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Degenerations of Complex Associative Algebras of Dimension Three via Lie and Jordan Algebras

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Abstract

Let $\boldsymbol{\Lambda}_3(\mathbb{C})$ $(=\mathbb{C}^{27})$ be the space of structure vectors of 3-dimensional algebras over $\mathbb{C}$ considered as a G-module via the action of $G = GL(3, \mathbb{C})$ on $\boldsymbol{\Lambda}_3(\mathbb{C})$ "by change of basis". We determine the complete degeneration picture inside the algebraic subset $\mathcal{A}_3^s$ of $\boldsymbol{\Lambda}_3(\mathbb{C})$ consisting of associative algebra structures via the corresponding information on the algebraic subsets $\mathcal{L}_3$ and $\mathcal{J}_3$ of $\boldsymbol{\Lambda}_3(\mathbb{C})$ of Lie and Jordan algebra structures respectively. This is achieved with the help of certain G-module endomorphisms ϕ_1, ϕ_2 of $\boldsymbol{\Lambda}_3(\mathbb{C})$ which map $\mathcal{A}_3^s$ onto algebraic subsets of $\mathcal{L}_3$ and $\mathcal{J}_3$ respectively.

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1 Introduction

The notion of degeneration or contraction arises in various physical investigations. It was first introduced by Segal [22] and Inönü and Wigner [10, 11] in the case of Lie groups and Lie algebras in order to link certain properties of the classical mechanics, the relativistic mechanics and the quantum mechanics. The main idea involved was to try to obtain certain properties of one of the physical theories using the corresponding properties of another theory via a kind of limiting process. It turned out that the classical mechanics can be studied as a limit case of quantum mechanics as the Planck constant tends to zero. The symmetry group of relativistic mechanics (the Poincaré group) can be viewed as a contraction of the symmetry group of classical mechanics (the Galilean group) if we assume that the light velocity $c \to \infty$. The notion of degeneration also has applications in other branches of mathematics.

The present paper is concerned with the investigation of degenerations within certain classes of algebras. The first classification of complex low-dimensional associative algebras has been made by B. Peirce in 1870, see also [21]. Although this classification contains only the so called "pure algebras", the approach used by Peirce can be generalized to more general classes of algebras. Complex associative 3-dimensional algebras with a unit were classified by P. Gabriel in [5]. Moreover, in [5] Gabriel also constructs all degenerations within the class of 2-dimensional complex associative algebras and all degenerations within the classes of 3- and 4-dimensional complex associative algebras with a unit. Modern classification of all complex 3-dimensional associative algebras can be found in [4, 16].

Jordan algebras of dimension three over an algebraically closed field of characteristic not equal to 2 or 3 are classified in [15], where the authors also determine the irreducible components of the variety of 3-dimensional Jordan algebras (see also [7] for a description of the degenerations within the variety of complex 3-dimensional Jordan algebras). The degenerations within the variety of complex 3-dimensional Lie algebras have been constructed in [1, 18].

It will be convenient at this point to introduce some notation. Let $\Lambda_3(\mathbb{C})$ $(= \mathbb{C}^{27})$ be the space of structure vectors of complex 3-dimensional algebras. We can consider $\Lambda_3(\mathbb{C})$ as a $\mathbb{C}G$-module via the natural (linear) action of $G = GL(3, \mathbb{C})$ on $\Lambda_3(\mathbb{C})$ "by change of basis". We say that there is a degeneration from λ to μ (with $\lambda, \mu \in \Lambda_3(\mathbb{C})$) if μ belongs to the Zariski-closure of the G-orbit of λ.

The aim of the present paper is to determine the complete degeneration picture inside the algebraic subset $\mathcal{A}_3^s$ of $\Lambda_3(\mathbb{C})$ consisting of the associative algebra structures, via the corresponding information on the algebraic subsets $\mathcal{L}_3$ and $\mathcal{J}_3$ of $\Lambda_3(\mathbb{C})$ of Lie and Jordan algebras respectively. In order to achieve this, we define certain $\mathbb{C}G$-module endomorphisms ϕ_1, ϕ_2 of $\Lambda_3(\mathbb{C})$ which map $\mathcal{A}_3^s$ onto algebraic subsets of $\mathcal{L}_3$ and $\mathcal{J}_3$ respectively. A key role in our approach is played by the explicit computation of the B-orbit (where B is a Borel subgroup of G) of appropriate elements of $\Lambda_3(\mathbb{C})$ and the consideration of the intersection of the closure of these B-orbits with certain algebraic subsets of $\Lambda_3(\mathbb{C})$ which played some part in [12] and [20]. Locating various polynomials in the vanishing ideal of such orbits is one of the important ingredients in some of our arguments. This approach not only allowed us to rule out the possibility of degeneration in certain cases where this was not easy to achieve via the various necessary conditions for degeneration commonly used in literature, but also provided a very practical means of constructing degenerations in certain cases where a degeneration actually exists. The idea of explicitly computing an orbit and locating polynomials in the vanishing ideal was already used in [13] and this led, as a by-product, to the determination of various degenerations between 3-dimensional Lie algebras over an arbitrary field.

We are indebted to the anonymous referee for bringing to our attention the close connection (even though different terminology is being used) of the degeneration process in the paper with PI-theory and quiver theory, see [2, 3].

The paper is organized as follows: In Section 2 we include some preliminary lemmas and discuss some of their applications via which we give the flavour of the general techniques that will be used later on in the paper. In Sections 3 and 4 we recall some necessary conditions for degeneration and also the defining conditions for certain algebraic sets which will play a key role in the paper. Moreover, in Section 4 we discuss some basic properties of the maps ϕ_1 and ϕ_2. In Section 5 we recall certain results regarding the varieties $\mathcal{A}_3^s$, $\mathcal{L}_3$ and $\mathcal{J}_3$ in the framework that has been built in the earlier sections. Finally, in Section 6 which contains the main results of the paper, the degeneration picture inside the variety $\mathcal{A}_3^s$ is completely determined.

We remark that in [17], the authors aim to determine the irreducible components of the variety $\mathcal{A}_3^s$. Their approach is based entirely on the computation of various algebra invariants, which they use together with certain necessary conditions for degeneration, in order to rule out the possibility of degeneration between various algebras. For the purposes of their work they do not include any constructions of degenerations or comment further in the cases where the possibility of degeneration is open. There are, however, some inaccuracies in their computations of certain algebra invariants, sometimes leading to inaccuracies regarding the possibility of degeneration.

2 Preliminaries

Throughout this paper, $\mathbb{F}$ denotes an arbitrary infinite field and n a positive integer. We also let $G = GL(n, \mathbb{F})$. We fix V to be a finite dimensional $\mathbb{F}$-vector space with $\dim_{\mathbb{F}} V = n$. We call $\mathfrak{g}$ an algebra structure on V if $\mathfrak{g}$ is an $\mathbb{F}$-algebra having V as its underlying vector space, so $\mathfrak{g}$ is a not necessarily associative algebra which has multiplication defined via a suitable $\mathbb{F}$-bilinear map

$$[\,,\,]_{\mathfrak{g}} \colon V \times V \to V : \ (u, v) \mapsto [u, v]_{\mathfrak{g}},$$

for $u, v \in V$. We denote by $\mathbf{A}$ the set of all algebra structures on V.

If $(u_1, \ldots, u_n)$ is an ordered $\mathbb{F}$-basis of V, the multiplication in $\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}$ is completely determined by the structure constants $\alpha_{ijk} \in \mathbb{F}$ ($1 \leqslant i, j, k \leqslant n$) given by $[u_i, u_j] = \sum_{k=1}^{n} \alpha_{ijk} u_k$. We will regard this set of structure constants α_{ijk} as an ordered n^3-tuple by imposing an ordering on the ordered triples (i, j, k), for example we could choose the lexicographic ordering. We call the ordered n^3-tuple $\boldsymbol{\alpha} = (\alpha_{ijk}) \in \mathbb{F}^{n^3}$ the structure vector of $\mathfrak{g} \in \mathbf{A}$ relative to the $\mathbb{F}$-basis $(u_1, \ldots, u_n)$ of V. Also denote by $\boldsymbol{\Lambda}\,(= \mathbb{F}^{n^3})$ the set of all $\lambda = (\lambda_{ijk}) \in \mathbb{F}^{n^3}$ such that λ occurs as the structure vector of some $\mathfrak{g} \in \mathbf{A}$ relative to some ordered basis o V. Clearly, the structure vector $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ occurs as the structure vector of both $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathbf{A}$ (relative to suitable $\mathbb{F}$-bases of V) if, and only if, the algebra structures $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are $\mathbb{F}$-isomorphic (that is, there exists a bijective $\mathbb{F}$-linear map $\Psi \colon V \to V$ such that $\Psi([x, y]_{\mathfrak{g}_1}) = [\Psi(x), \Psi(y)]_{\mathfrak{g}_2}$, for all $x, y \in V$). In what follows, we will write $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ to denote that two algebras $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathbf{A}$ are $\mathbb{F}$-isomorphic.

The set $\boldsymbol{\Lambda}\,(= \mathbb{F}^{n^3})$ forms an $\mathbb{F}$-vector space via the usual (componentwise) addition and scalar multiplication. We will use symbol **abc** to denote the member $\boldsymbol{\lambda}\,(= (\lambda_{ijk}))$ of $\boldsymbol{\Lambda}$ having $\lambda_{abc} = 1$ and all other λ_{ijk} equal to 0. We will refer to the $\mathbb{F}$-basis of $\boldsymbol{\Lambda}$ consisting of the n^3 structure vectors of this form as the standard basis of $\boldsymbol{\Lambda}$.

We can also regard $\mathbf{A}$ as an $\mathbb{F}$-vector space: For $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathbf{A}$, with $\mathfrak{g}_1 = (V, [\,,\,]_1)$, $\mathfrak{g}_2 = (V, [\,,\,]_2)$, and for $\alpha \in \mathbb{F}$, define

$$\mathfrak{g}_1 + \mathfrak{g}_2 = (V, [\,,\,]) \in \mathbf{A}$$

where

$$[u, v] = [u, v]_1 + [u, v]_2 \quad \text{and} \quad \alpha\mathfrak{g}_1 = (V, [\,,\,]_\alpha)$$

where $[u, v]_\alpha = \alpha[u, v]_1$, for all $u, v \in V$.

For the rest of the paper it will also be convenient to fix an ordered $\mathbb{F}$-basis $(e_1, \ldots, e_n)$ of V which we will call the standard basis of V.

We can then obtain an isomorphism of $\mathbb{F}$-vector spaces $\Theta : \mathbf{A} \to \boldsymbol{\Lambda}$ where, for $\mathfrak{g} \in \mathbf{A}$ we define $\Theta(\mathfrak{g})$ $(\in \boldsymbol{\Lambda})$ to be the structure vector of $\mathfrak{g}$ relative to the standard basis $(e_1, \ldots, e_n)$ of V. Clearly, for $\mathbf{abc} \in \boldsymbol{\Lambda}$ as above, we have that $\Theta^{-1}(\mathbf{abc})$ is the member $\mathfrak{g}$ of $\mathbf{A}$ having $[e_a, e_b]_\mathfrak{g} = e_c$ as its only non-zero commutation relation.

With the help of the isomorphism $\Theta : \mathbf{A} \to \boldsymbol{\Lambda}$ we define the map

$$\Omega : \boldsymbol{\Lambda} \times G \to \boldsymbol{\Lambda} : (\boldsymbol{\lambda}, g) \mapsto \boldsymbol{\lambda} g \quad (\boldsymbol{\lambda} \in \boldsymbol{\Lambda}, g = (g_{ij}) \in G)$$

where $\boldsymbol{\lambda} g \in \boldsymbol{\Lambda}$ is the structure vector of $\Theta^{-1}(\boldsymbol{\lambda}) \in \mathbf{A}$ relative to the $\mathbb{F}$-basis $(v_1, \ldots, v_n)$ of V given by $v_j = \sum_{i=1}^{n} g_{ij} e_i$, for $1 \leqslant j \leqslant n$ (in particular, $g \in G$ is the transition matrix from the basis $(e_i)_{i=1}^n$ to the basis $(v_i)_{i=1}^n$ of V).

It is easy to observe that the map Ω defines a linear right action of G on $\boldsymbol{\Lambda}$ and that the resulting orbits of this action correspond precisely to the isomorphism classes of n-dimensional $\mathbb{F}$-algebras. Also note that the map Ω gives $\boldsymbol{\Lambda}$ the structure of a right $\mathbb{F}G$-module. The orbit of $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ with respect to the above G-action will be denoted by $O(\boldsymbol{\lambda})$, (so $O(\boldsymbol{\lambda}) = \boldsymbol{\lambda} G$).

Next, we recall briefly some basic facts on algebraic sets.

Let $\mathbb{F}[\mathbf{X}]$ be the ring $\mathbb{F}[X_{ijk} : 1 \leqslant i, j, k \leqslant n]$ of polynomials in the indeterminates X_{ijk} $(1 \leqslant i, j, k \leqslant n)$ over $\mathbb{F}$. For each $\boldsymbol{\lambda} = (\lambda_{ijk}) \in \boldsymbol{\Lambda}$ we can define the evaluation map $\mathbf{ev}_{\boldsymbol{\lambda}} : \mathbb{F}[\mathbf{X}] \to \mathbb{F}$ to be the unique ring homomorphism $\mathbb{F}[\mathbf{X}] \to \mathbb{F}$ such that $X_{ijk} \mapsto \lambda_{ijk}$, for $1 \leqslant i, j, k \leqslant n$, and which is the identity on $\mathbb{F}$. A subset W of $\boldsymbol{\Lambda}$ is algebraic (and thus closed in the Zariski topology on $\boldsymbol{\Lambda}$) if there exists a subset $S \subseteq \mathbb{F}[\mathbf{X}]$ such that

$$W = \{\boldsymbol{\lambda} = (\lambda_{ijk}) \in \boldsymbol{\Lambda} : \mathbf{ev}_{\boldsymbol{\lambda}}(f) = 0, \text{ for all } f \in S\}.$$

The Zariski closure of a subset Y of $\boldsymbol{\Lambda}$ will be denoted by $\overline{Y}$. By a closed subset of $\boldsymbol{\Lambda}$ we will always mean a Zariski-closed subset of $\boldsymbol{\Lambda}$.

Finally, for $U \subseteq \mathbf{\Lambda}$, the vanishing ideal $\mathbf{I}(U)$ of U is defined by

$$\mathbf{I}(U) = \{f \in \mathbb{F}[\mathbf{X}] : \ \mathbf{ev}_{\boldsymbol{\lambda}}(f) = 0, \text{ for all } \boldsymbol{\lambda} \in U\}.$$

Definition 1 Let $\mathfrak{g}, \mathfrak{h} \in \mathbf{A}$. We say that $\mathfrak{g}$ *degenerates* to $\mathfrak{h}$ if $\Theta(\mathfrak{h}) \in \overline{O(\Theta(\mathfrak{g}))}$. Moreover, we say that $\mathfrak{g}$ *properly degenerates* to $\mathfrak{h}$ if $\Theta(\mathfrak{h}) \in \overline{O(\Theta(\mathfrak{g}))} - O(\Theta(\mathfrak{g}))$. Observe that, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbf{\Lambda}$, we have that $O(\boldsymbol{\mu}) \subseteq \overline{O(\boldsymbol{\lambda})}$ whenever $\mu \in \overline{O(\boldsymbol{\lambda})}$, see [12]. We write $\boldsymbol{\lambda} \to \boldsymbol{\mu}$ (and $\Theta^{-1}(\boldsymbol{\lambda}) \to \Theta^{-1}(\boldsymbol{\mu})$) if for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbf{\Lambda}$ we have that $\boldsymbol{\mu} \in \overline{O(\boldsymbol{\lambda})}$. Similarly, we write $\boldsymbol{\lambda} \not\to \boldsymbol{\mu}$ (and $\Theta^{-1}(\boldsymbol{\lambda}) \not\to \Theta^{-1}(\boldsymbol{\mu})$) if $\boldsymbol{\mu} \notin \overline{O(\boldsymbol{\lambda})}$.

