h}_n$.

PROOF — Assume the hypothesis. By extending a basis of $\operatorname{ann} \mathfrak{g}$ to a basis of V we get an $\mathbb{F}$-subspace decomposition

$$V = \mathbb{F}\text{-sp}(b_1, b_2) \oplus \operatorname{ann} \mathfrak{g};$$

in particular, the elements b_1, b_2 of V do not belong to $\operatorname{ann} \mathfrak{g}$, which also gives that $[b_1, b_2] \neq 0_V$. From the hypothesis,

$$[b_1, b_2] \in \operatorname{ann} \mathfrak{g}$$

since

$$[[b_1, b_2], x] = 0_V = [x, [b_1, b_2]]$$

for all $x \in V$. By completing $\{[b_1, b_2]\}$ to a basis of $\operatorname{ann} \mathfrak{g}$ we obtain a basis of V of the form $(e_i)_{i=1}^n$, where

$$e_1 = b_1, \ e_2 = b_2, \ e_3 = [b_1, b_2]$$

and

$$\text{ann}\, \mathfrak{g} = \mathbb{F}\text{-sp}(e_3, \ldots, e_n)$$

(clearly $[b_1, b_2] \notin \mathbb{F}\text{-sp}(b_1, b_2)$ since $[b_1, b_2] \in \text{ann}\, \mathfrak{g} \setminus \{0_V\}$). Finally, invoking the fact that $\mathfrak{g} \in \mathcal{K}_n^a(\mathbb{F})$ we get that the structure vector of $\mathfrak{g}$ relative to the basis $(e_i)_{i=1}^n$ is precisely η which is enough to complete the proof.   □

Lemma 4.10 *The only proper degeneration of $\mathfrak{h}_n$ is to the abelian Lie algebra $\mathfrak{a}_n$. In other words, $\overline{O(\eta)} = O(\eta) \cup \{0\}$.*

PROOF — Let $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$, with $\mathfrak{g} \neq \mathfrak{a}_n$, be a proper degeneration of $\mathfrak{h}_n$. Clearly, $\mathfrak{g} \in \mathcal{L}_n^a(\mathbb{F})$ since $\mathfrak{h}_n \in \mathcal{L}_n^a(\mathbb{F})$ and as we have seen, $\mathcal{L}_n(\mathbb{F})$ is an algebraic subset of $\mathbb{F}^{n^3}$. Our assumption $\mathfrak{g} \neq \mathfrak{a}_n$ ensures that

$$\dim_\mathbb{F} Z(\mathfrak{g}) \leqslant n - 2$$

(see Remark 3.12). On the other hand, $\dim_\mathbb{F} Z(\mathfrak{g}) \geqslant n - 2$ in view of Lemma 3.17 and Remark 4.8 (i) since there is a degeneration from $\mathfrak{h}_n$ to $\mathfrak{g}$. We conclude that

$$\dim_\mathbb{F} Z(\mathfrak{g}) = n - 2.$$

Moreover, we have $\mathfrak{g} \in \mathcal{B}_n^a(\mathbb{F})$ since $\mathfrak{h}_n \in \mathcal{B}_n^a(\mathbb{F})$ and $\mathcal{B}_n(\mathbb{F})$ is also an algebraic subset of $\mathbb{F}^{n^3}$. Invoking Lemma 4.9 we then get that

$$\mathfrak{g} \simeq \mathfrak{h}_n,$$

a contradiction. The desired result now follows from the fact that any algebra structure in $\mathcal{A}_n(\mathbb{F})$ degenerates to $\mathfrak{a}_n$ (see Example 3.10 (i)). □

We can now provide a proof of the first of our two main results.

PROOF OF THEOREM 1.1 — Combine Lemma 4.5, Corollary 4.7 and Lemma 4.10.   □

We next look at some consequences of the results so far in this section.

4.1 On the composition series of the $\mathbb{F}G$-module $\mathcal{K}_n(\mathbb{F})$

Definition 4.11   Let

$$\mathfrak{g} = (V, [,]_\mathfrak{g}) \in \mathcal{A}_n(\mathbb{F}).$$

Fix $x \in V$. We define the adjoint map in $\mathfrak{g}$ (relative to x) by

$$\mathrm{ad}_x : V \to V : y \mapsto [x, y]_{\mathfrak{g}}, \quad (y \in V).$$

Then ad_x is an $\mathbb{F}$-linear map. We say that the algebra structure $\mathfrak{g}$ is unimodular if

$$\mathrm{trace}(\mathrm{ad}_x) = 0_{\mathbb{F}}$$

for each $x \in V$.

