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On Sublattices of the Lattice of all ω-Composition Formations of Finite Groups *

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Abstract

It is proved that the lattice of all ω-local formations is a complete sublattice of the lattice of all ω-composition formations of finite groups.

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Keywords: finite group; formation of groups; ω-local formation; ω-composition formation; complete lattice of formations

1 Introduction

Throughout this paper, all groups are finite. Moreover, we will use ω to denote a non-empty set of primes and $\omega' = \mathbb{P} \setminus \omega$.

Recall that a *variety of groups* may be defined as a non-empty class of groups closed under taking homomorphic images and subcartesian products (see Chap. 1, Sec. 5, Remark 15.53 of [13]). In the universe of all finite groups the definition of a variety leads us to the concept of the formation: a class of finite groups $\mathfrak{F}$ which is closed under taking homomorphic images and finite subdirect products is called a *formation* (Gaschütz [6]). It is well known that the lattice of all varieties of groups is complete, modular but

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is not distributive [13]. Moreover, Skiba proved (see p.91 of [21] or p.137 of [26]) that the lattice of all varieties of locally finite groups is a sublattice of the lattice of all hereditary (in the sence of Mal'cev; see [12]) *formations.* This circumstance confirms the importance of studying of the lattices of formations.

The most useful for applications of the formation theory (in particular, in the theory of formal languages; see [2],[3],[4]) and in the theory of lattices of group classes (see [1],[5],[7],[8],[9],[10],[11],[14],[15],[16],[17],[18], [19],[21],[23],[24],[25],[26],[29],[30],[31],[32],[33],[35]) are the so-called local and Baer-local formations and their generalizations (ω-local and ω-composition formations).

Recall that the formation $\mathfrak{F}$ is said to be: *ω-local* or *ω-saturated* if $G \in \mathfrak{F}$ whenever $G/O_p(\Phi(G)) \in \mathfrak{F}$ for any prime $p \in \omega$; *ω-composition* or *solubly ω-saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(O_p(G)) \in \mathfrak{F}$ for any prime $p \in \omega$. A *local formation* is an ω-local formation, where $\omega = \mathbb{P}$ is the set of all primes. Analogously, a *composition* or *Baer-local formation* is an ω-composition formation, where $\omega = \mathbb{P}$.

It is well known that the set of all ω-local formations $\mathcal{L}^\omega$ and the set of all ω-composition formations $\mathcal{C}^\omega$ partially ordered by set inclusion are complete lattices (Skiba and Shemetkov in [22] and [27]). Moreover, it was proved [28] that the lattice of all local formations $\mathcal{L}$ is a complete sublattice of the lattice of all Baer-local formations $\mathcal{C}$.

We prove the following generalization of this result in this paper.

Theorem *The lattice $\mathcal{L}^\omega$ is a complete sublattice of the lattice $\mathcal{C}^\omega$.*

All unexplained notations and terminology are standard. The reader is referred to [1],[5],[21],[20],[22],[26],[27], if necessary.

2 Preliminaries

Recall that $\pi(G)$ denotes the set of all prime divisors of the order of a group G For any collection of groups $\mathfrak{X}$ we denote by $\mathrm{Com}(\mathfrak{X})$ the class of all simple abelian groups A such that $A \simeq H/K$, where H/K is a composition factor of $G \in \mathfrak{X}$.

Recall that the subgroup $C^p(G)$ is the intersection of the centralizers of all the abelian p-chief factors of G ($C^p(G) = G$ if G has no abelian p-chief factors).

The symbols $\mathfrak{G}$, $\mathfrak{N}_p$, $\mathfrak{G}_{p'}$, $\mathfrak{G}_\omega$ and $\mathfrak{S}_\omega$ denote the class of all groups, the class of all p-groups, the class of all p'-groups, the class of all ω-groups and the class of all soluble ω-groups, respectively. For every group class $\mathfrak{F} \supseteq (1)$, by $G_\mathfrak{F}$ we denote the product of all normal $\mathfrak{F}$-subgroups of a group G. In particular, we write

$$O_p(G) = G_{\mathfrak{N}_p}, \ O_\omega(G) = G_{\mathfrak{G}_\omega}, \ R_\omega(G) = G_{\mathfrak{S}_\omega}, \ O_{p',p}(G) = G_{\mathfrak{G}_{p'}\mathfrak{N}_p}.$$