A well-known result is that $\mathfrak{g}$ degenerates to the Abelian algebra (the zero algebra), for all $\mathfrak{g} \in \mathbf{A}$. In this paper, we will consider (and also compare) degenerations within certain classes of algebras.

For the rest of this section, we prove some preliminary lemmas and discuss certain of their applications which involve techniques that will be used in Section 6 where the main results of the paper are proved.

Lemma 2 *Let $f \colon \mathbb{F} \to \mathbf{\Lambda}$ be a continuous function in the Zariski topology. Also let U be a finite subset of $\mathbb{F}$ and let $S = \mathbb{F} - U$. Then $f(u) \in \overline{f(S)}$, for all $u \in U$.*

Proof — The hypothesis that the function f is continuous ensures that $f(\overline{S}) \subseteq \overline{f(S)}$. Hence, it suffices to show that $\overline{S} = \mathbb{F}$. Suppose, on the contrary, that $\overline{S} \subsetneqq \mathbb{F}$. Then $\overline{S} = \mathbb{F} - U'$ where $\varnothing \subsetneqq U' \subseteq U$ which, in turn, gives $\mathbb{F} = \overline{S} \cup U'$. This is a contradiction as $\overline{S}$ and U' are both closed subsets of $\mathbb{F}$ with $\varnothing \subsetneqq \overline{S} \subsetneqq \mathbb{F}$ and $\varnothing \subsetneqq U' \subsetneqq \mathbb{F}$, and $\mathbb{F}$ is irreducible (see, for example, [6, Example 1.1.13]). We conclude that $\overline{S} = \mathbb{F}$. □

Example 3 (an application of Lemma 2) Let $n = 3$ and suppose that $\operatorname{char} \mathbb{F} \neq 2$. Let $\boldsymbol{\lambda} = \mathbf{221} + \mathbf{331} + 2(\mathbf{321}) \in \mathbf{\Lambda}$ and, for all $t \in \mathbb{F} - \{0\}$, let

$$g(t) = \begin{pmatrix} -t & 0 & 0 \\ 0 & 0 & -1 \\ 0 & t & 1 \end{pmatrix} \in GL(3, \mathbb{F}).$$

Then $\boldsymbol{\lambda} g(t) = \mathbf{231} - \mathbf{321} - t(\mathbf{221})$ with $t \neq 0$. Moreover, the map

$$f \colon \mathbb{F} \to \mathbf{\Lambda}\ (= \mathbb{F}^{27}) : t \mapsto \mathbf{231} - \mathbf{321} - t(\mathbf{221}),$$

is continuous in the Zariski topology. Set $S = \mathbb{F} - \{0\}$, it is clear that $f(S) \subseteq \lambda G$. Invoking Lemma 2 we get that

$$\mathbf{231} - \mathbf{321}\,(= f(0)) \in \overline{f(S)} \subseteq \overline{\lambda G}.$$

Lemma 4 (compare with [8], Proposition 1.7) *Let* P *be a parabolic subgroup of* G *with* $\mathbb{F}$ *algebraically closed. Also let* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$*. Then*

$$(\overline{\boldsymbol{\lambda} P})G = \overline{\boldsymbol{\lambda} G}\,(= \overline{O(\boldsymbol{\lambda})}).$$

Proof — Assume the hypothesis. Clearly,

$$\boldsymbol{\lambda} P \subseteq \boldsymbol{\lambda} G \subseteq \overline{\boldsymbol{\lambda} G}$$

so $\overline{\boldsymbol{\lambda} P} \subseteq \overline{\boldsymbol{\lambda} G}$. It follows that $(\overline{\boldsymbol{\lambda} P})G \subseteq \overline{\boldsymbol{\lambda} G}$ since $\overline{\boldsymbol{\lambda} G}$ is a union of G-orbits by [6, Proposition 2.5.2 (a)]. Next, we let $C = \overline{\boldsymbol{\lambda} P}$. Then C is closed and it is also P-invariant, since C is a union of P-orbits again by [6, Proposition 2.5.2 (a)]. Clearly, $\boldsymbol{\lambda} G \subseteq (\overline{\boldsymbol{\lambda} P})G = CG$. Moreover, from [6, Corollary 3.2.12 (a)], $CG\,(\subseteq \boldsymbol{\Lambda})$ is a closed set. Consequently, $\overline{\boldsymbol{\lambda} G} \subseteq CG = (\overline{\boldsymbol{\lambda} P})G$. We conclude that $(\overline{\boldsymbol{\lambda} P})G = \overline{\boldsymbol{\lambda} G}$. □

Remark 5 Let $\mathbb{F}$ be algebraically closed. Also let B be a Borel subgroup of G. Then, let us note that:

(i) The conclusion of Lemma 4 still holds with B in the place of P since every Borel subgroup of G is parabolic by Borel's theorem, see for example, [6, Theorem 3.4.3];

(ii) suppose U is a subset of $\boldsymbol{\Lambda}$ which is also a union of G-orbits. If $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$, we obtain $U \cap \overline{O(\boldsymbol{\lambda})} = \varnothing$ whenever $U \cap \overline{\boldsymbol{\lambda} B} = \varnothing$. To prove this, suppose that $U \cap \overline{O(\boldsymbol{\lambda})} \neq \varnothing$ and $\boldsymbol{\nu} \in U \cap \overline{O(\boldsymbol{\lambda})}$. Since $\boldsymbol{\nu} \in \overline{O(\boldsymbol{\lambda})}$, Lemma 4 ensures that $\boldsymbol{\nu} \in O(\boldsymbol{\mu})$, with $\boldsymbol{\mu} \in \overline{\boldsymbol{\lambda} B}$. Hence $\boldsymbol{\mu} \in O(\boldsymbol{\nu})$. Since U is a union of G-orbits and $\boldsymbol{\nu} \in U$, we have $O(\boldsymbol{\nu}) \subseteq U$ and hence $\boldsymbol{\mu} \in U$. This leads to $U \cap \overline{\boldsymbol{\lambda} B} \neq \varnothing$, since $\boldsymbol{\mu} \in U \cap \overline{\boldsymbol{\lambda} B}$.

Example 6 (An application of Lemma 4) Let $n = 3$ and let $\mathbb{F} = \mathbb{C}$. Also let $\beta \in \mathbb{F} - \{-1\}$ and set $\boldsymbol{\lambda} = \mathbf{231} + \beta(\mathbf{321}) \in \boldsymbol{\Lambda}$. We aim to show that $\overline{O(\boldsymbol{\lambda})} \cap \mathcal{K}_3 = \{\mathbf{0}\}$, where $\mathcal{K}_3\,(\subseteq \boldsymbol{\Lambda})$ is given by

$$\Theta^{-1}(\mathcal{K}_3) = \{\mathfrak{g} = (V, [,]) \in \mathbf{A}\colon [u, u] = 0_V \text{ for all } u \in V\}.$$

In order to establish this we will use Lemma 4.

Let $b \in B$ where B is the Borel subgroup of all upper triangular matrices in $G = GL(n, \mathbb{F})$, so $b = (b_{ij})$ where $b_{ij} = 0$ whenever $i > j$, and $b_{11}b_{22}b_{33} \neq 0$. Then

$$\boldsymbol{\lambda} b = \Big(\frac{b_{22}b_{33}}{b_{11}}\Big)\mathbf{231} + \beta\Big(\frac{b_{22}b_{33}}{b_{11}}\Big)\mathbf{321} + (1+\beta)\Big(\frac{b_{23}b_{33}}{b_{11}}\Big)\mathbf{331}.$$

Our first goal is to show that $\overline{\boldsymbol{\lambda} B} \cap \mathcal{K}_3 = \{\mathbf{0}\}$.

It follows easily from the expression for $\boldsymbol{\lambda} b$ obtained above that the following polynomials all belong to $\mathbf{I}(\boldsymbol{\lambda} B)$, the vanishing ideal of the B-orbit $\boldsymbol{\lambda} B$:

$$X_{321} - \beta X_{231}, X_{232}, X_{233}, X_{322}, X_{323}$$

and

$$X_{11i}, X_{12i}, X_{13i}, X_{21i}, X_{22i}, X_{31i} \quad \text{for} \quad 1 \leqslant i \leqslant 3.$$

Now let $\boldsymbol{\mu} = (\mu_{ijk}) \in (\overline{\boldsymbol{\lambda} B}) \cap \mathcal{K}_3$. Since $\boldsymbol{\mu} \in \mathcal{K}_3$, we must have $\mu_{iik} = 0$, for $1 \leqslant i, k \leqslant 3$. Moreover, in view of the fact that in $\Theta^{-1}(\boldsymbol{\mu})$ we have $[e_2 + e_3, e_2 + e_3] = 0_V$, we also obtain that $\mu_{321} + \mu_{231} = 0$. Since $\boldsymbol{\mu} \in \overline{\boldsymbol{\lambda} B}$ and $X_{321} - \beta X_{231} \in \mathbf{I}(\boldsymbol{\lambda} B)$ we must also have that $\mu_{321} - \beta\mu_{231} = 0$. This gives $(1+\beta)\mu_{321} = 0$ with $\beta \neq -1$, so $\mu_{321} = 0 = \mu_{231}$. Finally, invoking the fact that $\mathbf{ev}_{\boldsymbol{\mu}}(f) = 0$ for each of the remaining polynomials f in $\mathbf{I}(\boldsymbol{\lambda} B)$ listed above we conclude that $\boldsymbol{\mu} = \mathbf{0}$. Hence $\overline{\boldsymbol{\lambda} B} \cap \mathcal{K}_3 = \{\mathbf{0}\}$. Remark 5 (ii) with $U = \mathcal{K}_3 - \{\mathbf{0}\}$ now ensures that $U \cap \overline{O(\boldsymbol{\lambda})} = \varnothing$ and hence $\overline{O(\boldsymbol{\lambda})} \cap \mathcal{K}_3 = \{\mathbf{0}\}$ as required. This observation will turn out to be useful in Section 6.

Remark 7 (i) An argument involving the B-orbit may not only be used in order to rule out the possibility of degeneration as in Example 6, but it may also be used in order to construct degenerations: Keeping the hypothesis and notation of Example 6, define $b(t) \in B$ for each $t \in \mathbb{F} - \{0\}$ by setting

$$b_{23} = 1 \quad \text{and} \quad b_{22} = b_{33} = b_{11} = t$$

(arbitrary values can be assigned to b_{12} and b_{13}). We then have that $\boldsymbol{\lambda} b(t) = t\,\mathbf{231} + (\beta t)\,\mathbf{321} + (1+\beta)\,\mathbf{331}$, $t \in \mathbb{F} - \{0\}$. Let

$$f\colon \mathbb{F} \to \boldsymbol{\Lambda}\colon t \mapsto t\,\mathbf{231} + (\beta t)\,\mathbf{321} + (1+\beta)\,\mathbf{331}$$

Also let $S = \mathbb{F} - \{0\}$. Then f is a continuous map in the Zariski topology satisfying $f(S) \subseteq O(\lambda)$. Invoking Lemma 2 we get that

$$(1+\beta)\mathbf{331}\,(= f(0)) \in \overline{f(S)} \subseteq \overline{O(\boldsymbol{\lambda})}.$$

Our hypothesis that $1+\beta \neq 0$ now ensures that $\mathbf{331} \in \overline{O(\boldsymbol{\lambda})}$.

(ii) The arguments in Example 3 and in item (i) of this remark also provide a method of constructing 1-parameter contractions, if one is working in the metric topology with $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, by letting $t \to 0$.

Lemma 8 *Let* $1 \leqslant r < n$ *and let* $\boldsymbol{\lambda}\,(= (\lambda_{ijk})) \in \boldsymbol{\Lambda}$ *satisfy* $\lambda_{ijk} = 0$ *whenever* $1 \leqslant i,j \leqslant r$ *and* $r+1 \leqslant k \leqslant n$. *Define* $\boldsymbol{\mu}\,(= (\mu_{ijk})) \in \boldsymbol{\Lambda}$ *by*

$$\mu_{ijk} = \begin{cases} \lambda_{ijk} & \textit{if } (1 \leqslant i,j \leqslant r \;\; \textit{and} \;\; 1 \leqslant k \leqslant n) \\ & \textit{or } (1 \leqslant i \leqslant r \;\; \textit{and} \;\; r+1 \leqslant j,k \leqslant n)\,, \\ & \textit{or } (1 \leqslant j \leqslant r \;\; \textit{and} \;\; r+1 \leqslant i,k \leqslant n) \\ \\ 0 & \textit{otherwise}. \end{cases}$$

Then $\boldsymbol{\mu} \in \overline{O(\boldsymbol{\lambda})}$.

Proof — Assume the hypothesis. For each $t \in \mathbb{F} - \{0\}$, let $g(t) \in G$ be the diagonal matrix having coefficient 1 in the first r entries and coefficient t in the last $n-r$ entries. Also let $\boldsymbol{\nu}(t)\,(= (\nu_{ijk}(t))) \in \boldsymbol{\Lambda}$ be defined, for each $t \in \mathbb{F}$, by

$$\nu_{ijk}(t) = \begin{cases} \lambda_{ijk} & \text{if } (1 \leqslant i,j \leqslant r \;\; \text{and} \;\; 1 \leqslant k \leqslant n) \\ & \text{or } (1 \leqslant i \leqslant r \;\; \text{and} \;\; r+1 \leqslant j,k \leqslant n)\,, \\ & \text{or } (1 \leqslant j \leqslant r \;\; \text{and} \;\; r+1 \leqslant i,k \leqslant n) \\ \\ t^2\,\lambda_{ijk} & \text{if } r+1 \leqslant i,j \leqslant n \;\; \text{and} \;\; 1 \leqslant k \leqslant r, \\ \\ t\,\lambda_{ijk} & \text{otherwise}. \end{cases}$$

Then $\boldsymbol{\nu}(t) = \boldsymbol{\lambda}\,g(t)$ for each $t \in \mathbb{F} - \{0\}$. We now define

$$f\colon \mathbb{F} \to \boldsymbol{\Lambda} : t \mapsto \boldsymbol{\nu}(t).$$

Then f is a continuous function in the Zariski topology. Moreover, by setting $S = \mathbb{F} - \{0\}$, we see that $f(S) \subseteq O(\boldsymbol{\lambda})$. Finally, invoking Lemma 2 we get that $\boldsymbol{\mu}\,(= f(0)) \in \overline{f(S)} \subseteq \overline{O(\boldsymbol{\lambda})}$. □

Remark 9 Keeping the notation and hypothesis of Lemma 8, we see that $\mathfrak{b} = \mathbb{F}\text{-span}(e_1, \ldots, e_r)$ is in fact a subalgebra of $\Theta^{-1}(\boldsymbol{\lambda})$. Moreover, $\Theta^{-1}(\boldsymbol{\mu})$ is a "semi-direct" sum of the algebra $\mathfrak{b}$ with an Abelian ideal of dimension $n - r$.

Example 10 (An application of Lemma 8) Let $n = 3$ and let

$$\boldsymbol{\lambda} = \mathbf{121} + \mathbf{211} + \mathbf{222} + \mathbf{323} \in \boldsymbol{\Lambda}.$$

We aim to show that $\mathbf{213} \in \overline{O(\boldsymbol{\lambda})}$. It will be convenient to consider the structure vector $\boldsymbol{\nu} = \boldsymbol{\lambda} g$ where

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \in GL(3, \mathbb{F})\,.$$

We have $\boldsymbol{\nu} = \mathbf{121} + \mathbf{211} + \mathbf{213} + \mathbf{222} + \mathbf{323} \in \boldsymbol{\Lambda}$. Invoking Lemma 8 with $r = 1$ we get that $\mathbf{213} \in \overline{O(\boldsymbol{\nu})} = \overline{O(\boldsymbol{\lambda})}$ (since $O(\boldsymbol{\nu}) = O(\boldsymbol{\lambda})$).

Finally, for this section, we include the following preparatory lemma which we will need in Section 6.