Remark 4.12 Suppose that

$$(u_1, \ldots, u_n)$$

is an $\mathbb{F}$-basis of V and that $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F})$. Observe that if

$$\lambda = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$$

is the structure vector of $\mathfrak{g}$ relative to $(u_1, \ldots, u_n)$, then

$$\mathrm{trace}(\mathrm{ad}_{u_i}) = \sum_{j=1}^{n} \lambda_{ijj}.$$

Let $x = \alpha_1 u_1 + \ldots + \alpha_n u_n \in V$ ($\alpha_i \in \mathbb{F}$). We then have that

$$\mathrm{trace}(\mathrm{ad}_x) = \sum_{i=1}^{n} \alpha_i \mathrm{trace}(\mathrm{ad}_{u_i}).$$

It follows that $\mathfrak{g}$ is unimodular if, and only if, $\mathrm{trace}(\mathrm{ad}_{u_i}) = 0_{\mathbb{F}}$ for $1 \leqslant i \leqslant n$, and this last condition holds true if, and only if,

$$\sum_{j=1}^{n} \lambda_{ijj} = 0_{\mathbb{F}}$$

for $1 \leqslant i \leqslant n$. □

Definition 4.13 Define $\mathcal{U}_n(\mathbb{F})$ to be the set

$$\left\{ \lambda = (\lambda_{ijk}) \in \mathcal{K}_n(\mathbb{F}) : \sum_{j=1}^{n} \lambda_{ijj} = 0_{\mathbb{F}} \text{ for } i = 1, \ldots, n \right\}.$$

It is then immediate that $\mathcal{U}_n(\mathbb{F})$ is an $\mathbb{F}$-subspace of $\mathbb{F}^{n^3}$. More-over, $\mathcal{U}_n(\mathbb{F})$ is also a union of orbits in view of Remark 4.12. We can thus regard $\mathcal{U}_n(\mathbb{F})$ as an $\mathbb{F}G$-submodule of $\mathbb{F}^{n^3}$.

Next we compute $\dim_{\mathbb{F}} \mathcal{U}_n(\mathbb{F})$. Assume first that

$$\lambda = (\lambda_{ijk}) \in \mathcal{K}_n(\mathbb{F}).$$

This forces

$$\lambda_{iik} = 0 \quad \text{and} \quad \lambda_{ijk} = -\lambda_{jik}$$

for $1 \leqslant i, j, k \leqslant n$. We see that this results in λ having

$$\frac{1}{2}n^2(n-1) \, (= \dim_{\mathbb{F}} \mathcal{K}_n(\mathbb{F}))$$

"free" positions, the components in the remaining positions of λ be-ing forced. With T as in Definition 3.5 we can take for example

$$T_1 = \{(i, j, k) \in T : \ i < j\}$$

as our set of "free" positions for λ. The fact that

$$\lambda_{ijk} = -\lambda_{jik}$$

ensures that $T_2 \cup T_3$, where

$$T_2 = \{(i, j, k) \in T : i < j \text{ and } k \notin \{i, j\}\}$$

and

$$T_3 = \{(i, j, k) \in T : i \neq k \text{ and } k = j\},$$

can also be taken as the set of "free" positions for $\lambda \in \mathcal{K}_n(\mathbb{F})$. We now make the further assumption that $\lambda \in \mathcal{U}_n(\mathbb{F})$. Note that T_2 has exactly

$$\frac{1}{2}n(n-1)(n-2)$$

members and no new constraints are imposed on the corresponding coefficients by our further assumption (giving

$$\frac{1}{2}n(n-1)(n-2)$$

"free" positions for $\lambda \in \mathcal{U}_n(\mathbb{F})$ from set T_2). The set T_3 has exactly $n(n-1)$ members which can be conveniently partitioned into n disjoint classes each having $(n-1)$ members. These are the classes

$$\{(1,j,j): j \neq 1\}, \{(2,j,j): j \neq 2\}, \ldots, \{(n,j,j): j \neq n\}.$$

Now for λ to be an element of $\mathcal{U}_n(\mathbb{F})$ what is required is precisely the condition that the sum of the coefficients of λ corresponding to the positions within each one of the above classes equals $0_\mathbb{F}$. This gives $n(n-2)$ "free" positions for λ from set T_3. We conclude that

$$\dim_\mathbb{F} \mathcal{U}_n(\mathbb{F}) = \tfrac{1}{2}n(n-1)(n-2) + n(n-2) = \tfrac{1}{2}n(n-2)(n+1)$$

$$= \dim_\mathbb{F} \mathcal{K}_n(\mathbb{F}) - n = \dim_\mathbb{F} \mathcal{K}_n(\mathbb{F}) - \dim_\mathbb{F} \rho(\mathbb{F}G).$$