Let f be a function of the form

$$f : \omega \cup \{\omega'\} \to \Big\{\text{formations of groups}\Big\}. \tag{1}$$

According to [22] and [27] we consider, respectively, two classes of groups

$$LF_\omega(f) = \big(G \mid G/O_\omega(G) \in f(\omega') \text{ and } G/O_{p',p}(G) \in f(p) \\ \text{for all } p \in \omega \cap \pi(G)\big)$$

and

$$CF_\omega(f) = \big(G \mid G/R_\omega(G) \in f(\omega') \text{ and } G/C^p(G) \in f(p) \\ \text{for all } p \in \omega \cap \pi(\mathrm{Com}(G))\big).$$

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = LF_\omega(f)$ for a function f of the form (1), then $\mathfrak{F}$ is said to be ω-*local* and f is said to be an ω-*local satellite* of $\mathfrak{F}$ (see [22]). If $\mathfrak{F} = LF_\omega(f)$ and $f(a) \subseteq \mathfrak{F}$ for all

$$a \in \omega \cup \{\omega'\},$$

then f is called an *inner* ω-*local satellite* of $\mathfrak{F}$. The symbol $\mathfrak{N}_p F(p)$ denotes the set of all groups A such that $A^{F(p)}$ is a p-group. According to Remark 1 of p.118 in [22], for any ω-local formation $\mathfrak{F}$, there exists a unique formation function F of the form (1) such that $\mathfrak{F} = LF_\omega(F)$ and $F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F}$ for all $p \in \omega$. The formation function F is called the *canonical* ω-*local satellite* of $\mathfrak{F}$.

If $\mathfrak{F}$ is a formation such that $\mathfrak{F} = CF_\omega(f)$ for a function f of the form (1), then $\mathfrak{F}$ is said to be ω-*composition* and f is said to be an ω-*composition satellite* of $\mathfrak{F}$ (see [27]). If $\mathfrak{F} = CF_\omega(f)$ and $f(a) \subseteq \mathfrak{F}$ for all $a \in \omega \cup \{\omega'\}$, then f is called an *inner* ω-*composition satellite* of $\mathfrak{F}$. According to Remark 1 of p.902 in [27], for any ω-composition formation $\mathfrak{F}$, there exists a unique formation function F of the form (1) such that $\mathfrak{F} = CF_\omega(F)$ and $F(p) = \mathfrak{N}_p F(p) \subseteq \mathfrak{F}$ for all $p \in \omega$. The formation function F is called the *canonical* ω-*composition satellite* of $\mathfrak{F}$.

Let Θ be a complete lattice of formations. A formation function f of the form (1) is called Θ-*valued* if all its values belong to the lattice Θ. We denote by Θ^{ω_l} the set of all formations having an ω-local Θ-valued satellite (see [22]); analogously, we denote by Θ^{ω_c} the set of all formations having an ω-composition Θ-valued satellite (see [27]).

By $\mathcal{L}^\omega$ we denote the set of all ω-local formations; by $\mathcal{C}^\omega$ we denote the set of all ω-composition formations. With respect to inclusion $\subseteq$, the sets $\mathcal{L}^\omega$ and $\mathcal{C}^\omega$ are complete lattices (see p.119 of [22] and p.904 of [27]). In the lattice $\mathcal{L}^\omega$ ($\mathcal{C}^\omega$, respectively) for arbitrary non-empty set

$$\Sigma = \{\mathcal{H}_i \mid i \in \Lambda\}$$

of its elements,

$$\bigcap_{i\in\Lambda}\mathcal{H}_i$$

is the greatest lower bound for Σ in $\mathcal{L}^\omega$ (in $\mathcal{C}^\omega$, respectively);

$$\mathcal{L}^\omega\mathrm{form}\Big(\bigcup_{i\in\Lambda}\mathcal{H}_i\Big)$$

is the least upper bound for Σ in $\mathcal{L}^\omega$ (in $\mathcal{C}^\omega$

$$\mathcal{C}^\omega\mathrm{form}\Big(\bigcup_{i\in\Lambda}\mathcal{H}_i\Big)$$

is the least upper bound, respectively). Here the symbol $\mathcal{L}^\omega\mathrm{form}(\mathfrak{X})$ $\big(\mathcal{C}^\omega\mathrm{form}(\mathfrak{X})\big)$ denotes the intersection of all ω-local (ω-composition) formations containing a collection of groups $\mathfrak{X}$.