Lemma 11 *Let $\kappa \in \mathbb{C} - \{-2, 2\}$ and let $\boldsymbol{\lambda} = \mathbf{221} + \mathbf{331} + \kappa(\mathbf{321}) \in \boldsymbol{\Lambda}$. Also let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^2 + \kappa x + 1 \in \mathbb{C}[x]$. Then, the following are satisfied:*

(i) $\alpha^2 \neq 1$,

(ii) $\kappa + 2\alpha \neq 0$,

(iii) $\dfrac{\alpha(\kappa\alpha + 2)}{\kappa + 2\alpha} = -\alpha^2$,

(iv) $\mathbf{231} + -\alpha^2(\mathbf{321}) \in O(\boldsymbol{\lambda})$.

Proof — Assume the hypothesis.

(i) Since α is a root of $x^2 + \kappa x + 1$, we get

$$x^2 + \kappa x + 1 = (x - \alpha)(x - \beta)$$

for some $\beta \in \mathbb{C}$ with $\alpha\beta = 1$. In particular β is also a root of $x^2 + \kappa x + 1$ and $\alpha \neq 0$, $\beta \neq 0$. Moreover, $\alpha \neq \beta$ since the roots of the polynomial $x^2 + \kappa x + 1$ over $\mathbb{C}$ are given by

$$\frac{1}{2}(-\kappa \pm \sqrt{\kappa^2 - 4})$$

and $\kappa^2 \neq 4$ by assumption. If $\alpha^2 = 1$, we would get

$$\alpha = \alpha(\alpha\beta) = \alpha^2\beta = \beta$$

which is a contradiction. We conclude that $\alpha^2 \neq 1$.

(ii) From $\alpha^2 + \kappa\alpha + 1 = 0$ we obtain $\kappa = \alpha^{-1}(-1 - \alpha^2)$ (recall that $\alpha \neq 0$). Hence

$$\kappa + 2\alpha = \alpha^{-1}(-1 - \alpha^2 + 2\alpha^2) = \alpha^{-1}(-1 + \alpha^2) \neq 0,$$

since $\alpha^2 \neq 1$ from item (i) of this lemma.

(iii) From item (ii) we get, $\alpha(\kappa\alpha + 2) = \alpha(-1 - \alpha^2 + 2) = \alpha(1 - \alpha^2)$. Hence,

$$\frac{\alpha(\kappa\alpha + 2)}{\kappa + 2\alpha} = \frac{\alpha(1 - \alpha^2)}{\alpha^{-1}(-1 + \alpha^2)} = -\alpha^2.$$

(iv) Let

$$g = \begin{pmatrix} \kappa + 2\alpha & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 1 & \alpha \end{pmatrix} \in M_3(\mathbb{C}).$$

Clearly $g \in GL(3, \mathbb{C})$ since $\det(g) = (\kappa + 2\alpha)(\alpha^2 - 1) \neq 0$. Then

$$\lambda g = \mathbf{231} + \frac{\alpha(\kappa\alpha + 2)}{\kappa + 2\alpha}(\mathbf{321}) = \mathbf{231} + -\alpha^2(\mathbf{321}).$$

Hence, $\mathbf{231} + -\alpha^2(\mathbf{321}) \in O(\boldsymbol{\lambda})$ as required.

The proof is complete. □

3 Necessary conditions for degeneration

Necessary conditions in the study of degenerations have been used extensively by many authors. In this section we recall certain necessary conditions for degeneration which we will need later on.

Definition 12 Let $\mathfrak{g} = (V, [,]) \in \mathbf{A}$. We define the following $\mathbb{F}$-vector spaces:

(i) $\operatorname{ann}_R \mathfrak{g} = \{c \in V \colon [a, c] = 0_V \text{ for all } a \in V\}$, (the right annihilator of $\mathfrak{g}$),

(ii) $\operatorname{ann}_L \mathfrak{g} = \{c \in V \colon [c, a] = 0_V \text{ for all } a \in V\}$, (the left annihilator of $\mathfrak{g}$),

(iii) $\operatorname{Der}_{(\alpha,\beta,\gamma)} \mathfrak{g} = \{\phi \in \operatorname{End}_{\mathbb{F}} V \colon \alpha\phi[u, v] = \beta[\phi(u), v] + \gamma[u, \phi(v)]$ for all $u, v \in V\}$, for each ordered triple $(\alpha, \beta, \gamma) \in \mathbb{F}^3$, and

(iv) $\operatorname{Der} \mathfrak{g} = \operatorname{Der}_{(1,1,1)} \mathfrak{g}$ (the Lie algebra of derivations of $\mathfrak{g}$).

In the following proposition we collect some well-known facts regarding degenerations. We supply the proofs for completeness.

Proposition 13 *Let $\mathfrak{g}, \mathfrak{h} \in \mathbf{A}$ and suppose $\mathfrak{g}$ degenerates to $\mathfrak{h}$. Then, the following are satisfied:*

(i) $\dim_{\mathbb{F}} \operatorname{ann}_R \mathfrak{g} \leqslant \dim_{\mathbb{F}} \operatorname{ann}_R \mathfrak{h}$.

(ii) $\dim_{\mathbb{F}} \operatorname{ann}_L \mathfrak{g} \leqslant \dim_{\mathbb{F}} \operatorname{ann}_L \mathfrak{h}$.

(iii) $\dim_{\mathbb{F}} \operatorname{Der}_{(\alpha,\beta,\gamma)} \mathfrak{g} \leqslant \dim_{\mathbb{F}} \operatorname{Der}_{(\alpha,\beta,\gamma)} \mathfrak{h}$.

(iv) *If $\mathbb{F} = \mathbb{C}$ and $\mathfrak{g}$ properly degenerates to $\mathfrak{h}$, then*

$$\dim_{\mathbb{F}} \operatorname{Der} \mathfrak{g} < \dim_{\mathbb{F}} \operatorname{Der} \mathfrak{h}.$$

Proof — For each $\boldsymbol{\lambda} = (\lambda_{ijk}) \in \boldsymbol{\Lambda}$, let $\widetilde{c}(\boldsymbol{\lambda}) \in \mathbb{F}^{n \times n^2}$ be the $(n \times n^2)$-matrix over $\mathbb{F}$ whose columns are precisely all the vectors of the form $(\lambda_{i1j}, \lambda_{i2j}, \dots, \lambda_{inj})^{tr}$, for $1 \leqslant i, j \leqslant n$ (in some fixed order).

(i) Define

$$\psi(\boldsymbol{\lambda}) \colon \mathbb{F}^{1 \times n} \to \mathbb{F}^{n \times n^2} : \widehat{\alpha} \mapsto \widehat{\alpha}\widetilde{c}(\boldsymbol{\lambda}),$$

$\widehat{\alpha} \in \mathbb{F}^{1 \times n}$ (the space of $(1 \times n)$-matrices over $\mathbb{F}$). Then $\psi(\boldsymbol{\lambda})$ is an $\mathbb{F}$-linear map and it is easy to see that for $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ we have $\alpha_1 e_1 + \ldots + \alpha_n e_n \in \operatorname{ann}_R \Theta^{-1}(\boldsymbol{\lambda})$ if, and only if, $(\alpha_1, \dots, \alpha_n) \in \ker \psi(\boldsymbol{\lambda})$. Hence,

$$\begin{aligned}\dim_{\mathbb{F}}(\operatorname{ann}_R \Theta^{-1}(\boldsymbol{\lambda})) &= \dim_{\mathbb{F}} \ker \psi(\boldsymbol{\lambda}) \\ &= n - \dim_{\mathbb{F}} \operatorname{im} \psi(\boldsymbol{\lambda}) \\ &= n - \operatorname{rank} \widetilde{c}(\boldsymbol{\lambda}).\end{aligned}$$

It follows that $\operatorname{rank}\widetilde{c}(\boldsymbol{\lambda}) = \operatorname{rank}\widetilde{c}(\boldsymbol{\mu})$ whenever $\boldsymbol{\mu} \in O(\boldsymbol{\lambda})$ since the algebras $\Theta^{-1}(\boldsymbol{\lambda})$ and $\Theta^{-1}(\boldsymbol{\mu})$ are isomorphic (and so $\dim_{\mathbb{F}} \operatorname{ann}_R \Theta^{-1}(\boldsymbol{\lambda}) = \dim_{\mathbb{F}} \operatorname{ann}_R \Theta^{-1}(\boldsymbol{\mu})$). We thus have that for each non-negative integer t the set $\{\boldsymbol{\nu} \in \boldsymbol{\Lambda} \colon \operatorname{rank}\widetilde{c}(\boldsymbol{\nu}) \leqslant t\}$ is a Zariski-closed set in $\boldsymbol{\Lambda}$ which is also a union of orbits (compare, for example, with [12, Remark 3.15]). We conclude that $\operatorname{rank}\widetilde{c}(\boldsymbol{\nu}) \leqslant \operatorname{rank}\widetilde{c}(\boldsymbol{\lambda})$ whenever $\boldsymbol{\nu} \in \overline{O(\boldsymbol{\lambda})}$. This leads to $\dim_{\mathbb{F}} \operatorname{ann}_R \Theta^{-1}(\boldsymbol{\nu}) \geqslant \dim_{\mathbb{F}} \operatorname{ann}_R \Theta^{-1}(\boldsymbol{\lambda})$ whenever $\boldsymbol{\nu} \in \overline{O(\boldsymbol{\lambda})}$ as required.

(ii) We can use same argument as in item (i) of this proposition, but in the place of the matrix $\widetilde{c}(\boldsymbol{\lambda})$ we now consider the matrix $\widetilde{a}(\boldsymbol{\lambda}) \in \mathbb{F}^{n \times n^2}$ where the columns of $\widetilde{a}(\boldsymbol{\lambda})$ are precisely the vectors of the form $(\lambda_{1ij}, \lambda_{2ij}, \ldots, \lambda_{nij})^{tr}$, for $1 \leqslant i, j \leqslant n$ (see also the proof of [12, Lemma 3.16]).

(iii) Let $\boldsymbol{\lambda}\,(= (\lambda_{ijk})) \in \boldsymbol{\Lambda}$ and let $\phi \in \operatorname{End}_{\mathbb{F}} V$. Also let α, β and γ be (not necessarily distinct) elements of $\mathbb{F}$. For $1 \leqslant r, s \leqslant n$, we denote by $\phi_{(r,s)}\,(\in \mathbb{F})$ the coefficient of e_r when we express $\phi(e_s)$ as a $\mathbb{F}$-linear combination of the elements of the standard basis $(e_1, \ldots, e_n)$ of V. Suppose further that the algebra $\Theta^{-1}(\boldsymbol{\lambda})$ has multiplication given by $[\,,\,]$. Recall that in Section 2 we have imposed an ordering on the n^3 ordered triples (i, j, k), $1 \leqslant i, j, k \leqslant n$. For the discussion that follows it will also be convenient to impose an ordering on the n^2 ordered pairs (r, s) $1 \leqslant r, s \leqslant n$. Via this ordering we can consider the row-matrix $\widehat{\phi} \in \mathbb{F}^{1 \times n^2}$ whose coefficients are the $\phi_{(r,s)}$, for $1 \leqslant r, s \leqslant n$. The map $\phi \mapsto \widehat{\phi}$ from $\operatorname{End}_{\mathbb{F}} V$ to $\mathbb{F}^{1 \times n^2}$ is then an isomorphism of $\mathbb{F}$-spaces.

By expanding (for $1 \leqslant i, j \leqslant n$) the expression

$$\alpha\phi([e_i, e_j]) - \beta[\phi(e_i), e_j] - \gamma[e_i, \phi(e_j)]\;(\in V)$$

as a linear combination of the elements of the standard basis $(e_1, \ldots, e_n)$ of V, we can see that the coefficient of e_k $(1 \leqslant k \leqslant n)$ in this expression is given by

$$\sum_{r,s=1}^{n} \phi_{(r,s)} \delta_{(r,s)}^{(i,j,k)}\;(\in \mathbb{F})$$

for some coefficients $\delta_{(r,s)}^{(i,j,k)} \in \mathbb{F}$ which depend on i, j, k, r and s. The coefficients $\delta_{(r,s)}^{(i,j,k)}$ are indeed expressions of the form $\sum \xi_{abc}\lambda_{abc}$ with the only allowed values of the elements ξ_{abc} of $\mathbb{F}$ coming from the list 0, α, β, γ (for example, when $n = 3$, we obtain that $\delta_{(1,1)}^{(1,2,1)} = \alpha\lambda_{121} - \beta\lambda_{121}$ and $\delta_{(1,2)}^{(1,2,1)} = \alpha\lambda_{122} - \gamma\lambda_{111}$).

For each $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ we can now define matrix $\widetilde{d}(\boldsymbol{\lambda}) \in \mathbb{F}^{n^2 \times n^3}$ as the matrix having the n^5 elements $\delta_{(r,s)}^{(i,j,k)}$ as its coefficients: the column-index (respectively, row-index) of the coefficient $\delta_{(r,s)}^{(i,j,k)}$ in the matrix $\widetilde{d}(\boldsymbol{\lambda})$ is given by the ordering we have fixed on the triples (i, j, k) (respectively, the pairs (r, s)). In particular, the coefficients in the (i, j, k)-column of $\widetilde{d}(\boldsymbol{\lambda})$ are the $\delta_{(r,s)}^{(i,j,k)}$, for $1 \leqslant r, s \leqslant n$.

It is then easy to observe that $\phi \in \mathrm{Der}_{(\alpha,\beta,\gamma)}\Theta^{-1}(\boldsymbol{\lambda})$ if, and only if, $\widehat{\phi}\,\widetilde{d}(\boldsymbol{\lambda}) = \mathbf{0}_{1\times n^3}$, the $(1 \times n^3)$ zero matrix. It follows that $\dim \mathrm{Der}_{(\alpha,\beta,\gamma)}\Theta^{-1}(\boldsymbol{\lambda}) = n^2 - \operatorname{rank} \widetilde{d}(\boldsymbol{\lambda})$ (since this equals to the dimension of the kernel of the linear map $\widehat{u} \mapsto \widehat{u}\,\widetilde{d}(\boldsymbol{\lambda})$ from $\mathbb{F}^{1\times n^2}$ to $\mathbb{F}^{1\times n^3}$). Comparing, for example, with [12, Result 3.13 and Remark 3.15] we see that the set

$$\{\boldsymbol{\mu} \in \boldsymbol{\Lambda} \colon \operatorname{rank} \widetilde{d}(\boldsymbol{\mu}) \leqslant \operatorname{rank} \widetilde{d}(\boldsymbol{\lambda})\}$$

is Zariski-closed. Now, note that $\operatorname{rank} \widetilde{d}(\boldsymbol{\lambda}) = \operatorname{rank} \widetilde{d}(\boldsymbol{\lambda}')$ whenever $\boldsymbol{\lambda}' \in O(\boldsymbol{\lambda})$, since $\Theta^{-1}(\boldsymbol{\lambda}) \simeq \Theta^{-1}(\boldsymbol{\lambda}')$. Hence,

$$O(\boldsymbol{\lambda}) \subseteq \{\boldsymbol{\mu} \in \boldsymbol{\Lambda} \colon \operatorname{rank} \widetilde{d}(\boldsymbol{\mu}) \leqslant \operatorname{rank} \widetilde{d}(\boldsymbol{\lambda})\}.$$

It follows that

$$\dim \mathrm{Der}_{(\alpha,\beta,\gamma)}\Theta^{-1}(\boldsymbol{\mu}) \geqslant \dim \mathrm{Der}_{(\alpha,\beta,\gamma)}\Theta^{-1}(\boldsymbol{\lambda})$$

whenever $\boldsymbol{\mu} \in \overline{O(\boldsymbol{\lambda})}$.

(iv) The statement follows, for example, by combining [6, Propositions 1.5.2 and 2.5.3] and [9, Proposition at p.60 and Corollary at p.88] (see also [19, Example 2, p. 23]).

