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Some Remarks on Anchor of Irreducible Characters

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Abstract

In this paper, we show that the direct product of the anchors of two irreducible characters is the anchor of the tensor product of their irreducible characters. We prove that the anchor of any irreducible character of a p-group G is G itself. We show that the degree of any irreducible character χ divides the index of the center of its anchor in a nilpotent group G. We study and explain the relationship between the anchor of an irreducible character of a subgroup of G which does not contain a Sylow p-subgroup of G and the anchor of an irreducible character of G.

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1 Introduction

Throughout this paper, p is a prime number and $(k, \mathcal{R}, F)$ is a p-modular system which consists of a complete discrete valuation ring $\mathcal{R}$ with field of fractions k of characteristic 0 and the residue field $F = \mathcal{R}/J(\mathcal{R})$ of characteristic p, where $J(\mathcal{R})$ is the Jacobson radical of the ring $\mathcal{R}$. Let G be a finite group and $\mathrm{Irr}(G)$ be the set of ordinary irreducible characters of G which corresponds to the set of simple kG-modules. For $\psi \in \mathrm{Irr}(G)$, we consider e_ψ to be the unique central primitive idempotent in kG such that $\psi(e_\psi) \neq 0$. The algebra $\mathcal{R}Ge_\psi$ is a primitive G-interior $\mathcal{R}$-algebra since $Z(\mathcal{R}Ge_\psi)$ is a subring of $Z(kGe_\psi)$.

We organize this paper as follows. Section 2 contains the preliminaries of the relative trace map and the Brauer map on G-algebras over $\mathcal{R}$. We define the defect group of a G-algebra over $\mathcal{R}$. We introduce the relationship between the Brauer map and the defect groups which appeared in [2, Lemma 3.1 (1)]. We present some results on the tensor product of G-algebras, and the formula for the kernel of the Brauer map $Br^{A_1 \otimes A_2}_{K_1 \times K_2}$ which were proved in [1]. Section 3 is devoted to the study of basic facts of the anchor of an irreducible character of G, see [11]. The anchor of an irreducible character ψ of G is defined as the defect group of the primitive G-interior $\mathcal{R}$-algebra $\mathcal{R}Ge_\psi$ for any irreducible character ψ of G. We prove Proposition 6 that says the anchor of an irreducible character is a p-subgroup of G. In Proposition 7, we recast the proof that the largest normal p-subgroup of G is contained in the anchor of an irreducible character. We study Proposition 8 which explains the relationship between the anchor of an irreducible character ψ of G, the defect group of the block $\mathfrak{B}$ of $\mathcal{R}G$ and the vertex of indecomposable $\mathcal{R}G$-lattice affording ψ.

For the finite groups that appear in this paper, we assume in general that k and F are splitting fields. A few places do not require this assumption. In particular, Propositions 7 and 8 do not need this assumption.

The fact that the defect group of the primitive idempotent $e_1 \otimes e_2$ of $(A_1 \otimes_{\mathcal{R}} A_2)^{G_1 \times G_2}$ can be expressed as a direct product of defect groups of e_i in $A_i^{G_i}$ for $i = 1, 2$, was proved in [1]. We shall use the same idea in that the case of the defect group of the primitive G-interior $\mathcal{R}$-algebra $\mathcal{R}Ge_\chi$, for $\chi \in \mathrm{Irr}(G)$. One of our main results is to prove that the direct product of the anchors of χ_i, where $\chi_i \in \mathrm{Irr}(G_i)$ for $i = 1, 2$, is the anchor of the character $\chi_1 \otimes \chi_2$ which is in Section 4. The second of our main results is to prove that if G is a p-group then the anchor of the irreducible character of G is G itself. The third result is to show that the degree of any irreducible character χ divides the index $|G : Z(A_\chi)|$, for a p-group G. The generalization of the same result in the case of nilpotent groups appears in Section 4. A suitable example for this case is provided. We study and explain the relationship between the anchor of an irreducible character of a subgroup of G which does not contain a Sylow p-subgroup of G and the anchor of an irreducible character of G in Theorem 17.