We are now ready to construct a spanning set for $\mathcal{U}_n(\mathbb{F})$ which we will need in the next lemma. For $1 \leqslant r,s,t \leqslant n$ with $r < s$ and $t \notin \{r,s\}$, define $\widehat{\lambda}(r,s,t)$ to be the structure vector $(\widehat{\lambda}_{ijk}) \in \mathcal{U}_n(\mathbb{F})$ where

$$\widehat{\lambda}_{ijk} = \begin{cases} 1_\mathbb{F}, & \text{if } (i,j,k) = (r,s,t), \\ -1_\mathbb{F}, & \text{if } (i,j,k) = (s,r,t), \\ 0_\mathbb{F}, & \text{otherwise.} \end{cases}$$

Moreover, for $1 \leqslant r,s,t \leqslant n$ with $s \neq t$ and $r \notin \{s,t\}$ define $\widetilde{\lambda}(r,s,t)$ to be the structure vector $(\widetilde{\lambda}_{ijk})$ where

$$\widetilde{\lambda}_{ijk} = \begin{cases} 1_\mathbb{F}, & \text{if } (i,j,k) = (r,s,s) \text{ or } (i,j,k) = (t,r,t), \\ -1_\mathbb{F}, & \text{if } (i,j,k) = (r,t,t) \text{ or } (i,j,k) = (s,r,s), \\ 0_\mathbb{F}, & \text{otherwise.} \end{cases}$$

It is then an easy consequence of the preceding discussion that the union

$$\{\widehat{\lambda}(r,s,t): 1 \leqslant r,s,t \leqslant n \text{ with } r < s \text{ and } t \notin \{r,s\}\}$$

$$\cup \{\widetilde{\lambda}(r,s,t): 1 \leqslant r,s,t \leqslant n \text{ with } s \neq t \text{ and } r \notin \{s,t\}\}$$

is in fact a spanning set for $\mathcal{U}_n(\mathbb{F})$.

Lemma 4.14 $\mathcal{U}_n(\mathbb{F}) = \eta(\mathbb{F}G)$.

PROOF — Note first that $\eta = \widehat{\lambda}(1, 2, 3)$. It follows that $\eta \in \mathcal{U}_n(\mathbb{F})$ and hence

$$\eta(\mathbb{F}G) \subseteq \mathcal{U}_n(\mathbb{F}).$$

To prove that $\mathcal{U}_n(\mathbb{F}) \subseteq \eta(\mathbb{F}G)$ it is enough to show that the above spanning set we have constructed for $\mathcal{U}_n(\mathbb{F})$ is contained in $\eta(\mathbb{F}G)$. It will be convenient in what follows to identify a permutation in the symmetric group S_n with the permutation matrix in $G = GL(n, \mathbb{F})$ obtained by applying this permutation on the rows of the identity matrix. It is then easy to check that for $g \in S_n$ and

$$\lambda = (\lambda_{ijk}) \in \mathbb{F}^{n^3},$$

we have

$$\lambda g = (\lambda'_{ijk}) \in \mathbb{F}^{n^3},$$

where

$$\lambda'_{ijk} = \lambda_{g(i),g(j),g(k)}.$$

So, starting with

$$\eta = \widehat{\lambda}(1, 2, 3)$$

we can obtain (inside $\eta(\mathbb{F}G)$) all elements of $\mathcal{U}_n(\mathbb{F})$ of the form

$$\widehat{\lambda}(r, s, t),$$

with $1 \leqslant r, s, t \leqslant n$ and such that $r < s$ and $t \notin \{r, s\}$, we have described above.