The proof is complete. □

Clearly, if for some $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ we have $\overline{O(\boldsymbol{\lambda})} \subseteq S$, with S an algebraic subset of $\boldsymbol{\Lambda}$, then $\boldsymbol{\mu} \in S$, for all $\boldsymbol{\mu} \in \overline{O(\boldsymbol{\lambda})}$. In the next section, we recall from [12] and [20] some particular algebraic subsets of $\boldsymbol{\Lambda}$ which played a key role in those papers and which will also play some part in the present paper.

4 Certain algebraic subsets of $\boldsymbol{\Lambda}$ and the maps ϕ_1, ϕ_2

Following the notation in [20], we define the subsets $\mathcal{K}$, $\mathcal{C}$, $\mathcal{M}^*$ and $\mathcal{M}^{**}$ of $\boldsymbol{\Lambda}$ by

$$\begin{aligned}
\Theta^{-1}(\mathcal{K}) &= \{\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}\colon [u,u] = 0_V \text{ for all } u \in V\},\\
\Theta^{-1}(\mathcal{C}) &= \{\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}\colon [u,v] = [v,u] \text{ for all } u,v \in V\},\\
\Theta^{-1}(\mathcal{M}^*) &= \{\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}\colon [u,v] \in \mathbb{F}\text{-span}(u,v) \text{ for all } u,v \in V\},\\
\Theta^{-1}(\mathcal{M}^{**}) &= \{\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}\colon [u,u] \in \mathbb{F}\text{-span}(u) \text{ for all } u \in V\}.
\end{aligned}$$

Then (see [20, Sections 3.1, 5.1 (Eq. (7)) and 6.1 (Eq. (12))]) we have

$$\begin{aligned}
\mathcal{K} &= \{(\lambda_{ijk}) \in \boldsymbol{\Lambda}\colon \lambda_{iii} = \lambda_{iij} = \lambda_{ijk} + \lambda_{jik} = \lambda_{iji} + \lambda_{jii} = 0\},\\
\mathcal{C} &= \{(\lambda_{ijk}) \in \boldsymbol{\Lambda}\colon \lambda_{ijj} = \lambda_{jij} \text{ and } \lambda_{ijk} = \lambda_{jik}\},\\
\mathcal{M}^* &= \left\{(\lambda_{ijk}) \in \boldsymbol{\Lambda}\colon \begin{array}{l} \lambda_{iij} = \lambda_{ijk} = 0,\ \lambda_{ijj} = \lambda_{ikk},\ \lambda_{jij} = \lambda_{kik} \\ \text{and } \lambda_{iii} = \lambda_{ijj} + \lambda_{jij} \end{array}\right\},\\
\mathcal{M}^{**} &= \{(\lambda_{ijk}) \in \boldsymbol{\Lambda}\colon \lambda_{ijk} + \lambda_{jik} = \lambda_{iij} = 0 \text{ and } \lambda_{iii} = \lambda_{ijj} + \lambda_{jij}\},
\end{aligned}$$

where the following convention is in force for the description of the last four sets: Different letters in the subscripts for the components of a structure vector represent different numerical values, but all such choices of subscripts are allowed. In particular, $\mathcal{K}$, $\mathcal{C}$, $\mathcal{M}^*$ and $\mathcal{M}^{**}$ are Zariski-closed subsets of $\boldsymbol{\Lambda}$. Moreover, these sets are all unions of orbits with respect to the action of G on $\boldsymbol{\Lambda}$ we are considering.

We also define the subsets $\mathcal{B}$ and $\mathcal{T}$ of $\boldsymbol{\Lambda}$ by

$$\begin{aligned}
\Theta^{-1}(\mathcal{B}) &= \{\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}\colon [[u,v],w] = 0_V \text{ for all } u,v,w \in V\},\\
\Theta^{-1}(\mathcal{T}) &= \{\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}\colon \operatorname{trace} \operatorname{ad}_u = 0 \text{ for all } u \in V\},
\end{aligned}$$

where $\mathrm{ad}_u\colon V \to V : v \mapsto [u,v]$, $(v \in V)$ is the adjoint map. Then, (see for example [12, Remark 2.7 and Remark 4.12]), we have that

$$\mathcal{B} = \{\boldsymbol{\lambda} = (\lambda_{ijk}) \in \boldsymbol{\Lambda}\colon \sum_{l=1}^{n} \lambda_{ijl}\lambda_{lkm} = 0 \text{ for } 1 \leqslant i,j,k,m \leqslant n\}$$

and

$$\mathcal{T} = \{\boldsymbol{\lambda} = (\lambda_{ijk}) \in \boldsymbol{\Lambda}\colon \sum_{j=1}^{n} \lambda_{ijj} = 0 \text{ for } 1 \leqslant i \leqslant n\}.$$

In particular, both $\mathcal{B}$ and $\mathcal{T}$ are algebraic subsets of $\boldsymbol{\Lambda}$ which are also unions of orbits.

Finally, we introduce the subsets $\mathcal{A}^s$, $\mathcal{L}$ and $\mathcal{J}$ of $\boldsymbol{\Lambda}$ by

$$\Theta^{-1}(\mathcal{A}^s) = \{\mathfrak{g} \in \mathbf{A}\colon [[u,v],w] = [u,[v,w]] \text{ for all } u,v,w \in V\},$$

$$\Theta^{-1}(\mathcal{L}) = \left\{\mathfrak{g} \in \Theta^{-1}(\mathcal{K})\colon \begin{array}{l} [[u,v],w] + [[v,w],u] + [w,u],v] = 0_V \\ \text{for all } u,v,w \in V \end{array}\right\},$$

$$\Theta^{-1}(\mathcal{J}) = \{\mathfrak{g} \in \Theta^{-1}(\mathcal{C})\colon [[[u,u],v],u] = [[u,u],[v,u]] \text{ for all } u,v \in V\},$$

which correspond respectively to the sets of Associative, Lie and Jordan algebra structures in $\mathbf{A}$.

The sets $\mathcal{A}^s$, $\mathcal{L}$ and $\mathcal{J}$ of $\mathbf{A}$ are also algebraic and consist of unions of orbits. For defining conditions of these sets via polynomial equations see for example [14, Proposition 1 at p.4] for the sets $\mathcal{A}^s$ and $\mathcal{L}$, and [15, Section 2] for $\mathcal{J}$.

These last three sets will be of central importance in this paper as our goal is to determine the degeneration picture in $\mathcal{A}^s$ for the special case $n = 3$ and $\mathbb{F} = \mathbb{C}$ via the degeneration pictures in $\mathcal{L}$ and $\mathcal{J}$. In order to achieve this we will need first to introduce certain maps ϕ_1, ϕ_2 satisfying $\phi_1(\mathcal{A}^s) \subseteq \mathcal{L}$ and $\phi_2(\mathcal{A}^s) \subseteq \mathcal{J}$. For this we need $\mathrm{char}\,\mathbb{F} \neq 2$ and we therefore make this assumption for the rest of this section.

For an algebra $\mathfrak{g} = (V, [\,,\,]) \in \mathbf{A}$, the opposite algebra $\widetilde{\mathfrak{g}}$ has product $\widetilde{[\,,\,]}$ defined by $\widetilde{[u,v]} = [v,u]$, for all $u,v \in V$. If $\Theta(\mathfrak{g}) = \boldsymbol{\lambda}$ with $\boldsymbol{\lambda} = (\lambda_{ijk})$, we will write $\Theta(\widetilde{\mathfrak{g}}) = \widetilde{\boldsymbol{\lambda}}$ with $\widetilde{\boldsymbol{\lambda}} = (\widetilde{\lambda}_{ijk})$. Obviously $(\widetilde{\widetilde{\boldsymbol{\lambda}}}) = \boldsymbol{\lambda}$ and $\widetilde{\lambda}_{ijk} = \lambda_{jik}$, for all i,j,k. In [20, Lemma 3.2] it is shown that $(\widetilde{\boldsymbol{\lambda}})g = (\widetilde{\boldsymbol{\lambda} g})$, for all $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ and for all $g \in G$.

We now define the maps ϕ, ϕ_1 and $\phi_2 \colon \boldsymbol{\Lambda} \to \boldsymbol{\Lambda}$, respectively, by

$$\boldsymbol{\lambda} \mapsto \widetilde{\boldsymbol{\lambda}}, \quad \boldsymbol{\lambda} \mapsto \frac{1}{2}(\boldsymbol{\lambda} - \widetilde{\boldsymbol{\lambda}}), \quad \text{and} \quad \boldsymbol{\lambda} \mapsto \frac{1}{2}(\boldsymbol{\lambda} + \widetilde{\boldsymbol{\lambda}})$$

Clearly the maps ϕ, ϕ_1 and ϕ_2 are regular and hence continuous in the Zariski topology. Moreover, we have

$$\phi(\mathcal{A}^s) = \mathcal{A}^s, \ \phi(\mathcal{L}) = \mathcal{L}, \ \phi(\mathcal{J}) = \mathcal{J}, \phi_1(\mathcal{A}^s) \subseteq \mathcal{L}, \ \text{and} \ \phi_2(\mathcal{A}^s) \subseteq \mathcal{J}.$$

We collect some further observations regarding the maps ϕ, ϕ_1 and ϕ_2 in the following remark.

Remark 14 Let ψ be any of the maps ϕ, ϕ_1 or ϕ_2. Also let $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$.

(i) It follows from the fact that $(\widetilde{\boldsymbol{\lambda}})g = (\widetilde{\boldsymbol{\lambda} g})$, for any $g \in G$, and the linearity of the G-action we are considering, that ψ is an $\mathbb{F}G$-module endomorphism of $\boldsymbol{\Lambda}$ (see [20, Lemma 3.2]). In particular,

$$\psi(O(\boldsymbol{\lambda})) = \{\psi(\boldsymbol{\lambda} g) \colon g \in G\} = \{\psi(\boldsymbol{\lambda})g \colon g \in G\} = O(\psi(\boldsymbol{\lambda})).$$

(ii) If S is subset of $\boldsymbol{\Lambda}$ which is a union of G-orbits, then the subset $\psi^{-1}(S)$ of $\boldsymbol{\Lambda}$ is also a union of G-orbits since, for $g \in G$ and $\boldsymbol{\nu} \in \psi^{-1}(S)$, we have $\psi(\boldsymbol{\nu} g) = \psi(\boldsymbol{\nu})g \in S$.

(iii) From the continuity of ψ we get

$$\psi(\overline{O(\boldsymbol{\lambda})}) \subseteq \overline{O(\psi(\boldsymbol{\lambda}))} \, (= \overline{\psi(O(\boldsymbol{\lambda}))}).$$

(iv) Since ϕ is invertible with $\phi^{-1} = \phi$, we must have

$$\phi\,(\overline{O(\boldsymbol{\lambda})}) = \overline{O(\phi(\boldsymbol{\lambda}))} \, (= \overline{O(\widetilde{\boldsymbol{\lambda}})}).$$

(v) If $\mathbb{F}$ is algebraically closed and $\widetilde{\boldsymbol{\lambda}} \notin O(\boldsymbol{\lambda})$, then $\widetilde{\boldsymbol{\lambda}} \notin \overline{O(\boldsymbol{\lambda})}$ (and $\boldsymbol{\lambda} \notin \overline{O(\widetilde{\boldsymbol{\lambda}})}$). To see this, note that if $\widetilde{\boldsymbol{\lambda}} \in \overline{O(\boldsymbol{\lambda})} - O(\boldsymbol{\lambda})$, then

$$\boldsymbol{\lambda} \, (= \phi(\widetilde{\boldsymbol{\lambda}})) \in \phi(\overline{O(\boldsymbol{\lambda})}) = \overline{O(\widetilde{\boldsymbol{\lambda}})},$$

which is impossible as orbits are open in their closure (see, for example, [6, Proposition 2.5.2 (a)]).

5 3-Dimensional complex associative, Lie and Jordan algebras

We fix $\mathbb{F} = \mathbb{C}$ and $n = 3$ for the rest of the paper. We also denote by $\mathbb{K}$ the subset of $\mathbb{C}$ given by

$$\Big\{x + yi \colon x, y \in \mathbb{R} \text{ with } x > 0 \text{ or } (x = 0 \text{ and } y > 0)\Big\}.$$

$\mathfrak{a}$	Non-zero commutation relations relative to the standard basis of V	ϕ_1	ϕ_2
$\mathfrak{a}_0$	—	$\mathfrak{a}_0$	$\mathfrak{a}_0$
$\mathfrak{l}_1$	$e_2e_3 = -e_3e_2 = e_1$	$\mathfrak{l}_1$	$\mathfrak{a}_0$
$\mathfrak{c}_1$	$e_3e_3 = e_1$	$\mathfrak{a}_0$	$\mathfrak{c}_1$
$\mathfrak{c}_2$	$e_1e_1 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e_1e_3 = e_3e_1 = e_3$	$\mathfrak{a}_0$	$\mathfrak{c}_2$
$\mathfrak{c}_3$	$e_2e_2 = e_1$, $e_3e_3 = e_1$	$\mathfrak{a}_0$	$\mathfrak{c}_3$
$\mathfrak{c}_4$	$e_1e_1 = e_1$	$\mathfrak{a}_0$	$\mathfrak{c}_4$
$\mathfrak{c}_5$	$e_1e_1 = e_2$, $e_1e_2 = e_2e_1 = e_3$	$\mathfrak{a}_0$	$\mathfrak{c}_5$
$\mathfrak{c}_6$	$e_1e_1 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e_1e_3 = e_3e_1 = e_3$, $e_2e_2 = e_3$	$\mathfrak{a}_0$	$\mathfrak{c}_6$
$\mathfrak{c}_7$	$e_1e_1 = e_1$, $e_1e_2 = e_2e_1 = e_2$	$\mathfrak{a}_0$	$\mathfrak{c}_7$
$\mathfrak{c}_8$	$e_1e_1 = e_1$, $e_2e_2 = e_3$	$\mathfrak{a}_0$	$\mathfrak{c}_8$
$\mathfrak{c}_9$	$e_1e_1 = e_1$, $e_2e_2 = e_2$, $e_1e_3 = e_3e_1 = e_3$	$\mathfrak{a}_0$	$\mathfrak{c}_9$
$\mathfrak{c}_{10}$	$e_1e_1 = e_1$, $e_2e_2 = e_2$	$\mathfrak{a}_0$	$\mathfrak{c}_{10}$
$\mathfrak{c}_{11}$	$e_1e_1 = e_1$, $e_2e_2 = e_2$, $e_3e_3 = e_3$	$\mathfrak{a}_0$	$\mathfrak{c}_{11}$
$\mathfrak{a}_1$	$e_1e_1 = e_1$, $e_1e_2 = e_2e_1 = e_2$, $e_1e_3 = e_3e_1 = e_3$, $e_2e_3 = -e_3e_2 = e_2$, $e_3^2 = e_1$	$\mathfrak{m}_4$	$\mathfrak{J}_2$
$\mathfrak{a}_2$	$e_2e_3 = e_1$	$\mathfrak{l}_1$	$\mathfrak{c}_3$
$\mathfrak{a}_3(\kappa)$	$e_2e_2 = e_1$, $e_3e_2 = \kappa e_1$, $e_3e_3 = e_1$, where $\kappa \in \mathbb{K}$ (different values of $\kappa \in \mathbb{K}$ correspond to non-isomorphic algebras)	$\mathfrak{l}_1$	$\mathfrak{c}_3$
$\mathfrak{a}_4$	$e_3e_1 = e_1$, $e_3e_2 = e_2$, $e_3e_3 = e_3$	$\mathfrak{m}_2$	$\mathfrak{J}_5$
$\mathfrak{a}_5$	$e_1e_3 = e_1$, $e_2e_3 = e_2$, $e_3e_3 = e_3$	$\mathfrak{m}_2$	$\mathfrak{J}_5$
$\mathfrak{a}_6$	$e_2e_1 = e_1$, $e_2e_2 = e_2$, $e_3e_2 = e_3$	$\mathfrak{m}_3$	$\mathfrak{J}_5$
$\mathfrak{a}_7$	$e_1e_2 = e_2e_1 = e_1$, $e_2e_2 = e_2$, $e_3e_2 = e_3$	$\mathfrak{m}_4$	$\mathfrak{J}_4$
$\mathfrak{a}_8$	$e_1e_2 = e_2e_1 = e_1$, $e_2e_2 = e_2$, $e_2e_3 = e_3$	$\mathfrak{m}_4$	$\mathfrak{J}_4$
$\mathfrak{a}_9$	$e_3e_2 = e_2$, $e_3e_3 = e_3$	$\mathfrak{m}_4$	$\mathfrak{J}_8$
$\mathfrak{a}_{10}$	$e_2e_3 = e_2$, $e_3e_3 = e_3$	$\mathfrak{m}_4$	$\mathfrak{J}_8$
$\mathfrak{a}_{11}$	$e_1e_1 = e_1$, $e_3e_2 = e_2$, $e_3e_3 = e_3$	$\mathfrak{m}_4$	$\mathfrak{J}_3$
$\mathfrak{a}_{12}$	$e_1e_1 = e_1$, $e_2e_3 = e_2$, $e_3e_3 = e_3$	$\mathfrak{m}_4$	$\mathfrak{J}_3$

Table 1. Non-isomorphic 3-dimensional complex associative algebras.