We will use the following results in the proof of the theorem.

Lemma 2.1 (see Lemma 4 of [22]) *If* $\mathfrak{F} = LF_\omega(f)$ *and* $G/O_p(G) \in f(p) \cap \mathfrak{F}$ *for some* $p \in \omega$, *then* $G \in \mathfrak{F}$.

Lemma 2.2 (see Chapter IV, Proposition 1.5 of [5]) *Let* R/S *be a normal section of a group* G *in a formation* $\mathfrak{F}$, *and let* K *be a normal subgroup of* G *contained in* $C_G(R/S)$. *With respect to the following action of* G/K *on* R/S :

$$(rS)^{gK} = g^{-1}rgS, \qquad r \in R, \quad g \in G$$

form the semidirect product

$$H = (R/S) \rtimes (G/K).$$

Then $H \in \mathfrak{F}$.

Let Θ be a complete lattice of formations and let $\mathfrak{X} \subseteq \mathfrak{F} \in \Theta$ be a collection of groups. We write $\Theta\mathrm{form}\mathfrak{X}$ to denote the intersection of all formations of Θ containing all groups of $\mathfrak{X}$. For any collection of formations $\{\mathfrak{F}_i \mid i \in I\}$ of Θ we write

$$\vee_\Theta(\mathfrak{F}_i \mid i \in I) = \Theta\mathrm{form}\Big(\bigcup_{i\in I}\mathfrak{F}_i\Big).$$

Let $\{f_i \mid i \in I\}$ be a collection of Θ-valued functions of the form (1). Then we denote by $\vee_\Theta(f_i \mid i \in I)$ a function f such that

$$f(a) = \Theta\mathrm{form}\Big(\bigcup_{i\in I}f_i(a)\Big)$$

for all $a \in \omega \cup \{\omega'\}$.

A complete lattice of formations Θ^{ω_l} is called *inductive* (see pp.151–152 of [26]), if for any collection $\{\mathfrak{F}_i \mid i \in I\}$ of formations $\mathfrak{F}_i$ of Θ^{ω_l} and for any collection $\{f_i \mid i \in I\}$ of inner Θ-valued ω-local satellites f_i, where f_i is an ω-local satellite of $\mathfrak{F}_i$, we have

$$\vee_{\Theta^{\omega_l}}(\mathfrak{F}_i \mid i \in I) = LF_\omega\big(\vee_\Theta(f_i \mid i \in I)\big).$$

Lemma 2.3 (see Lemma 4 of [9]) *The lattice $\mathcal{L}^\omega$ is inductive.*

Analogously, a complete lattice of formations Θ^{ω_c} is called *inductive* [26], if for any collection

$$\{\mathfrak{F}_i \mid i \in I\}$$

of formations $\mathfrak{F}_i$ of Θ^{ω_c} and for any collection

$$\{f_i \mid i \in I\}$$

of inner Θ-valued ω-composition satellites f_i, where f_i is an ω-composition satellite of $\mathfrak{F}_i$, we have

$$\vee_{\Theta^{\omega_c}}(\mathfrak{F}_i \mid i \in I) = CF_\omega\big(\vee_\Theta(f_i \mid i \in I)\big).$$

Lemma 2.4 (see Theorem of [30] and Theorem 2.1 of [34]) *The lattice $\mathcal{C}^\omega$ is inductive.*

3 Proof of the Theorem

Let $\{\mathfrak{F}_i \mid i \in I\}$ be a collection of ω-local formations and let F_i be the canonical ω-local satellite of $\mathfrak{F}_i$. Let

$$\mathfrak{F} = \vee_{\mathcal{L}^\omega}(\mathfrak{F}_i \mid i \in I) \quad \text{and} \quad \mathfrak{H} = \vee_{\mathcal{C}^\omega}(\mathfrak{F}_i \mid i \in I).$$