We next consider the action on η by the elementary matrix g' obtained from the identity matrix by adding the second column of the identity matrix to its third column. Then

$$\eta g' = \widehat{\lambda}(1, 2, 3)g' = \widehat{\lambda}(1, 2, 3) - \widehat{\lambda}(1, 3, 2) - \widetilde{\lambda}(1, 2, 3),$$

so

$$\widetilde{\lambda}(1, 2, 3) \in \eta(\mathbb{F}G).$$

Considering now $\widetilde{\lambda}(1, 2, 3)$ and acting successively by the permutations

$$(3\,4), (4\,5), \ldots, (n-1\,n)$$

we obtain the elements

$$\widetilde{\lambda}(1,2,4),\ \widetilde{\lambda}(1,2,5),\ldots,\widetilde{\lambda}(1,2,n)$$

inside $\eta(\mathbb{F}G)$. By setting g'' to be the n-cycle $(n\,n{-}1\,\ldots\,2\,1) \in S_n$, we see that

$$\{\widetilde{\lambda}(2,s,t): 1 \leqslant s,t \leqslant n \text{ with } s \neq t \text{ and } 2 \notin \{s,t\}\} \subseteq \eta(\mathbb{F}G).$$

Finally, it is easy to see that further applications of g'' on this last subset of elements of $\eta(\mathbb{F}G)$ generate the remaining elements of the spanning set we have constructed above. $\qquad\square$

Below we discuss the composition series of $\mathcal{K}_n(\mathbb{F})$ as an $\mathbb{F}G$-module. For this we need to consider the cases

$$\operatorname{char}\mathbb{F} \!\not|\,(n-1) \quad \text{and} \quad \operatorname{char}\mathbb{F}|(n-1)$$

separately. Recall our assumption that $\mathbb{F}$ is an arbitrary infinite field and that $n \geqslant 3$. We include a preliminary remark first.

Remark 4.15 A useful observation is that $\rho \in \mathcal{U}_n(\mathbb{F})$ if, and only if,

$$\operatorname{char}\mathbb{F}|(n-1).$$

This is easily seen by considering the basis

$$(v_1^*,\ldots,v_n^*)$$

of V we have fixed relative to which ρ is the structure vector of

$$\mathfrak{r}_n = (V,[,]_{\mathfrak{r}_n}) \in \mathcal{K}_n^a(\mathbb{F})$$

where

$$[v_i^*,v_n^*] = v_i^*$$

for $1 \leqslant i \leqslant n-1$ (see Remark 4.2). Then

$$\operatorname{trace}(\operatorname{ad}_{v_i^*}) = 0_{\mathbb{F}}$$

for $1 \leqslant i \leqslant n-1$ and $\operatorname{trace}(\operatorname{ad}_{v_n^*}) = -(n-1)\cdot 1_{\mathbb{F}}$.

Also note that $\rho(\mathbb{F}G)$ is an irreducible $\mathbb{F}G$-submodule of $\mathcal{K}_n(\mathbb{F})$ for any field $\mathbb{F}$, since, as sets, $\rho(\mathbb{F}G) = O(\rho) \cup \{0\}$. $\qquad\square$

(i): *case* char $\mathbb{F} \nmid (n-1)$.
Then
$$\rho(\mathbb{F}G) \cap \eta(\mathbb{F}G) = \{\mathbf{0}\}$$
in view of the above remark (recall $\eta(\mathbb{F}G) = \mathcal{U}_n(\mathbb{F})$ always). Moreover, our assumption on char $\mathbb{F}$ ensures that $\eta(\mathbb{F}G)$ is an irreducible $\mathbb{F}G$-module. Indeed, suppose M is a nonzero $\mathbb{F}G$-submodule of $\eta(\mathbb{F}G)$. Then there exists
$$\lambda \in M \setminus \{\mathbf{0}\}.$$
Since $\lambda \notin \rho(\mathbb{F}G)$, invoking Corollary 4.7 we see that
$$\eta \in \overline{O(\lambda)}.$$
But $\overline{O(\lambda)} \subseteq M$ since M, as an $\mathbb{F}G$-submodule of $\mathbb{F}^{n^3}$, is a Zariski-closed subset of $\mathbb{F}^{n^3}$ containing $O(\lambda)$. It follows that $\eta \in M$. We conclude that
$$M = \eta(\mathbb{F}G)$$
since
$$\eta(\mathbb{F}G) \subseteq M \subseteq \eta(\mathbb{F}G).$$
This establishes that $\eta(\mathbb{F}G)$ is irreducible when char $\mathbb{F} \nmid (n-1)$. So, in this case, $\rho(\mathbb{F}G)$ and $\eta(\mathbb{F}G)$ are both irreducible $\mathbb{F}G$-submodules of $\mathcal{K}_n(\mathbb{F})$. Moreover, their dimensions as $\mathbb{F}$-spaces add up to $\dim_{\mathbb{F}} \mathcal{K}_n(\mathbb{F})$ (see the discussion after Definition 4.13) giving
$$\mathcal{K}_n(\mathbb{F}) = \rho(\mathbb{F}G) \oplus \eta(\mathbb{F}G).$$
We conclude that in this case, $\mathcal{K}_n(\mathbb{F})$ has precisely two composition series, namely $\{\mathbf{0}\} \subseteq \rho(\mathbb{F}G) \subseteq \mathcal{K}_n(\mathbb{F})$ and $\{\mathbf{0}\} \subseteq \eta(\mathbb{F}G) \subseteq \mathcal{K}_n(\mathbb{F})$. This is because any irreducible $\mathbb{F}G$-submodule of $\mathcal{K}_n(\mathbb{F})$ which is not equal to $\rho(\mathbb{F}G)$ must necessarily be $\eta(\mathbb{F}G)$ by similar argument as above since such a module contains an element μ with $\eta \in \overline{O(\mu)}$.