Comments on Table 1

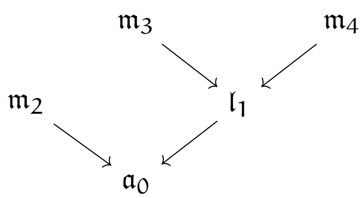
(i) In the first column of Table 1 we list a complete set of non-isomorphic 3-dimensional complex associative algebras based on the classification obtained in [16].

(ii) Entry $\mathfrak{g}$ (with $\mathfrak{g} \in \mathbf{A}$) in the column headed ϕ_1 (respectively, ϕ_2) and in the row corresponding to algebra $\mathfrak{a} \in \Theta^{-1}(\mathcal{A}^s)$ means that $\phi_1(\Theta(\mathfrak{a})) \in O(\Theta(\mathfrak{g}))$ (respectively, $\phi_2(\Theta(\mathfrak{a})) \in O(\Theta(\mathfrak{g}))$).

(iii) A complete list of non-isomorphic algebras $\mathfrak{g} \in \mathbf{A}$ with $\Theta(\mathfrak{g})$ lying in the set $\phi_1(\mathcal{A}^s)$ consists of the two associative Lie algebras (the Abelian algebra $\mathfrak{a}_0$ and the Heisenberg algebra $\mathfrak{l}_1$) and the following three non-associative Lie algebras (only non-zero commutation relations are listed):

$$\begin{aligned}
&\mathfrak{m}_2 : e_1e_3 = -e_3e_1 = e_1, e_2e_3 = -e_3e_2 = e_2,\\
&\mathfrak{m}_3 : e_1e_3 = -e_3e_1 = e_1, e_2e_3 = -e_3e_2 = -e_2,\\
&\mathfrak{m}_4 : e_1e_2 = -e_2e_1 = e_1.
\end{aligned}$$

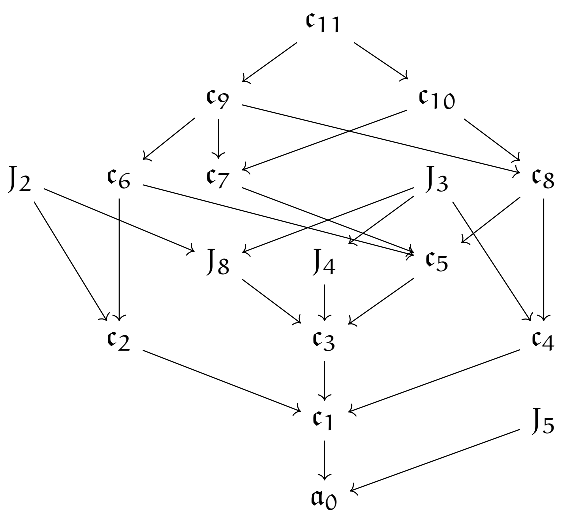
In Picture 1 below we include all possible degenerations inside the set $\phi_1(\mathcal{A}^s)$ ($\subseteq \mathcal{L}$) as these were determined in [1, 18]. In particular, the results in [1, 18] show that the set $\phi_1(\mathcal{A}^s)$ is an algebraic subset of $\mathbf{\Lambda}$. We remark here that although the arguments in [1, 18] are made with respect to the metric topology (as the authors are interested in contractions of Lie algebras), they carry out easily to arguments in the Zariski topology (compare with Example 3 and Remark 7 (ii) of the present paper). In fact, it was shown in [8] that the closures in the Zariski topology and in the standard topology of the orbit of a point of an affine variety over $\mathbb{C}$ under the action of an algebraic group coincide.



Picture 1

(iv) A complete list of all non-isomorphic algebras $\mathfrak{g} \in \mathbf{A}$ with $\Theta(\mathfrak{g}) \in \phi_2(\mathcal{A}^s)$ consists of the associative commutative algebras $\mathfrak{a}_0$ and $\mathfrak{c}_i$ $(1 \leqslant i \leqslant 11)$ together with the following non-associative Jordan algebras (only non-zero commutation relations are listed):

$$J_2 : e_1e_1 = e_1, e_2e_2 = e_2, e_1e_3 = e_3e_1 = \frac{1}{2}e_3, e_2e_3 = e_3e_2 = \frac{1}{2}e_3,$$
$$J_3 : e_1e_1 = e_1, e_2e_2 = e_2, e_1e_3 = e_3e_1 = \frac{1}{2}e_3,$$
$$J_4 : e_1e_1 = e_1, e_1e_2 = e_2e_1 = \frac{1}{2}e_2, e_1e_3 = e_3e_1 = e_3,$$
$$J_5 : e_1e_1 = e_1, e_1e_2 = e_2e_1 = \frac{1}{2}e_2, e_1e_3 = e_3e_1 = \frac{1}{2}e_3,$$
$$J_8 : e_1e_1 = e_1, e_1e_3 = e_3e_1 = \frac{1}{2}e_3.$$



Picture 2

In Picture 2 we include all possible degenerations inside the set $\phi_2(\mathcal{A}^s)$ $(\subseteq \mathcal{J})$ as these were determined in [15, 7]. In particular, the results of those papers show that $\phi_2(\mathcal{A}^s)$ is an algebraic subset of $\mathcal{J}$.

(v) In Table 2 below we collect information on certain algebra invariants, computed for some of the algebras listed in Table 1.

$\mathfrak{a}$	$\mathfrak{a}_0$	$\mathfrak{a}_2$	$\mathfrak{a}_3(\kappa)$	$\mathfrak{l}_1$	$\mathfrak{c}_1$	$\mathfrak{c}_3$	$\mathfrak{a}_7$	$\mathfrak{a}_8$	$\mathfrak{a}_9$	$\mathfrak{a}_{10}$	$\mathfrak{a}_{11}$
$\dim \operatorname{ann}_L \mathfrak{a}$	3	2	1	1	2	1	0	1	2	1	1
$\dim \operatorname{ann}_R \mathfrak{a}$	3	2	1	1	2	1	1	0	1	2	0
$\dim \operatorname{Der}$	9	4	4	6	5	4	3	3	3	3	2
$\dim \operatorname{Der}_{(1,0,1)}$	9	5	3	3	5	3	2	5	5	3	5

Table 2 (with $\kappa \in \mathbb{K}$).

Notation At this point it will be convenient to introduce some more notation.

(i) We will say that two algebras $\mathfrak{g}_1, \mathfrak{g}_2 \in A$ form a $\{\mathfrak{g}, \widetilde{\mathfrak{g}}\}$-pair if $\Theta(\widetilde{\mathfrak{g}_2}) \in O(\Theta(\mathfrak{g}_1))$ but $\Theta(\mathfrak{g}_2) \notin O(\Theta(\mathfrak{g}_1))$. In such a case we also have that $\Theta(\widetilde{\mathfrak{g}_1}) \in O(\Theta(\mathfrak{g}_2))$ and $\Theta(\mathfrak{g}_1) \notin O(\Theta(\mathfrak{g}_2))$. Note that, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{\Lambda}$, we have that $\phi(\boldsymbol{\mu}) \in O(\boldsymbol{\lambda})$ if, and only if, $\boldsymbol{\lambda} \in O(\phi(\boldsymbol{\mu}))\,(= \phi(O(\boldsymbol{\mu})))$, and this last statement holds if, and only if, $\phi(\boldsymbol{\lambda}) \in O(\boldsymbol{\mu})$ since $\phi^2 = \mathrm{id}_{\boldsymbol{\Lambda}}$.

(ii) For the sake of simplicity, in various occasions in the rest of the paper we will use, for $\mathfrak{g} \in \mathbf{A}$, symbol $\mathfrak{g}^o$ to mean $\overline{O(\Theta(\mathfrak{g}))}$, in particular, $\mathfrak{g}^o \subseteq \boldsymbol{\Lambda}$. In this notation we have $\overline{\mathfrak{a}_0^o} = \mathfrak{a}_0^o = \{\mathbf{0}\} \subseteq \boldsymbol{\Lambda}$.

In the following remark we collect some observations regarding the algebras in Table 1.

Remark 15 (i) The following is a complete (up to isomorphism) list of pairs of algebras from Table 1 that form $\{\mathfrak{g}, \widetilde{\mathfrak{g}}\}$-pairs:

$$\{\mathfrak{a}_4, \mathfrak{a}_5\}, \quad \{\mathfrak{a}_7, \mathfrak{a}_8\}, \quad \{\mathfrak{a}_9, \mathfrak{a}_{10}\}, \quad \{\mathfrak{a}_{11}, \mathfrak{a}_{12}\}.$$

(ii) We have the following intersections of $\mathcal{A}^s$ with some of our familiar algebraic sets:

$$\mathcal{A}^s \cap \mathcal{K} = \{\mathbf{0}\} \cup \mathfrak{l}_1^o,$$

$$\mathcal{A}^s \cap \mathcal{M}^{**} = \{\mathbf{0}\} \cup \mathfrak{a}_4^o \cup \mathfrak{a}_5^o \cup \mathfrak{a}_6^o \cup \mathfrak{l}_1^o,$$

$$\mathcal{A}^s \cap \mathcal{M}^{*} = \{\mathbf{0}\} \cup \mathfrak{a}_4^o \cup \mathfrak{a}_5^o,$$

$$\mathcal{A}^s \cap \mathcal{C} = \{\mathbf{0}\} \cup \Big(\bigcup_{1 \leqslant i \leqslant 11} \mathfrak{c}_i^o \Big),$$

$$\mathcal{A}^s \cap \mathcal{T} = \{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{c}_1^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_5^o \cup \mathfrak{a}_2^o \cup \Big(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o \Big),$$

$$\mathcal{A}^s \cap (\mathcal{C} \cap \mathcal{T}) = \{\mathbf{0}\} \cup \mathfrak{c}_1^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_5^o,$$

$$\mathcal{A}^s \cap \mathcal{B} = \{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{c}_1^o \cup \mathfrak{c}_3^o \cup \mathfrak{a}_2^o \cup \Big(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o \Big).$$

By setting

$$\widetilde{\mathcal{T}} = \phi(\mathcal{T}) \Big(= \{\boldsymbol{\lambda} = (\lambda_{ijk}) \in \boldsymbol{\Lambda} \colon \sum_{j=1}^{n} \lambda_{jij} = 0 \text{ for all } 1 \leqslant i \leqslant n\} \Big),$$

we see that $\mathcal{A}^s \cap \mathcal{T} = \mathcal{A}^s \cap \widetilde{\mathcal{T}}$ and that $\mathcal{A}^s \cap \mathcal{B} \subseteq \mathcal{A}^s \cap (\mathcal{T} \cap \widetilde{\mathcal{T}})$.

(iii) (a) Note that $\Theta(\mathfrak{l}_1) \in O(\boldsymbol{\eta})$ and $\Theta(\mathfrak{c}_1) \in O(\boldsymbol{\delta})$ with

$$\boldsymbol{\eta} = \mathbf{123} - \mathbf{213} \quad \text{and} \quad \boldsymbol{\delta} = \mathbf{112}$$

Also recall from [12] that

$$\overline{O(\boldsymbol{\eta})} = \{\mathbf{0}\} \cup O(\boldsymbol{\eta}) \quad \text{and} \quad \overline{O(\boldsymbol{\delta})} = \{\mathbf{0}\} \cup O(\boldsymbol{\delta}).$$

(b) If $\boldsymbol{\lambda} \in \mathcal{M}^{**} - \mathcal{M}^*$, then $\boldsymbol{\eta} \in \overline{O(\boldsymbol{\lambda})}$ (see [12, Lemma 4.4]).

(c) If $\boldsymbol{\lambda} \in \boldsymbol{\Lambda} - \mathcal{M}^{**}$, then $\boldsymbol{\delta} \in \overline{O(\boldsymbol{\lambda})}$ (see [12, Lemma 5.4]).

(iv) Clearly $\mathcal{C} \cap \mathcal{A}^s = \{\boldsymbol{\lambda} \in \mathcal{A}^s \colon \boldsymbol{\lambda} = \widetilde{\boldsymbol{\lambda}}\}$, so for $\boldsymbol{\mu} \in \mathcal{C} \cap \mathcal{A}^s$ we have that $\boldsymbol{\mu} = \phi_2(\boldsymbol{\mu}) \in \mathcal{J}$. Invoking the fact that $\mathcal{J} \subseteq \mathcal{C}$, we get $\mathcal{C} \cap \mathcal{A}^s = \mathcal{J} \cap \mathcal{A}^s$. It follows that the degeneration picture inside the algebraic subset $\mathcal{C} \cap \mathcal{A}^s$ of $\mathcal{A}^s$ coincides with the degeneration picture inside the algebraic subset $\mathcal{J} \cap \mathcal{A}^s$ of $\mathcal{J}$.

6 Degenerations of 3-dimensional complex associative algebras

We continue with our hypothesis that $\mathbb{F} = \mathbb{C}$ and $n = 3$. Also recall that $\mathbb{K} = \big\{x + yi \colon x, y \in \mathbb{R} \text{ with } x > 0 \text{ or } (x = 0 \text{ and } y > 0)\big\} \subseteq \mathbb{C}$.

Our aim in this section is to determine the complete degeneration picture inside $\mathcal{A}^s$. Our general plan in order to achieve this is to make use of the observations in Remark 15 together with sets of the form

$$T = \phi_1^{-1}(T_1) \cap \phi_2^{-1}(T_2) \cap \mathcal{A}^s,$$

where the sets T_1, T_2 ($T_1 \subseteq \mathcal{L}$, $T_2 \subseteq \mathcal{J}$) are algebraic sets which are also unions of G-orbits under the action of G on $\mathbf{\Lambda}$ we are considering. Since the maps ϕ_1 and ϕ_2 are continuous in the Zariski topology, we can see that such sets T are also algebraic and, moreover, Remark 14 (ii) ensures that they consist of a union of G-orbits (note that the intersection of subsets of $\mathbf{\Lambda}$ which are unions of G-orbits, if it is non-empty, is also a union of G-orbits). In order to obtain the complete degeneration picture inside $\mathcal{A}^s$, we will need to employ additional arguments, depending for example on necessary conditions for degeneration or arguments using the ideas involved in the preliminary lemmas in Section 2 of this paper.

The process will be completed through a sequence of steps. The corresponding sets T_1 and T_2 at each step of the process are chosen by "filtering out" closed sets as we go from bottom to top in Pictures 1 and 2 respectively.

Step 1 We consider the set

$$S_1 = \phi_1^{-1}(\{\mathbf{0}\} \cup \mathfrak{m}_2^o) \cap \phi_2^{-1}(\{\mathbf{0}\} \cup J_5^o) \cap \mathcal{A}^s = \{\mathbf{0}\} \cup \mathfrak{a}_4^o \cup \mathfrak{a}_5^o = \mathcal{A}^s \cap \mathcal{M}^*,$$

a closed set which is also a union of orbits from the above discussion. From Remark 14 (v) and Remark 15 (i),(ii) we get that both the sets $\{\mathbf{0}\} \cup \mathfrak{a}_4^o$ and $\{\mathbf{0}\} \cup \mathfrak{a}_5^o$ are closed (alternatively, we could use the results in [12]).