It is clear that

$$\bigcap_{i \in I} \mathfrak{F}_i$$

is an ω-local formation and this formation is the greatest lower bound for $\{\mathfrak{F}_i \mid i \in I\}$ in $\mathcal{L}^\omega$. On the other hand, clearly, $\mathfrak{F}$ is the least upper bound for $\{\mathfrak{F}_i \mid i \in I\}$ in $\mathcal{L}^\omega$ and $\mathfrak{H}$ is the least upper bound for $\{\mathfrak{F}_i \mid i \in I\}$ in $\mathcal{C}^\omega$. Therefore, in fact, we need only prove that $\mathfrak{F} = \mathfrak{H}$. The inclusion $\mathfrak{H} \subseteq \mathfrak{F}$ is evident. Hence, we need only show that $\mathfrak{F} \subseteq \mathfrak{H}$.

Let $\mathfrak{H}_i = CF_\omega(H_i)$, where H_i is an ω-composition satellite such that

$$H_i(a) = \begin{cases} \mathfrak{F}_i, & \text{if } a = \omega', \\ F_i(a), & \text{if } a = p \in \omega. \end{cases}$$

We show that $\mathfrak{F}_i = \mathfrak{H}_i$ for all $i \in I$.

Suppose $\mathfrak{H}_i \not\subseteq \mathfrak{F}_i$. Let G be a group of minimal order in $\mathfrak{H}_i \setminus \mathfrak{F}_i$. Then G is a monolithic group and $R = G^{\mathfrak{F}_i}$ is the monolith of G.

If $R_\omega(G) = 1$, then

$$G \simeq G/1 = G/R_\omega(G) \in H_i(\omega') = \mathfrak{F}_i,$$

a contradiction.

Hence, $R_\omega(G) \neq 1$, i.e., R is a p-group for some $p \in \omega$. Since $\mathfrak{F}_i$ is ω-saturated, it follows that $R \not\subseteq \Phi(G)$. Consequently, there exists a maximal subgroup M of G such that $R \not\leq M$. Hence, $M_G = 1$, i.e., G is a primitive group. Therefore, $R = C_G(R) = O_{p',p}(G)$. Hence, $R = O_p(G) = C^p(G)$. Consequently,

$$G/O_{p',p}(G) = G/C^p(G) = G/O_p(G) \in H_i(p) = F_i(p).$$

Hence, by Lemma 2.1, we have $G \in \mathfrak{F}_i$, a contradiction. Therefore, $\mathfrak{H}_i \subseteq \mathfrak{F}_i$.

Now we show that $\mathfrak{F}_i \subseteq \mathfrak{H}_i$. Assume it is false. Let G be a group of minimal order in $\mathfrak{F}_i \setminus \mathfrak{H}_i$. Then G is a monolithic group and $R = G^{\mathfrak{H}_i}$ is the monolith of G.

Let $O_\omega(G) = 1$. Then $R_\omega(G) = 1$ and $\omega \cap \pi(\mathrm{Com}(G)) = \omega \cap \pi(\mathrm{Com}(G/R))$. Let

$$p \in \omega \cap \pi(\mathrm{Com}(G)).$$

If R is non-abelian, then

$$G \simeq G/1 = G/R_\omega(G) \in H_i(\omega') = \mathfrak{F}_i.$$

We have $G/R \in \mathfrak{H}_i$, by the choice of G. Moreover, $C^p(G/R) = C^p(G)/R$. Consequently,

$$(G/R)/C^p(G/R) = (G/R)/(C^p(G)/R) \simeq G/C^p(G) \in H_i(p).$$

Thus, $G/C^p(G) \in H_i(p)$ for all $p \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{H}_i$, a contradiction.

Let R be an abelian ω'-group. Note that

$$G \simeq G/1 = G/R_\omega(G) \in H_i(\omega') = \mathfrak{F}_i$$

since $R_\omega(G) = 1$. Since $G/R \in \mathfrak{H}_i$ and $C^p(G/R) = C^p(G)/R$, we have

$$G/C^p(G) \in H_i(p)$$

for all $p \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{H}_i$, a contradiction.