(ii): *case* char $\mathbb{F} | (n-1)$.
Then $\rho \in \eta(\mathbb{F}G)$ and hence
$$\rho(\mathbb{F}G) \subseteq \eta(\mathbb{F}G).$$
Since $\rho(\mathbb{F}G)$ is always irreducible (regardless of the characteristic of $\mathbb{F}$) we get, again by similar argument as above, that $\rho(\mathbb{F}G)$ is the

only irreducible $\mathbb{F}G$-submodule of $\mathcal{K}_n(\mathbb{F})$ when $\operatorname{char}\mathbb{F}|(n-1)$ and, in addition, that every composition series for $\mathcal{K}_n(\mathbb{F})$ necessarily begins with

$$\{0\} \subseteq \rho(\mathbb{F}G) \subseteq \eta(\mathbb{F}G).$$

5 A proof of Theorem 1.2

We assume that $\mathbb{F}$ is an arbitrary infinite field and that $n \geqslant 2$ throughout this section.

Definition 5.1 Following [18] we introduce the algebra structures $\mathfrak{d}_n$ and $\mathfrak{e}_n(\alpha)$, for $\alpha \in \mathbb{F}$, as follows. Let

$$\delta_n = (\delta_{ijk}) \in \mathbb{F}^{n^3}$$

be the structure vector which has $\delta_{112} = 1_{\mathbb{F}}$ as its only nonzero component. Also, for $\alpha \in \mathbb{F}$, let

$$\varepsilon_n(\alpha) = (\varepsilon_{ijk}(\alpha)) \in \mathbb{F}^{n^3}$$

be the structure vector which has

$$\varepsilon_{111}(\alpha) = 1_{\mathbb{F}}, \; \varepsilon_{1ii}(\alpha) = \alpha$$

for $2 \leqslant i \leqslant n$ and

$$\varepsilon_{i1i}(\alpha) = (1_{\mathbb{F}} - \alpha)$$

for $2 \leqslant i \leqslant n$ as its only components which can possibly be nonzero. Finally define the algebra structures

$$\mathfrak{d}_n, \mathfrak{e}_n(\alpha) \in \mathcal{A}_n(\mathbb{F})$$

by

$$\mathfrak{d}_n = \Theta^{-1}(\delta_n)$$

and

$$\mathfrak{e}_n(\alpha) = \Theta^{-1}(\varepsilon_n(\alpha))$$

with Θ as in Remark 2.4.

We now collect some immediate consequences of this definition.

Remark 5.2   (i) Fix $\alpha \in \mathbb{F}$ and let $(\beta_1, \ldots, \beta_n) \in \mathbb{F}^n$. Also let

$$\mathfrak{g}(\alpha; \beta_1, \ldots, \beta_n) = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$$

be the algebra structure which, relative to our basis $(v_i^*)_{i=1}^n$, has the following as the only components which can possibly be nonzero:

$$[v_i^*, v_i^*] = \beta_i v_i^*, \ (1 \leqslant i \leqslant n)$$

and

$$[v_i^*, v_j^*] = \alpha \beta_i v_j^* + (1 - \alpha)\beta_j v_i^*$$

(for $1 \leqslant i, j \leqslant n$, $i \neq j$). It is easy to check that

$$O(\varepsilon_n(\alpha)) = \{\Theta(\mathfrak{g}(\alpha; \beta_1, \ldots, \beta_n)) : (\beta_1, \ldots, \beta_n) \neq (0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}})\}.$$