Step 2 We consider the set

$$\begin{aligned} S_2 &= \phi_1^{-1}(\{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{m}_3^o) \cap \phi_2^{-1}(\{\mathbf{0}\} \cup J_5^o) \cap \mathcal{A}^s \\ &= \{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{a}_6^o \subseteq \mathcal{A}^s \cap \mathcal{M}^{**}. \end{aligned}$$

From Remark 15 (ii),(iii) we get that $\Theta(\mathfrak{l}_1) \in \overline{\mathfrak{a}_6^o}$ since

$$\Theta(\mathfrak{a}_6) \in \mathcal{M}^{**} - \mathcal{M}^*.$$

Note that

$$\mathcal{A}^s \cap \mathcal{M}^{**} = (\{\mathbf{0}\} \cup \mathfrak{a}_4^o \cup \mathfrak{a}_5^o) \cup (\{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{a}_6^o),$$

a union of two closed sets. Moreover, the set $\{\mathbf{0}\} \cup \mathfrak{l}_1^o$ is closed, again by Remark 15 (iii).

Step 3 We consider the set

$$\begin{aligned} S_3 &= \phi_1^{-1}\left(\{\mathbf{0}\} \cup \mathfrak{l}_1^o\right) \cap \phi_2^{-1}\left(\{\mathbf{0}\} \cup \mathfrak{c}_1^o \cup \mathfrak{c}_3^o\right) \cap \mathcal{A}^s \\ &= \{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{c}_1^o \cup \mathfrak{c}_3^o \cup \mathfrak{a}_2^o \cup \Big(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o \Big) \\ &= \mathcal{A}^s \cap \mathcal{B} \subseteq \mathcal{A}^s \cap \mathcal{T}. \end{aligned}$$

The sets $\{\mathbf{0}\} \cup \mathfrak{l}_1^o$ $(= \mathcal{A}^s \cap \mathcal{K})$ and $\{\mathbf{0}\} \cup \mathfrak{c}_1^o \cup \mathfrak{c}_3^o$ $(= \mathcal{A}^s \cap \mathcal{B} \cap \mathcal{C} \cap \mathcal{T})$ are both closed, see Remark 15 (ii). Hence $\{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{c}_1^o \cup \mathfrak{c}_3^o$ is a closed subset of $\mathcal{A}^s$. Moreover, from Remark 15 (ii),(iii) we get that $\mathfrak{c}_3$, $\mathfrak{a}_2$ and $\mathfrak{a}_3(\kappa)$, for all $\kappa \in \mathbb{K}$, all degenerate to $\mathfrak{c}_1$ since $\Theta(\mathfrak{c}_1) \in O(\boldsymbol{\delta})$.

From the information about $\dim \operatorname{ann}_L$ in Table 2, we also get that $\Theta(\mathfrak{a}) \notin \overline{\mathfrak{a}_2^o}$, for all $\mathfrak{a} \in \{\mathfrak{l}_1, \mathfrak{c}_3\} \cup \{\mathfrak{a}_3(\kappa) \colon \kappa \in \mathbb{K}\}$. In particular,

$$\overline{\mathfrak{a}_2^o} = \{\mathbf{0}\} \cup \mathfrak{c}_1^o \cup \mathfrak{a}_2^o.$$

Also from Table 2, looking now at $\dim \operatorname{Der}$ of the corresponding algebras, we see that $\mathfrak{a}_3(\kappa) \not\to \mathfrak{a}_2$ and $\mathfrak{a}_3(\kappa) \not\to \mathfrak{c}_3$, for all $\kappa \in \mathbb{K}$, and, moreover, there is no degeneration between any two members of the infinite family $\mathfrak{a}_3(\kappa)$ with $\kappa \in \mathbb{K}$.

Next, we investigate whether there is a degeneration from $\mathfrak{a}_3(\kappa)$ to $\mathfrak{l}_1$, at least for some $\kappa \in \mathbb{K}$, and for this it will be convenient to consider the cases $\kappa = 2$ and $\kappa \in \mathbb{K} - \{2\}$ separately.

(a) We assume $\kappa = 2$: Let

$$\boldsymbol{\lambda} = \mathbf{221} + \mathbf{331} + 2(\mathbf{321}) \in \boldsymbol{\Lambda}.$$

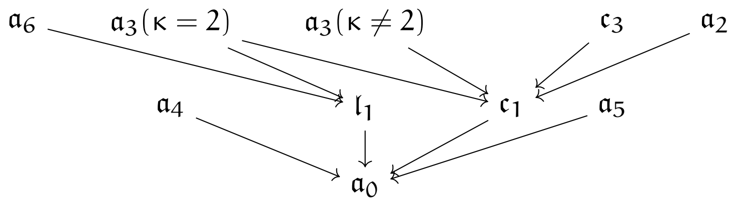
Then $\boldsymbol{\lambda} = \Theta(\mathfrak{a}_3(\kappa))$, so in Example 3 we established, in fact, that $\mathfrak{a}_3(\kappa) \to \mathfrak{l}_1$ in the case $\kappa = 2$.

(b) We assume $\kappa \in \mathbb{K} - \{2\}$: Let

$$\boldsymbol{\nu} = \Theta(\mathfrak{a}_3(\kappa)) = \mathbf{221} + \mathbf{331} + \kappa(\mathbf{321}) \in \boldsymbol{\Lambda}.$$

Also let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^2 + \kappa x + 1 \in \mathbb{C}[x]$. Comparing with Lemma 11, we see that our hypothesis on κ ensures that $\alpha^2 \neq 1$ and that $\boldsymbol{\lambda} = \mathbf{231} + -\alpha^2(\mathbf{321}) \in O(\boldsymbol{\nu})$. Invoking now Example 6 with $\beta = -\alpha^2$ $(\neq -1)$ we get that $\overline{O(\boldsymbol{\nu})} \cap \mathcal{K} = \overline{O(\boldsymbol{\lambda})} \cap \mathcal{K} = \{\mathbf{0}\}$. We conclude that $\mathfrak{a}_3(\kappa) \not\to \mathfrak{l}_1$ whenever $\kappa \in \mathbb{K} - \{2\}$ (to see this it might be useful to recall that $\overline{O(\boldsymbol{\nu})} \subseteq \mathcal{A}^s$ since $\boldsymbol{\nu} \in \mathcal{A}^s$, and also that $\mathcal{K} \cap \mathcal{A}^s = \{\mathbf{0}\} \cup \mathfrak{l}_1^o$).

This completes the degeneration picture inside the closed set S_3:



Picture 3 (with $\kappa \in \mathbb{K}$)

Step 4 We consider the set

$$S_4 = \phi_1^{-1}\left(\{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{m}_4^o\right) \cap \phi_2^{-1}\left(\{\mathfrak{J}_8^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_1^o \cup \{\mathbf{0}\}\}\right) \cap \mathcal{A}^s$$
$$= \mathfrak{a}_9^o \cup \mathfrak{a}_{10}^o \cup \mathfrak{a}_2^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_1^o \cup \mathfrak{l}_1^o \cup \{\mathbf{0}\} \cup \Big(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o\Big).$$

We can make the following observations.

(i) We know from Remark 15(i) that $\{\mathfrak{a}_9, \mathfrak{a}_{10}\}$ is a $\{\mathfrak{g}, \widetilde{\mathfrak{g}}\}$-pair, so $\mathfrak{a}_9 \not\to \mathfrak{a}_{10}$ and $\mathfrak{a}_{10} \not\to \mathfrak{a}_9$ in view of Remark 14 (v). This can also be seen by either considering $\dim \operatorname{ann}_L$ and $\dim \operatorname{ann}_R$ or $\dim \operatorname{Der}$ of these algebras (see Table 2).

(ii) From Step 3, the subset

$$S_3 = \mathfrak{a}_2^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_1^o \cup \mathfrak{l}_1^o \cup \{\mathbf{0}\} \cup \Big(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o\Big)$$

of S_4 is closed and we know everything about the degenerations between the members of S_3. Hence, in order to determine all possible degenerations between the members of the closed set S_4 it suffices to determine all possible degenerations from $\mathfrak{a}_9$ (or $\mathfrak{a}_{10}$) to each of the members of S_3.

(iii) Let $\lambda, \mu \in \Lambda$ with

$$\lambda = \mathbf{322} + \mathbf{333} \quad \text{and} \quad \mu = \mathbf{232} + \mathbf{333}\,(= \widetilde{\lambda})$$

Then $\lambda = \Theta(\mathfrak{a}_9)$ and $\mu = \Theta(\mathfrak{a}_{10})$ (see Table 1).

For $t \in \mathbb{C} - \{0\}$, define matrices $b_1(t), b_2(t) \in GL(3, \mathbb{C})$ as follows

$$b_1(t) = \begin{pmatrix} t & 0 & -1 \\ 0 & 0 & 1 \\ 0 & t & 0 \end{pmatrix} \quad \text{and} \quad b_2(t) = \begin{pmatrix} t & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

Then $\lambda\, b_1(t) = \mathbf{231} + t(\mathbf{233}) + t(\mathbf{222})$. Invoking Lemma 2 (see also argument in Example 3), we get that $\mathfrak{a}_9 \to \mathfrak{a}_2 = \Theta^{-1}(\mathbf{231})$. Similarly, by computing $\mu\, b_2(t)$, we also observe that $\mathfrak{a}_{10} \to \mathfrak{a}_2$. These are examples of construction of degeneration via Lemma 8. Observe that the subspace $\mathbb{C}$-span$(-e_1 + e_2)$ is a subalgebra of both $\mathfrak{a}_9$ and $\mathfrak{a}_{10}$. Note that, the observation that the pair of algebras $\{\mathfrak{a}_9, \mathfrak{a}_{10}\}$ is a $\{\mathfrak{g}, \widetilde{\mathfrak{g}}\}$-pair together with the fact that $\Theta(\widetilde{\mathfrak{a}_2}) \in \mathfrak{a}_2^o$ ensure that $\mathfrak{a}_9 \to \mathfrak{a}_2$ if, and only if, $\mathfrak{a}_{10} \to \mathfrak{a}_2$.

(iv) From the transitivity of degenerations we also get (the facts already known from [12], see Remark 15 (iii)) that

$$\mathfrak{g} \to \mathfrak{c}_1, \ \mathfrak{g} \to \mathfrak{a}_0,$$

for $\mathfrak{g} \in \{\mathfrak{a}_9, \mathfrak{a}_{10}\}$, since $\mathfrak{g} \to \mathfrak{a}_2$ and $\mathfrak{a}_2 \to \mathfrak{c}_1$, $\mathfrak{a}_2 \to \mathfrak{a}_0$.

(v) Finally, we use the information from Table 2 on $\dim \operatorname{ann}_L$ and $\dim \operatorname{ann}_R$: $\dim \operatorname{ann}_L$ gives:

$$\mathfrak{a}_9 \not\to \mathfrak{l}_1, \ \mathfrak{a}_9 \not\to \mathfrak{c}_3, \ \mathfrak{a}_9 \not\to \mathfrak{a}_3(\kappa),$$

for all $\kappa \in \mathbb{K}$, while $\dim \operatorname{ann}_R$ gives:

$$\mathfrak{a}_{10} \not\to \mathfrak{l}_1, \ \mathfrak{a}_{10} \not\to \mathfrak{c}_3, \ \mathfrak{a}_{10} \not\to \mathfrak{a}_3(\kappa),$$

for all $\kappa \in \mathbb{K}$. Alternatively, we could use an obvious modification of the remark immediately above in order to obtain the corresponding information for $\mathfrak{a}_{10}$ given the information on $\mathfrak{a}_9$.

The observations (i)–(v) above supply sufficient information in order to complete the degeneration picture inside set S_4.

Step 5 We consider the set

$$\begin{aligned} S_5 &= \phi_1^{-1}\left(\{\mathbf{0}\}\cup \mathfrak{l}_1^o \cup \mathfrak{m}_4^o\right)\cap \phi_2^{-1}\left(\mathfrak{J}_4^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_1^o \cup \{\mathbf{0}\}\right)\cap \mathcal{A}^s \\ &= \mathfrak{a}_7^o \cup \mathfrak{a}_8^o \cup \mathfrak{a}_2^o \cup \mathfrak{l}_1^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_1^o \cup \{\mathbf{0}\}\cup \Big(\bigcup_{\kappa\in\mathbb{K}} \mathfrak{a}_3(\kappa)^o\Big) \\ &= \mathfrak{a}_7^o \cup \mathfrak{a}_8^o \cup S_3. \end{aligned}$$

Now $\{\mathfrak{a}_7, \mathfrak{a}_8\}$ is a $\{\mathfrak{g}, \widetilde{\mathfrak{g}}\}$-pair and the set S_3 is closed. Comparing with the discussion in Step 4, we get that $\mathfrak{a}_7 \not\to \mathfrak{a}_8$, $\mathfrak{a}_8 \not\to \mathfrak{a}_7$ and, moreover, in order to complete the degeneration picture inside the closed set S_5 it suffices to determine whether any of the members of the set $S_3 = \mathcal{A}^s \cap \mathcal{B}$ belong to $\overline{\mathfrak{g}^o}$, for $\mathfrak{g} \in \{\mathfrak{a}_7, \mathfrak{a}_8\}$. We aim to use Lemma 4.

Let $\boldsymbol{\lambda}' = \mathbf{121} + \mathbf{211} + \mathbf{222} + \mathbf{323} \in \boldsymbol{\Lambda}$, so $\boldsymbol{\lambda}' = \Theta(\mathfrak{a}_7)$ from Table 1. Also let B be the Borel subgroup of all upper triangular matrices in $GL(3,\mathbb{C})$. It will be convenient to consider the basis (e_1', e_2', e_3') of the underlying vector space V, where $e_1' = e_1$, $e_2' = e_3$, $e_3' = e_2$. Note that the subspace $\mathbb{C}$-span(e_1', e_2') is a subalgebra of $\mathfrak{a}_7$ isomorphic to the 2-dimensional Abelian algebra so the commutation relations remain very simple (many of the coefficients remain zero) when we act by the subgroup B. The structure vector of $\mathfrak{a}_7$ relative to the basis (e_1', e_2', e_3') is $\boldsymbol{\lambda} = \mathbf{131} + \mathbf{311} + \mathbf{333} + \mathbf{232}$. For $b = (b_{ij}) \in B$, in particular $b_{ij} = 0$ if $i > j$ and $b_{11}b_{22}b_{33} \neq 0$, we then have

$$\begin{aligned} \boldsymbol{\lambda} b &= b_{33}\mathbf{131} + b_{33}\mathbf{311} + b_{33}\mathbf{232} + \Big(\frac{b_{12}b_{33}}{b_{11}}\Big)\mathbf{321} \\ &\quad + \Big(\frac{b_{13}b_{33}}{b_{11}}\Big)\mathbf{331} + b_{33}\mathbf{333} \in \boldsymbol{\lambda}' G. \end{aligned}$$

Clearly, from the above expression for $\boldsymbol{\lambda} b$, the following polynomials all belong to $\mathbf{I}(\boldsymbol{\lambda} B)$, the vanishing ideal of $\boldsymbol{\lambda} B$:

$$\begin{aligned} &X_{11i},\ X_{12i},\ X_{21i},\ X_{22i}, \quad (1 \leqslant i \leqslant 3) \\ &X_{231},\ X_{233},\ X_{322},\ X_{323},\ X_{332}, \\ &X_{132},\ X_{133},\ X_{312},\ X_{313}, \\ &X_{232} - X_{333},\ X_{131} - X_{333},\ X_{311} - X_{333}. \end{aligned}$$

Now let $\boldsymbol{\mu} = (\mu_{ijk}) \in \overline{\boldsymbol{\lambda}B}$. Then $\mathbf{ev}_{\boldsymbol{\mu}}(f) = 0$, for all $f \in \mathbf{I}(\boldsymbol{\lambda}B)$, hence

$$\mu_{11i} = \mu_{12i} = 0,\ \mu_{21i} = \mu_{22i} = 0, \quad (1 \leqslant i \leqslant 3)$$

$$\mu_{231} = \mu_{233} = \mu_{322} = \mu_{323} = \mu_{332} = 0$$

$$\mu_{132} = \mu_{133} = \mu_{312} = \mu_{313} = 0,$$

$$\mu_{232} = \mu_{333} = \mu_{131} = \mu_{311}.$$

It follows that the only coefficients μ_{ijk} which can possibly be non-zero are: μ_{131}, μ_{232}, μ_{311}, μ_{321}, μ_{331} and μ_{333}, with the additional restriction that $\mu_{232} = \mu_{333} = \mu_{131} = \mu_{311}$.