Consequently, $O_\omega(G) \neq 1$. Let $p \in \pi(R) \subseteq \omega$. If R is non-abelian, then $\pi(\mathrm{Com}(R)) = \varnothing$. Hence, $R_\omega(G) = 1$. Consequently,

$$G \simeq G/1 = G/R_\omega(G) \in H_i(\omega') = \mathfrak{F}_i.$$

Since $G/R \in \mathfrak{H}_i$ and $C^p(G/R) = C^p(G)/R$, it follows that $G/C^p(G) \in H_i(p)$ for all $p \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{H}_i$, a contradiction.

Consequently, R is an abelian p-group. We have $G/R \in \mathfrak{H}_i$, by the choice of G. Moreover, since $R \leqslant R_\omega(G)$, we have $R_\omega(G/R) = R_\omega(G)/R$. It follows that

$$G/R_\omega(G) \in H_i(\omega') = \mathfrak{F}_i.$$

Let $T = R \rtimes (G/C_G(R))$. Since $G \in \mathfrak{F}_i$, using Lemma 2.2, we have $T \in \mathfrak{F}_i$.

If $|T| < |G|$, then $T \in \mathfrak{H}_i$, by the choice of the group G. Hence,

$$G/C_G(R) \simeq T/R = T/C_T(R) = T/C^p(T) \in H_i(p).$$

Let $C^*/R = C^p(G/R)$. Since $G/R \in \mathfrak{H}_i$, we have

$$(G/R)/C^p(G/R) = (G/R)/(C^*/R) \simeq G/C^* \in H_i(p).$$

Moreover, from the above proved we know $G/C_G(R) \in H_i(p)$. Consequently,

$$G/(C^* \cap C_G(R)) = G/C^p(G) \in H_i(p).$$

If $q \neq p$, then, evidently,

$$C^q(G/R) = C^q(G)/R.$$

Since $G/R \in \mathfrak{H}_i$, it follows that

$$G/C^q(G) \in H_i(q)$$

for all $q \in (\omega \cap \pi(\mathrm{Com}(G))) \setminus \{p\}$. Thus,

$$G/C^r(G) \in H_i(r)$$

for all $r \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{H}_i$, a contradiction.

Therefore, $|T| = |G|$. Hence, $R = C_G(R)$. It follows that

$$R = C_G(R) = C^p(G) = O_{p',p}(G).$$

Therefore,

$$G/C^p(G) = G/O_{p',p}(G) \in F_i(p) = H_i(p).$$

Moreover, from the above proved we know

$$G/C^q(G) \in H_i(q)$$

for all $q \in (\omega \cap \pi(\mathrm{Com}(G))) \setminus \{p\}$. Thus,

$$G/C^r(G) \in H_i(r)$$

for all $r \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{H}_i$, a contradiction. Consequently, $\mathfrak{F}_i \subseteq \mathfrak{H}_i$. Thus, $\mathfrak{F}_i = \mathfrak{H}_i$ for all $i \in I$.

Since by Lemma 2.3 the lattice $\mathcal{L}^\omega$ is inductive, we have

$$\mathfrak{F} = \vee_{\mathcal{L}^\omega}(\mathfrak{F}_i \mid i \in I) = LF_\omega\big(\vee(F_i \mid i \in I)\big).$$

Since by Lemma 2.4 the lattice $\mathcal{C}^\omega$ is inductive, we have

$$\mathfrak{H} = \vee_{\mathcal{C}^\omega}(\mathfrak{F}_i \mid i \in I) = CF_\omega\big(\vee(H_i \mid i \in I)\big).$$

Now we are ready to prove the following equality $\mathfrak{F} = \mathfrak{H}$. Clearly, $\mathfrak{H} \subseteq \mathfrak{F}$. Suppose that $\mathfrak{F} \not\subseteq \mathfrak{H}$. Let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{H}$. Then G is a monolithic group and $R = G^{\mathfrak{H}}$ is the monolith of G.