It follows that, for each $\alpha \in \mathbb{F}$, the union

$$O(\varepsilon_n(\alpha)) \cup \{\mathbf{0}\}$$

is an $\mathbb{F}$-subspace of $\mathbb{F}^{n^3}$, and

$$\overline{O(\varepsilon_n(\alpha))} = O(\varepsilon_n(\alpha)) \cup \{\mathbf{0}\}.$$

Moreover, for α_1, $\alpha_2 \in \mathbb{F}$ with $\alpha_1 \neq \alpha_2$ we have

$$O(\varepsilon_n(\alpha_1)) \cap O(\varepsilon_n(\alpha_2)) = \varnothing.$$

Finally note that δ_n does not belong to any of these orbits.

   (ii) The algebra structure

$$\mathfrak{d}_n = (V, [,]_{\mathfrak{d}_n}) \in \mathcal{A}_n(\mathbb{F})$$

satisfies the commutativity relation $[x, y]_{\mathfrak{d}_n} = [y, x]_{\mathfrak{d}_n}$ for all $x, y \in V$. Moreover, $\mathfrak{d}_n \in \mathcal{B}_n^a(\mathbb{F})$ and $\dim_{\mathbb{F}}(\operatorname{ann} \mathfrak{d}_n) = \dim_{\mathbb{F}}(\operatorname{ann}_L \mathfrak{d}_n) = n - 1$. □

Lemma 5.3 $\overline{O(\delta_n)} = O(\delta_n) \cup \{\mathbf{0}\}.$

Proof — Let

$$\mathfrak{g} = (V, [,]_{\mathfrak{g}}) \in \mathcal{A}_n(\mathbb{F})$$

be any degeneration of $\mathfrak{d}_n$ with $\mathfrak{g} \neq \mathfrak{a}_n$. In view of the above remark we get that $\mathfrak{g} \in \mathcal{B}_n^a(\mathbb{F})$ and also that $[x, y]_{\mathfrak{g}} = [y, x]_{\mathfrak{g}}$ for all $x, y \in V$.

Invoking Lemma 3.17 we get in addition that

$$\dim_{\mathbb{F}}(\text{ann}\,\mathfrak{g}) = \dim_{\mathbb{F}}(\text{ann}_L\,\mathfrak{g}) = n - 1.$$

We can thus complete a basis $(e_i)_{i=2}^n$ of $\text{ann}\,\mathfrak{g}$ to a basis $(e_i)_{i=1}^n$ of V. Then

$$[e_1, e_1]_{\mathfrak{g}} \neq 0_V$$

(otherwise we would have $\text{ann}\,\mathfrak{g} = V$). The fact that $\mathfrak{g} \in \mathcal{B}_n^a(\mathbb{F})$ also ensures that

$$[e_1, e_1]_{\mathfrak{g}} \in \text{ann}\,\mathfrak{g} \quad \text{and} \quad [e_1, e_1]_{\mathfrak{g}} \notin \mathbb{F}\text{-sp}(e_1).$$

Hence we can consider a basis $(v_i)_{i=2}^n$ of $\text{ann}\,\mathfrak{g}$ with $v_2 = [e_1, e_1]_{\mathfrak{g}}$. Completing to a basis $(v_i)_{i=1}^n$ of V with $v_1 = e_1$ we get that $\mathfrak{g} \simeq \mathfrak{d}_n$. $\square$

For the remaining lemmas we use ideas similar to those of Proposition 2.2 in [18]. To obtain results over an arbitrary field we also make use of Lemma 3.9.

Lemma 5.4 *Let $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F})$ and suppose $\mathfrak{g}$ does not satisfy condition $(\ast\ast)$. Then $\mathfrak{g}$ degenerates to $\mathfrak{d}_n$.*

PROOF — Assume the hypothesis and let $x \in V$ be such that

$$[x, x] \notin \mathbb{F}\text{-sp}(x).$$

Clearly $x \neq 0_V$ and we can consider a basis $(e_i)_{i=1}^n$ of V with $e_1 = x$ and $e_2 = [x, x]$. Let

$$\lambda = (\lambda_{ijk})$$

be the structure vector of $\mathfrak{g}$ relative to this basis and set

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

with $q_1 = 1$ and $q_i = 2$ for $i \geqslant 2$. Invoking Lemma 3.9 we get the desired result. $\square$

Combining the results of this section so far with Lemma 4.4 and Corollary 4.6, in order to complete the proof of Theorem 1.2 it suffices to consider the case $\mathfrak{g} \in \mathcal{A}_n(\mathbb{F}) \setminus \mathcal{K}_n^a(\mathbb{F})$ with $\mathfrak{g}$ satisfying condition $(\ast)$.