At this point we make the further assumption that $\boldsymbol{\mu} \in \mathcal{B}$, so we have that $\boldsymbol{\mu} = (\mu_{ijk}) \in \overline{\boldsymbol{\lambda}B} \cap \mathcal{B}$. From the defining conditions of $\mathcal{B}$ (see Section 4), we must have that

$$\mu_{331}\mu_{133} + \mu_{332}\mu_{233} + \mu_{333}\mu_{333} = 0.$$

But $\mu_{133} = 0 = \mu_{233}$ from our previous considerations, so $\mu_{333}^2 = 0$ and hence $\mu_{333} = 0$. We conclude that all coefficients μ_{ijk} are equal to zero except possibly μ_{321} and μ_{331}. Next, we impose the further restriction that $\boldsymbol{\mu} \neq \mathbf{0}$. Hence it suffices to consider the following cases: (i) $\mu_{321} = 0$, $\mu_{331} \neq 0$ and (ii) $\mu_{321} \neq 0$. Note that there is no guarantee that any of cases (i), (ii) described above actually occur, as some other polynomials in $\mathbf{I}(\boldsymbol{\lambda}B)$ or some other defining conditions for $\mathcal{B}$ not already listed above could possibly impose even further restrictions on the coefficients μ_{ijk}. However, our goal at the moment is, by making use of Lemma 4, to exclude the possibility of degeneration from $\mathfrak{a}_7$ to various members of the set $\Theta^{-1}(\mathcal{A}^s \cap \mathcal{B}) = \Theta^{-1}(S_3)$.

Case (i): $\mu_{231} = 0$, $\mu_{331} \neq 0$. Clearly, $\Theta(\mathfrak{c}_1) \in O(\boldsymbol{\mu})$, so $\Theta^{-1}(\boldsymbol{\mu}) \simeq \mathfrak{c}_1$.

Case (ii): $\mu_{321} \neq 0$. Then $\Theta^{-1}(\boldsymbol{\mu}) \simeq \mathfrak{g}$ where $\Theta(\mathfrak{g}) = \mathbf{321} + \gamma(\mathbf{331})$, for some $\gamma \in \mathbb{C}$. We will consider subcases (iia) and (iib) given below:

Subcase (iia): $\gamma = 0$: Then clearly $\Theta^{-1}(\boldsymbol{\mu}) \simeq \mathfrak{a}_2$.

Subcase (iib): $\gamma \neq 0$: We consider $(\mathbf{321} + \gamma\mathbf{331})g$, with

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -\gamma^{-1} \end{pmatrix} \in GL(3, \mathbb{C})\,.$$

It is easy to compute that

$$(\mathbf{321} + \gamma\mathbf{331})g = -\frac{1}{\gamma}(\mathbf{321}),$$

so again $\Theta^{-1}(\mu) \simeq \mathfrak{a}_2$.

Summing up, we have shown up to this point that

$$\overline{\boldsymbol{\lambda}B} \cap \mathcal{B} \subseteq \{\mathbf{0}\} \cup \mathfrak{c}_1^o \cup \mathfrak{a}_2^o.$$

Let

$$\mathcal{U} = \mathfrak{l}_1^o \cup \mathfrak{c}_3^o \cup \Big(\bigcup_{\kappa\in\mathbb{K}} \mathfrak{a}_3(\kappa)^o\Big).$$

Then $\mathcal{U} \subseteq \mathcal{B}$ from Remark 15 (ii), hence

$$\mathcal{B} \cap \mathcal{U} = \mathcal{U} \quad \text{and} \quad \overline{\boldsymbol{\lambda}B} \cap \mathcal{U} = (\overline{\boldsymbol{\lambda}B} \cap \mathcal{B}) \cap \mathcal{U} = \varnothing.$$

It follows from Remark 5 (iv) that $\overline{O(\boldsymbol{\lambda})} \cap \mathcal{U} = \varnothing$. We conclude that

$$\mathfrak{a}_7 \not\to \mathfrak{l}_1,\ \mathfrak{a}_7 \not\to \mathfrak{c}_3 \text{ and } \mathfrak{a}_7 \not\to \mathfrak{a}_3(\kappa),$$

for all $\kappa \in \mathbb{K}$.

Recalling the expression for $\boldsymbol{\lambda}b$ obtained above and invoking Lemma 2 with $t \in \mathbb{C} - \{0\}$ (and an argument as in Example 3) we also get:

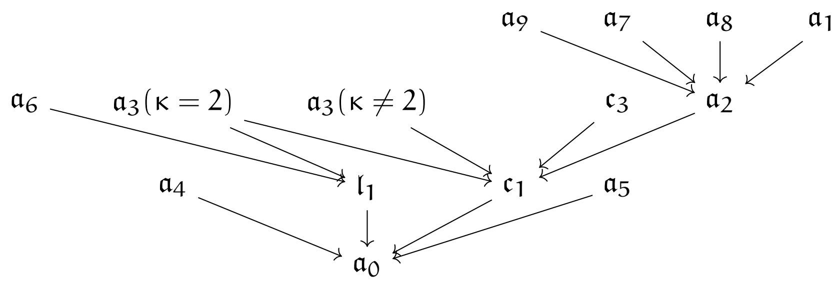
(i) $\mathfrak{a}_7 \to \mathfrak{a}_2$, by setting $b_{33} = t$, $b_{12} = t$, $b_{11} = t^2$ and $b_{13} = t^2$, and

(ii) $\mathfrak{a}_7 \to \mathfrak{c}_1$, by setting $b_{33} = t$, $b_{12} = t$, $b_{11} = t$ and $b_{13} = 1$.

Alternatively, we can also observe that in Example 10 a degeneration $\mathfrak{a}_7 \to \mathfrak{a}_2$ was in fact constructed since

$$\Theta(\mathfrak{a}_7) = \mathbf{121} + \mathbf{211} + \mathbf{222} + \mathbf{323}$$

and $\mathbf{213} \in \mathfrak{a}_2^o$. We can then invoke the transitivity of degenerations to establish that $\mathfrak{a}_7 \to \mathfrak{c}_1$ and $\mathfrak{a}_7 \to \mathfrak{a}_0$. Note that, these last two facts are already known since $\Theta(\mathfrak{a}_7) \in \boldsymbol{\Lambda} - \mathcal{M}^{**}$ (see Remark 15(ii),(iii)). This completes the degeneration picture inside the closed set S_5 since our results on degenerations still hold if we replace $\mathfrak{a}_7$ by $\mathfrak{a}_8$ in view of the fact that $\boldsymbol{\lambda} \in O(\widetilde{\boldsymbol{\lambda}})$, for all $\boldsymbol{\lambda} \in S_3$.



Picture 4 (with $\kappa \in \mathbb{K}$)

Step 6 We consider the set

$$S_6 = \phi_1^{-1}\left(\{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{m}_4^o\right) \cap \phi_2^{-1}\left(\mathfrak{J}_3^o \cup \mathfrak{J}_4^o \cup \mathfrak{J}_8^o \cup \mathfrak{c}_4^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_1^o \cup \{\mathbf{0}\}\right) \cap \mathcal{A}^s$$
$$= \mathfrak{a}_{11}^o \cup \mathfrak{a}_{12}^o \cup \mathfrak{a}_9^o \cup \mathfrak{a}_{10}^o \cup \mathfrak{a}_7^o \cup \mathfrak{a}_8^o \cup \mathfrak{c}_4^o \cup \mathfrak{a}_2^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_1^o \cup \mathfrak{l}_1^o \cup \{\mathbf{0}\}$$
$$\cup \left(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o\right)$$
$$= \mathfrak{a}_{11}^o \cup \mathfrak{a}_{12}^o \cup \mathfrak{c}_4^o \cup S_4 \cup S_5.$$

The set $S_4 \cup S_5$ is closed as it is the union of two closed sets and we know everything about the degeneration picture in this set from the previous steps of this process. We also know from the information given in Picture 2 that $\overline{\mathfrak{c}_4^o} = \{\mathbf{0}\} \cup \mathfrak{c}_1^o \cup \mathfrak{c}_4^o$.

Next, the algebras $\mathfrak{a}_{11}$ and $\mathfrak{a}_{12}$ form a $\{\mathfrak{g}, \widetilde{\mathfrak{g}}\}$-pair so there cannot be any degeneration between them. Moreover, invoking the facts that the pairs of algebras $\{\mathfrak{a}_7, \mathfrak{a}_8\}$ and $\{\mathfrak{a}_9, \mathfrak{a}_{10}\}$ also form $\{\mathfrak{g}, \widetilde{\mathfrak{g}}\}$-pairs while $\Theta(\widetilde{\mathfrak{a}}) \in \mathfrak{a}^o$, for all $\mathfrak{a} \in \mathbf{A}$ with

$$\Theta(\mathfrak{a}) \in S_6 - (\mathfrak{a}_7^o \cup \mathfrak{a}_8^o \cup \mathfrak{a}_9^o \cup \mathfrak{a}_{10}^o \cup \mathfrak{a}_{11}^o \cup \mathfrak{a}_{12}^o),$$

the set of algebras to which $\mathfrak{a}_{12}$ degenerates can easily be obtained once we know the set of algebras to which $\mathfrak{a}_{11}$ degenerates (compare with Remark 14 (iv)).

By considering now $\dim \mathrm{Der}_{(1,0,1)}$ (see Table 2), we can exclude the possibility of degeneration from $\mathfrak{a}_{11}$ to each of the members of the set

$$\{\mathfrak{a}_7, \mathfrak{a}_{10}, \mathfrak{c}_3, \mathfrak{l}_1\} \cup \{\mathfrak{a}_3(\kappa) \colon \kappa \in \mathbb{K}\}.$$

Moreover, the following holds:

Claim *Algebra* $\mathfrak{a}_{11}$ *degenerates to each one of the members of the set* $\{\mathfrak{a}_8, \mathfrak{a}_9, \mathfrak{c}_4, \mathfrak{a}_2, \mathfrak{c}_1, \mathfrak{a}_0\}$.
Proof of claim — From the transitivity of degenerations it suffices to prove that $\mathfrak{a}_{11} \to \mathfrak{c}_4$, $\mathfrak{a}_{11} \to \mathfrak{a}_9$ and $\mathfrak{a}_{11} \to \mathfrak{a}_8$ (compare with Picture 4).

We let $\boldsymbol{\lambda} = \Theta(\mathfrak{a}_{11}) = \mathbf{111} + \mathbf{322} + \mathbf{333}$, (see Table 1). We also define, for $t \in \mathbb{C} - \{0\}$, matrices $g_1(t)$, $g_2(t)$ and $g_3(t) \in GL(3, \mathbb{C})$ as follows:

$$g_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}, \ g_2(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } g_3(t) = \begin{pmatrix} t & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then,

(i) $\boldsymbol{\lambda} g_1(t) = \mathbf{111} + t(\mathbf{322}) + t(\mathbf{333})$,

(ii) $\boldsymbol{\lambda} g_2(t) = t(\mathbf{111}) + \mathbf{322} + \mathbf{333}$,

(iii) $\boldsymbol{\lambda} g_3(t) = t(\mathbf{111}) + \mathbf{121} + \mathbf{211} + \mathbf{222} + \mathbf{233}$.

Invoking Lemma 2 (and an argument as in Example 3) and comparing with the information given in Table 1, we get from the observations (i), (ii) and (iii) immediately above the existence, respectively, of degenerations $\mathfrak{a}_{11} \to \mathfrak{c}_4$, $\mathfrak{a}_{11} \to \mathfrak{a}_9$ and $\mathfrak{a}_{11} \to \mathfrak{a}_8$. This completes the degeneration picture inside the closed set S_6.

Step 7 Finally we consider the set

$$\begin{aligned} S_7 &= \phi_1^{-1}\left(\{\mathbf{0}\} \cup \mathfrak{l}_1^o \cup \mathfrak{m}_4^o\right) \cap \phi_2^{-1}\left(\mathfrak{J}_2^o \cup \mathfrak{J}_8^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_2^o \cup \mathfrak{c}_1^o \cup \{\mathbf{0}\}\right) \cap \mathcal{A}^s \\ &= \mathfrak{a}_1^o \cup \mathfrak{a}_2^o \cup \mathfrak{a}_9^o \cup \mathfrak{a}_{10}^o \cup \mathfrak{c}_3^o \cup \mathfrak{c}_2^o \cup \mathfrak{c}_1^o \cup \mathfrak{l}_1^o \cup \{\mathbf{0}\} \cup \Big(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o \Big). \end{aligned}$$

Let

$$\boldsymbol{\nu} = \Theta(\mathfrak{a}_1) = \mathbf{111} + \mathbf{122} + \mathbf{212} + \mathbf{133} + \mathbf{313} + \mathbf{232} - \mathbf{322} + \mathbf{331} \in \boldsymbol{\Lambda},$$

so $\boldsymbol{\nu}$ is the structure vector of algebra $\mathfrak{a}_1$ relative to the standard basis (e_1, e_2, e_3) of the underlying $\mathbb{C}$-vector space V. Consider now the basis (f_1, f_2, f_3) of V defined by

$$f_1 = \frac{1}{2}(e_1 - e_3), \ f_2 = \frac{1}{2}e_2, \ f_3 = \frac{1}{2}(e_1 + e_3)$$

and observe that the subspaces $\mathbb{C}$-span(f_2), $\mathbb{C}$-span(f_1, f_2) and $\mathbb{C}$-span(f_2, f_3) are all ideals of $\mathfrak{a}_1$. Let $\boldsymbol{\nu}'$ be the structure vector of $\mathfrak{a}_1$ relative to the basis (f_1, f_2, f_3). Then

$$\boldsymbol{\nu}' = \mathbf{111} + \mathbf{122} + \mathbf{232} + \mathbf{333}\, (\in O(\boldsymbol{\nu})).$$

Now define, for $t \in \mathbb{C} - \{0\}$, matrices $g_1(t)$, $g_2(t) \in GL(3, \mathbb{C})$ by

$$g_1(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

It follows that

$$\boldsymbol{\nu}' g_1(t) = t(\mathbf{111}) + t(\mathbf{122}) + \mathbf{232} + \mathbf{333} \in O(\boldsymbol{\nu})$$

and that

$$\boldsymbol{\nu}' g_2(t) = \mathbf{111} + \mathbf{122} + t(\mathbf{232}) + t(\mathbf{333}) \in O(\boldsymbol{\nu}),$$

for $t \in \mathbb{C} - \{0\}$. Invoking Lemma 2 (and an argument as in Example 3) together with the information on the commutation relations given in Table 1, we can see that the expression for $\boldsymbol{\nu}' g_1(t)$ (respectively, $\boldsymbol{\nu}' g_2(t)$) obtained above, leads to a degeneration $\mathfrak{a}_1 \to \mathfrak{a}_{10}$ (respectively, $\mathfrak{a}_1 \to \mathfrak{a}_9$). Observe that, by setting $\mathfrak{b} = \mathbb{C}$-span$(f_2, f_3)$ (respectively, $\mathfrak{b} = \mathbb{C}$-span$(f_1, f_2)$), the existence of the above degenerations can also be established using Lemma 8, (compare also with Remark 9). From the transitivity of degenerations we also get $\mathfrak{a}_1 \to \mathfrak{a}_2$, (and $\mathfrak{a}_1 \to \mathfrak{c}_1$, $\mathfrak{a}_1 \to \mathfrak{a}_0$), see Picture 4.