If $O_\omega(G) = 1$, then

$$\begin{gathered} G \simeq G/1 = G/O_\omega(G) \in \big(\vee(F_i \mid i \in I)\big)(\omega') = \\ = \vee(F_i(\omega') \mid i \in I) = \vee(\mathfrak{F}_i \mid i \in I) \subseteq \vee_{\mathcal{C}^\omega}(\mathfrak{F}_i \mid i \in I) = \mathfrak{H}, \end{gathered}$$

a contradiction.

Hence, $O_\omega(G) \neq 1$. Let $p \in \pi(R) \subseteq \omega$. If R is non-abelian, then

$$O_{p',p}(G) = 1.$$

Hence, since the canonical ω-local satellite F_i is inner,

$$\begin{gathered} G \simeq G/1 = G/O_{p',p}(G) \in \big(\vee(F_i \mid i \in I)\big)(p) = \\ = \vee(F_i(p) \mid i \in I) \subseteq \vee(\mathfrak{F}_i \mid i \in I) \subseteq \vee_{\mathcal{C}^\omega}(\mathfrak{F}_i \mid i \in I) = \mathfrak{H}. \end{gathered}$$

This contradicts the choice of the group G.

Hence, R is an abelian p-group. We have $G/R \in \mathfrak{H}$, by the choice of G. Moreover, since $R \leqslant R_\omega(G)$, we have $R_\omega(G/R) = R_\omega(G)/R$. It follows that

$$G/R_\omega(G) \in \big(\vee(H_i \mid i \in I)\big)(\omega').$$

Let $T = R \rtimes (G/C_G(R))$. Since $G \in \mathfrak{F}$, using Lemma 2.2, we have $T \in \mathfrak{F}$.

If $|T| < |G|$, then $T \in \mathfrak{H}$, by the choice of G. Consequently,

$$G/C_G(R) \simeq T/R = T/C_T(R) = T/C^p(T) \in \big(\vee(H_i \mid i \in I)\big)(p).$$

Let $C^*/R = C^p(G/R)$. Since $G/R \in \mathfrak{H}$, we have

$$G/C^* \in \big(\vee(H_i \mid i \in I)\big)(p).$$

Moreover, from the above proved we know

$$G/C_G(R) \in \big(\vee(H_i \mid i \in I)\big)(p).$$

Hence,

$$G/C^p(G) \in \big(\vee(H_i \mid i \in I)\big)(p).$$

If $q \neq p$, then, evidently, $C^q(G/R) = C^q(G)/R$. Since $G/R \in \mathfrak{H}$, it follows that

$$G/C^q(G) \in \big(\vee(H_i \mid i \in I)\big)(q)$$

for all $q \in \big(\omega \cap \pi(\mathrm{Com}(G))\big) \setminus \{p\}$. Thus,

$$G/C^r(G) \in \big(\vee(H_i \mid i \in I)\big)(r)$$

for all $r \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{H}$, a contradiction.

Therefore, $|T| = |G|$. Consequently, $R = C_G(R)$. It follows that

$$R = C_G(R) = O_p(G) = C^p(G) = O_{p',p}(G).$$

Therefore, since $\mathfrak{F}_i = \mathfrak{H}_i$ for all $i \in I$, we have

$$G/C^p(G) = G/O_{p',p}(G) \in \big(\vee(F_i \mid i \in I)\big)(p) = \vee(F_i(p) \mid i \in I) =$$

$$= \vee(H_i(p) \mid i \in I) = \big(\vee(H_i \mid i \in I)\big)(p).$$

Moreover, from the above proved we know

$$G/C^q(G) \in \big(\vee(H_i \mid i \in I)\big)(q)$$

for all $q \in \big(\omega \cap \pi(\mathrm{Com}(G))\big) \setminus \{p\}$. Thus,

$$G/C^r(G) \in \big(\vee(H_i \mid i \in I)\big)(r)$$

for all $r \in \omega \cap \pi(\mathrm{Com}(G))$. Hence, $G \in \mathfrak{H}$. Consequently, $\mathfrak{F} \subseteq \mathfrak{H}$. Thus, $\mathfrak{F} = \mathfrak{H}$, and the theorem is proved.

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