Lemma 5.5 *Let $\mathfrak{g} = (V, [,]) \in \mathcal{A}_n(\mathbb{F}) \setminus \mathcal{K}_n^a(\mathbb{F})$ and suppose $\mathfrak{g}$ satisfies condition $(\ast)$. Then $\mathfrak{g}$ degenerates to $\mathfrak{e}_n(\alpha)$ for some $\alpha \in \mathbb{F}$.*

PROOF — Assume the hypothesis. Since there exists $x \in V$ with

$$[x, x] \neq 0_V,$$

we can consider, by scaling if necessary, a basis

$$(e_i)_{i=1}^n$$

of V such that $[e_i, e_i] = e_i$ for $1 \leqslant k \leqslant n$ and $[e_i, e_i] = 0$ for $k + 1 \leqslant i \leqslant n$ (for some $k \geqslant 1$). Now let $\xi \in \mathbb{F}^*$ and fix i with $2 \leqslant i \leqslant k$. Considering the product

$$[e_1 + \xi e_i, e_1 + \xi e_i]$$

(which equals to $\alpha_\xi(e_1 + \xi e_i)$ for some $\alpha_\xi \in \mathbb{F}$) and combining with the fact that

$$[e_1, e_i] + [e_i, e_1] = \gamma_1 e_1 + \gamma_i e_i$$

for some (constant) $\gamma_1, \gamma_i \in \mathbb{F}$ we get that

$$\alpha_\xi = \gamma_i + \xi = 1_{\mathbb{F}} + \xi \gamma_1.$$

As a consequence we have that

$$(1_{\mathbb{F}} - \gamma_i) + (\gamma_1 - 1_{\mathbb{F}})\xi = 0_{\mathbb{F}}.$$

This is true for all $\xi \in \mathbb{F}^*$ from which we deduce that

$$\gamma_1 = \gamma_i = 1_{\mathbb{F}}.$$

We conclude that

$$[e_1, e_i] + [e_i, e_1] = e_1 + e_i$$

for $2 \leqslant i \leqslant k$. Moreover, setting $\xi = -1_{\mathbb{F}}$ in the above argument we get that

$$[e_1 - e_i, e_1 - e_i] = 0_{\mathbb{F}}$$

for $2 \leqslant i \leqslant k$.

Next we consider the basis $(e_i')_{i=1}^n$ of V where

$$e_1' = e_1, \ e_i' = e_1 - e_i$$

for $2 \leqslant i \leqslant k$ and $e_i' = e_i$ for $k + 1 \leqslant i \leqslant n$. By similar argument as

above (considering the product

$$[e_1' + \xi e_i', e_1' + \xi e_i']$$

for $2 \leqslant i \leqslant n$) we get that $[e_1', e_i'] + [e_i', e_1'] = e_i'$ for $2 \leqslant i \leqslant n$.

Now let
$$\boldsymbol{\lambda} = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$$

be the structure vector of $\mathfrak{g}$ relative to the basis (e_i') of V. In view of the above, we have that $\lambda_{111} = 1_{\mathbb{F}}$ and

$$\lambda_{i1i} + \lambda_{1ii} = 1_{\mathbb{F}}$$

for $2 \leqslant i \leqslant n$. But the λ_{i1i} all have the same value for $2 \leqslant i \leqslant n$ (compare with the proof of Corollary 4.6 — we use the fact that $\mathfrak{g}$ satisfies condition $(*)$). We can thus complete the proof by setting

$$\widehat{q} = (q_i)_{i=1}^n \in \mathbb{Z}^n$$

where $q_1 = 0$ and $q_i = 1$ for $2 \leqslant i \leqslant n$ in Lemma 3.9. $\qquad\square$

Finally, an easy adjustment of the above arguments ensures that for $n = 2$ the algebras $\mathfrak{d}_2$ and $\mathfrak{e}_2(\alpha)$ for $\alpha \in \mathbb{F}$ give a complete list of elements of

$$\mathcal{A}_2(\mathbb{F}) \setminus \mathcal{K}_2^{\mathfrak{a}}(\mathbb{F})$$

which have $\mathfrak{a}_2$ as their only proper degeneration.

REFERENCES

[1] A. Borel: "Linear Algebraic Groups", *Springer*, Berlin (1991).