Next, we consider the structure vector

$$\begin{aligned} \boldsymbol{\nu} g_2(t) = {} & \mathbf{111} + \mathbf{122} + \mathbf{212} + \mathbf{133} \\ & + \mathbf{313} + t(\mathbf{232}) - t(\mathbf{322}) + t^2(\mathbf{331}) \in O(\boldsymbol{\nu}). \end{aligned}$$

Applying Lemma 2 (compare also with Example 3) we get $\mathfrak{a}_1 \to \mathfrak{c}_2$.

Now let $S = \mathfrak{c}_3^o \cup \mathfrak{l}_1^o \cup \left(\bigcup_{\kappa \in \mathbb{K}} \mathfrak{a}_3(\kappa)^o \right)$. In order to complete the degeneration picture inside the set S_7 it remains to examine whether there is a degeneration from $\mathfrak{a}_1$ to any one of the members of the set S. It is useful to observe that

$$S \subseteq \mathcal{B} \cap \mathcal{A}^s \subseteq (\mathcal{T} \cap \widetilde{\mathcal{T}}) \cap \mathcal{A}^s$$

(see Remark 15 (ii)). In addition, $\dim \operatorname{ann}_L \mathfrak{a} = 1 = \dim \operatorname{ann}_R \mathfrak{a}$, for all $\mathfrak{a} \in \Theta^{-1}(S)$ (see Table 2).

For this investigation it will be convenient to consider $\boldsymbol{\lambda} \in O(\boldsymbol{\nu})$, where $\boldsymbol{\lambda} = \boldsymbol{\nu}' g$ with

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in GL(3, \mathbb{C}),$$

and then examine the structure vector λb where $b = (b_{ij}) \in GL(3, \mathbb{C})$ is upper triangular (so $b_{ij} = 0$ if $i > j$ and $b_{11}b_{22}b_{33} \neq 0$).

Now

$$\boldsymbol{\lambda} = \mathbf{111} + \mathbf{133} + \mathbf{323} + \mathbf{222} \in O(\boldsymbol{\nu})$$

and

$$\begin{aligned}
\lambda b = {} & b_{11}(\mathbf{111}) + b_{12}(\mathbf{121}) + \frac{b_{12}b_{23}}{b_{22}}(\mathbf{131}) + \frac{-b_{11}b_{23}}{b_{22}}(\mathbf{132}) \\
& + b_{11}(\mathbf{133}) + b_{12}(\mathbf{211}) + \frac{b_{12}(b_{12} - b_{22})}{b_{11}}(\mathbf{221}) \\
& + b_{22}(\mathbf{222}) + \frac{b_{12}b_{23}(b_{12} - b_{22})}{b_{11}b_{22}}(\mathbf{231}) + \frac{-b_{23}(b_{12} - b_{22})}{b_{22}}(\mathbf{232}) \\
& + b_{12}(\mathbf{233}) + b_{13}(\mathbf{311}) + \frac{b_{13}(b_{12} - b_{22})}{b_{11}}(\mathbf{321}) + b_{22}(\mathbf{323}) \\
& + \frac{b_{13}b_{23}(b_{12} - b_{22})}{b_{11}b_{22}}(\mathbf{331}) + \frac{-b_{13}b_{23}}{b_{22}}(\mathbf{332}) + (b_{13} + b_{23})(\mathbf{333}).
\end{aligned}$$

As a consequence, the following is a list of polynomials in $\mathbf{I}(\lambda B)$, where B is the subgroup of all upper triangular matrices in $GL(3, \mathbb{C})$:

$$\begin{aligned}
& X_{112},\ X_{113},\ X_{122},\ X_{123},\ X_{212},\ X_{213},\ X_{223},\ X_{312},\ X_{313},\ X_{322}, \\
& X_{121} - X_{211},\ X_{111} - X_{133},\ X_{222} - X_{323},\ X_{121} - X_{233}, \\
& X_{131}X_{311} + X_{332}X_{211},\ X_{331}X_{221} - X_{231}X_{321}, \\
& X_{331}X_{211} - X_{231}X_{311},\ X_{311}X_{221} - X_{211}X_{321}.
\end{aligned}$$

Observe tha $\overline{\boldsymbol{\lambda}B} \subseteq \mathcal{A}^s$ since $\boldsymbol{\lambda}B \subseteq \mathcal{A}^s$ and $\mathcal{A}^s$ is closed. Hence, from $\mathcal{B} \cap \mathcal{A}^s \subseteq (\mathcal{T} \cap \widetilde{\mathcal{T}}) \cap \mathcal{A}^s$, we get $(\overline{\boldsymbol{\lambda}B}) \cap \mathcal{B} \subseteq \mathcal{T} \cap \widetilde{\mathcal{T}}$. Now let $\boldsymbol{\mu} = (\mu_{ijk}) \in (\overline{\boldsymbol{\lambda}B}) \cap \mathcal{B}$. Then $\boldsymbol{\mu} \in \mathcal{T} \cap \widetilde{\mathcal{T}}$ so, for $1 \leqslant i \leqslant 3$, we have $\sum_{j=1}^3 \mu_{ijj} = 0$ and $\sum_{j=1}^3 \mu_{jij} = 0$ (see Section 4 and Remark 15 (ii)). Moreover, $\mathbf{ev}_{\boldsymbol{\mu}}(f) = 0$, for all polynomials $f \in \mathbf{I}(\boldsymbol{\lambda}B)$. This forces

$$\begin{aligned} \mu_{112} = \mu_{113} = \mu_{122} = \mu_{123} = \mu_{212} = \mu_{213} = \mu_{223} \\ = \mu_{312} = \mu_{313} = \mu_{322} = 0 \end{aligned}$$

and

$$\mu_{233} = \mu_{121} = \mu_{211}, \mu_{111} = \mu_{133}, \mu_{222} = \mu_{323}.$$

Invoking the conditions forced by $\boldsymbol{\mu} = (\mu_{ijk}) \in \mathcal{T}$, we also get

$$\mu_{111} = \mu_{133} = 0,$$

since $\mu_{122} = 0$ and $\mu_{111} = \mu_{133}$,

$$2\mu_{211} + \mu_{222} = 0,$$

since $\mu_{211} = \mu_{233}$, and

$$\mu_{311} + \mu_{333} = 0,$$

since $\mu_{322} = 0$. So the defining conditions for $\boldsymbol{\mu} = (\mu_{ijk})$ to belong to $\widetilde{\mathcal{T}}$ now give

$$2\mu_{222} + \mu_{211} = 0,$$

since $\mu_{222} = \mu_{323}$ and $\mu_{121} = \mu_{211}$, and

$$\mu_{131} + \mu_{232} + \mu_{333} = 0.$$

Also observe that the conditions

$$\begin{aligned} &2\mu_{211} + \mu_{222} = 0,\ 2\mu_{222} + \mu_{211} = 0, \\ &\mu_{222} = \mu_{323} \quad \text{and} \quad \mu_{211} = \mu_{121} = \mu_{233} \end{aligned}$$

force $\mu_{211} = \mu_{222} = \mu_{121} = \mu_{233} = \mu_{323} = 0$.

Next, we invoke the fact that $\boldsymbol{\mu} \in \mathcal{B}$, so for $1 \leqslant i, j, k, m \leqslant n$ we have $\sum_{l=1}^{n} \mu_{ijl}\mu_{lkm} = 0$ (see Section 4). From

$$\mu_{331}\mu_{133} + \mu_{332}\mu_{233} + \mu_{333}\mu_{333} = 0$$

we obtain $\mu_{333} = 0$ (since $\mu_{133} = 0 = \mu_{233}$), and hence $\mu_{311} = 0$ since $\mu_{311} + \mu_{333} = 0$.

Moreover, from $\mu_{131} + \mu_{232} + \mu_{333} = 0$ we now get $\mu_{131} + \mu_{232} = 0$. Also the fact that $X_{331}X_{221} - X_{231}X_{321} \in \mathbf{I}(\boldsymbol{\lambda}B)$ ensures that

$$\mu_{331}\mu_{221} - \mu_{231}\mu_{321} = 0.$$

Going back once more to the defining conditions for the algebraic set $\mathcal{B}$ and combining with various restrictions already obtained above we get the following further constraints on the coefficients μ_{ijk}:

$$\begin{aligned} &\mu_{221}\mu_{131} = 0,\ \mu_{221}\mu_{132} = 0,\ \mu_{131}^2 + \mu_{132}\mu_{231} = 0, \\ &\mu_{321}\mu_{131} = 0,\ \mu_{321}\mu_{132} = 0,\ \mu_{332}\mu_{221} = 0, \\ &\mu_{331}\mu_{131} + \mu_{332}\mu_{231} = 0, \mu_{331}\mu_{132} + \mu_{332}\mu_{232} = 0. \end{aligned} \tag{6.1}$$

Summing up, the assumption $\boldsymbol{\mu} = (\mu_{ijk}) \in (\overline{\boldsymbol{\lambda}B}) \cap \mathcal{B}$ gives the constraints that the only coefficients μ_{ijk} which can possibly be non-zero are

$$\mu_{131},\ \mu_{132},\ \mu_{221},\ \mu_{231},\ \mu_{232},\ \mu_{321},\ \mu_{331} \quad \text{and} \quad \mu_{332}$$

and these satisfy the conditions (6.1) together with the conditions

$$\mu_{131} + \mu_{232} = 0 \quad \text{and} \quad \mu_{331}\mu_{221} - \mu_{231}\mu_{321} = 0.$$

We will consider the cases (i) $\mu_{131} \neq 0$ and (ii) $\mu_{131} = 0$ separately.

Case (i): $\mu_{131} \neq 0$. Then $\mu_{232} = -\mu_{131}$ ($\neq 0$) and $\mu_{221} = \mu_{321} = 0$. Hence, all coefficients μ_{ijk} are zero except possibly

$$\mu_{131},\ \mu_{132},\ \mu_{231},\ \mu_{232},\ \mu_{331} \text{ and } \mu_{332}.$$

This gives $\mathbb{C}\text{-span}(e_1, e_2) \subseteq \text{ann}_R\Theta^{-1}(\boldsymbol{\mu})$. In particular, $\boldsymbol{\mu} \notin S$.

Case (ii): $\mu_{131} = 0$ (hence $\mu_{232} = 0$ as well).

In this case, all coefficients μ_{ijk} are zero except possibly

$$\mu_{132},\ \mu_{221},\ \mu_{231},\ \mu_{321},\ \mu_{331}, \quad \text{and} \quad \mu_{332}$$

and we have the constraints:

$$\mu_{221}\mu_{132} = 0,\ \mu_{132}\mu_{231} = 0,\ \mu_{321}\mu_{132} = 0$$
$$\mu_{332}\mu_{221} = 0,\ \mu_{332}\mu_{231} = 0,\ \mu_{331}\mu_{132} = 0,$$
$$\mu_{331}\mu_{221} - \mu_{231}\mu_{321} = 0.$$

We will consider the subcases (iia) $\mu_{132} \neq 0$ and (iib) $\mu_{132} = 0$ separately:

Subcase (iia): $\mu_{132} \neq 0$: Then, $\mu_{221} = \mu_{231} = \mu_{321} = \mu_{331} = 0$. Again we get that $\mathbb{C}\text{-span}(e_1, e_2) \subseteq \text{ann}_R\Theta^{-1}(\boldsymbol{\mu})$ so $\boldsymbol{\mu} \notin S$ in this subcase.

Subcase (iib): $\mu_{132} = 0$. In this subcase only

$$\mu_{221},\ \mu_{231},\ \mu_{321},\ \mu_{331} \quad \text{and} \quad \mu_{332}$$

can possibly be non-zero and we have the constraints:

$$\mu_{332}\mu_{221} = 0,\ \mu_{332}\mu_{231} = 0, \quad \text{and} \quad \mu_{331}\mu_{221} - \mu_{231}\mu_{321} = 0.$$

If $\mu_{332} \neq 0$, then $\mu_{221} = 0 = \mu_{231}$. Hence

$$\mathbb{C}\text{-span}(e_1, e_2) \subseteq \text{ann}_L\Theta^{-1}(\boldsymbol{\mu})$$

leading to $\boldsymbol{\mu} \notin S$. So we assume that $\mu_{332} = 0$ as well, leaving us with $\mu_{221}, \mu_{231}, \mu_{321}, \mu_{331}$ as the only coefficients which can possibly be non-zero and also satisfying the constraint

$$\mu_{331}\mu_{221} - \mu_{231}\mu_{321} = 0.$$

Now the matrix

$$\begin{pmatrix} \mu_{221} & \mu_{321} \\ \mu_{231} & \mu_{331} \end{pmatrix} \in M_2(\mathbb{C})$$

has determinant 0 (and hence has rank < 2) so there exists a matrix $(\beta\ \gamma) \in \mathbb{C}^{1\times 2} - \{(0\ 0)\}$ satisfying

$$(\beta\ \gamma)\begin{pmatrix} \mu_{221} & \mu_{321} \\ \mu_{231} & \mu_{331} \end{pmatrix} = (0\ 0).$$

Inside $\Theta^{-1}(\mu)$ we get:

$$[e_2, \beta e_2 + \gamma e_3] = \beta\mu_{221}e_1 + \gamma\mu_{231}e_1 = 0$$

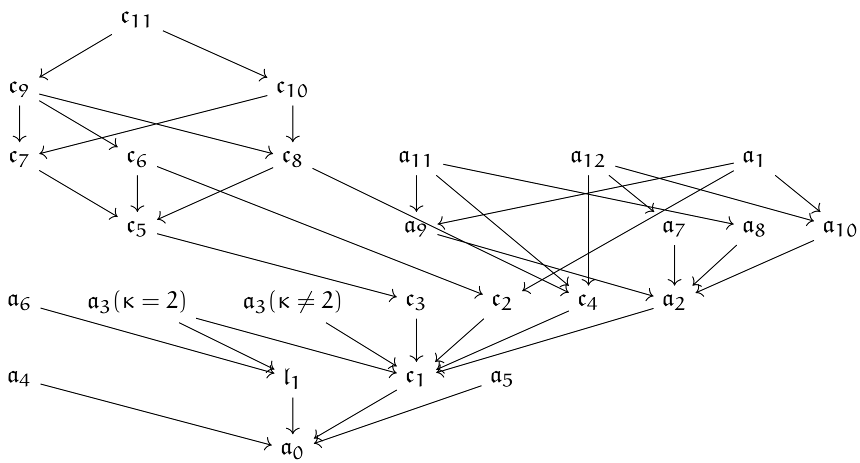
and

$$[e_3, \beta e_2 + \gamma e_3] = \beta\mu_{321}e_1 + \gamma\mu_{331}e_1 = 0.$$

Hence, $\mathbb{C}\text{-span}(e_1, \beta e_2 + \gamma e_3) \subseteq \text{ann}_R\Theta^{-1}(\mu)$ giving $\mu \notin S$ in this final subcase also.

We conclude that $(\overline{\lambda B}) \cap S \,(= (\overline{\lambda B}) \cap \mathcal{B} \cap S) = \emptyset$. From Remark 5 (ii) with $U = S$ we finally get that there is no degeneration from $\mathfrak{a}_1$ to any one of the members of the set $\{\mathfrak{c}_3, \mathfrak{l}_1\} \cup \{\mathfrak{a}_3(\kappa)\colon \kappa \in \mathbb{K}\}$ (information which cannot so easily be obtained by just considering various algebra invariants). This completes the degeneration picture inside the closed set S_7.

In view of Remark 15 (iv), in order to obtain the complete degeneration picture inside the algebraic set $\mathcal{A}^s$ it is enough to combine the information given in Picture 2 together with information obtained in Steps 1 to 7 of the above process (see Picture 5).



Picture 5. Degenerations of 3-dimensional complex associative algebras (with $\kappa \in \mathbb{K}$).

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