[2] D. Burde – C. Steinhoff: "Classification of orbit closures of 4-dimensional complex Lie algebras", *J. Algebra* 214 (1999), 729–739.

[3] C. Curtis: "Linear Algebra: an Introductory Approach", *Springer*, Berlin (1984).

[4] A. Fialowski – J. O'Halloran: "A comparison of deformations and orbit closure", *Comm. Algebra* 18 (1990), 4121–4140.

[5] M. Geck: "An Introduction to Algebraic Geometry and Algebraic Groups", *Oxford University Press*, Oxford (2003).

[6] M. Gerstenhaber: "On the deformation of rings and algebras", *Ann. Math.* 79 (1964), 59–103.

[7] V.V. Gorbatsevich: "Contractions and degenerations of finite-dimensional algebras", *Izv. Vyssh. Uchebn. Zaved., Mat.* 10 (1991), 19–27; translated in: *Soviet Math. (Iz. VUZ)* 35 (1991), 17–24.

[8] V.V. Gorbatsevich: "Anticommutative finite-dimensional algebras of the first three levels of complexity", *Algebra Anal.* 5 (1993), 100–118; translated in: *St. Petersbg. Math. J.* 5 (1994), 505–521.

[9] V.V. Gorbatsevich: "On the level of some solvable Lie algebras", *Sib. Mat. Z.* 39 (1998), 1013–1027; translated in: *Sib. Math. J.* 39 (1998), 872–883.

[10] F. Grunewald – J. O'Halloran: "Varieties of nilpotent Lie algebras of dimensions less then six", *J. Algebra* 112 (1988), 315–325.

[11] F. Grunewald – J. O'Halloran: "A characterization of orbit closure and applications", *J. Algebra* 116 (1988), 163–175.

[12] J. Humphreys: "Linear Algebraic Groups", *Springer*, Berlin (1998).

[13] E. Inönü – E.P. Wigner: "On the contraction of groups and their representations", *Proc. Nat. Acad. Sci. U.S.A.* 39 (1953), 510–524.

[14] E. Inönü – E.P. Wigner: "On a particular type of convergence to a singular matrix", *Proc. Nat. Acad. Sci. U.S.A.* 40 (1954) 119–121.

[15] N. Jacobson: "Lie Algebras", *Interscience*, New York (1962).

[16] I. Kaygorodov – Yu. Popov – Yu. Volkov: "Degenerations of binary Lie and nilpotent Malcev algebras", *Comm. Algebra* 46 (2018), 4928–4940.

[17] I. Kaygorodov – Yu. Volkov: "The variety of 2-dimensional algebras over an algebraically closed field"; *arXiv*:1701.08233.

[18] A.Kh. Khudoyberdiyev – B.A. Omirov: "The classification of algebras of level one", *Linear Algebra Appl.* 439 (2013), 3460–3463.

[19] J. Lauret: "Degenerations of Lie algebras and geometry of Lie groups", *Differ. Geom. Appl.* 18 (2003), 177–194.

[20] M. Levy-Nahas: "Deformation and contraction of Lie algebras", *J. Math. Phys.* 8 (1967), 1211–1223.

[21] M. Nesterenko – R. Popovych: "Contractions of low-dimensional Lie algebras", *J. Math. Phys.* 47 (2006), 123515, 45pp.

[22] V.L. Popov: "Two orbits: when is one in the closure of the other?", *Proc. Steklov Math. Inst.* 264 (2009), 146–158.

[23] I.S. Rakhimov: "On the degenerations of finite dimensional nilpotent complex Leibniz algebras", *Zapiski Nauch. Semin. POMI* 319 (2005), 1–7.

[24] C. Seeley: "Degenerations of 6-dimensional nilpotent Lie algebras over ℂ", *Comm. Algebra* 18 (1990), 3493–3503.

[25] C. Seeley: "Degenerations of central quotients", *Arch. Math. (Basel)* 56 (1991), 236–241.

[26] I.E. Segal: "A class of operator algebras determined by groups", *Duke Math. J.* 18 (1951), 221–265.

Nataliya M. Ivanova
Institute of Mathematics
National Academy of Sciences of Ukraine
3 Tereshchenkivska Str.
01601 Kyiv (Ukraine)
e-mail: ivanova.nataliya@gmail.com

Christakis A. Pallikaros
Department of Mathematics and Statistics
University of Cyprus
PO Box 20537, 1678 Nicosia (Cyprus)
e-mail: pallikar@ucy.ac.cy