2 Definitions and preliminaries

Let p be a prime number, let G be a finite group. Consider A to be a G-algebra over $\mathcal{R}$. Let Q to be a subgroup of G. We write

$$A^Q := \{a \in A : a^q = a \text{ for all } q \in Q\},$$

for the set of Q-fixed elements of A under the action by the finite group G. A^Q is a unitary subalgebra of A over $\mathcal{R}$. If $Q \subseteq L$ are subgroups of G, then $A^L \subseteq A^Q$. We define the relative trace map as the function

$$\mathrm{tr}_Q^L : A^Q \to A^L$$

such that $\mathrm{tr}_Q^L(a) = \sum_{t \in T} a^t$, for all $a \in A^Q$, where T is a transversal of Q in L. We write $A_Q^L = \mathrm{tr}_Q^L(A^Q)$. Then A_Q^L is an ideal of A^L. We define the Brauer map with respect to a p-subgroup K to be the canonical map

$$\mathrm{Br}_K^A : A^K \longrightarrow A(K);$$

such that

$$\mathrm{Br}_K^A(a) = a + I_K(A),$$

where $A(K) := A^K/I_K(A)$; $I_K(A) := \sum_{P<K} A_P^K + J(\mathcal{R})A^K$. The factor algebra $A(K)$ is called the Brauer quotient and it is clear that it is an $\overline{N}_G(K)$-algebra over $\mathcal{R}$, where $\overline{N}_G(K) = N_G(K)/K$.

The following definition is the Green approach to study defect group, see [6].

Definition 1 Suppose that A is a G-algebra over $\mathcal{R}$ and e is a primitive idempotent in A^G. We define the set $S := \{H \leqslant G : e \in A_H^G\}$. The minimal element of the set S is called a *defect group* of e in the G-algebra A over $\mathcal{R}$ and is denoted by $\mathrm{Def}_G^A(e)$.

The following result describes the relationship between the Brauer map and the defect groups. This result appeared in [2, Lemma 3.1 (1)]. For two subgroups H and K of G, the symbol $H \leqslant_G K$ means the subgroup H is contained in a G-conjugate of the subgroup K.

Lemma 2 *Using the same notation in the definition above. If* $\mathrm{Br}_D^A(e) \neq 0$ *then* $D \leqslant_G \mathrm{Def}_G^A(e)$, *where* D *is a fixed* p*-subgroup of* G.

Lemma 3 *If* A_i *is* G_i*–algebra over* $\mathcal{R}$ *and* $K_i \leqslant G_i$ *for* $i = 1, 2$. *Then*

$$(A_1 \otimes_{\mathcal{R}} A_2)_{K_1 \times K_2}^{G_1 \times G_2} \simeq (A_1)_{K_1}^{G_1} \otimes_{\mathcal{R}} (A_2)_{K_2}^{G_2}.$$

The following formula for the kernel of the Brauer map $\mathrm{Br}^{A_1\otimes A_2}_{K_1\times K_2}$ was proved in [1].

Proposition 4 *By the same hypotheses as above, we have:*

$$I_{K_1\times K_2}(A_1\otimes_{\mathcal{R}} A_2) = I_{K_1}(A_1)\otimes_{\mathcal{R}} A_2^{K_2} + A_1^{K_1}\otimes_{\mathcal{R}} I_{K_2}(A_2).$$

Recall that an interior G-algebra over $\mathcal{R}$ is an $\mathcal{R}$-algebra A endowed with a homomorphism of groups $\gamma : G \to A^{\times}$, where $A^{\times}$ is the group of invertible elements of A. Any interior G-algebra is a G-algebra, since there is a group homomorphism $A^{\times} \to \mathrm{Aut}_{\mathcal{R}}(A)$ which maps a to the inner automorphism $\mathrm{Inn}(a)$ of the algebra A. The group algebra $\mathcal{R}G$ of G over $\mathcal{R}$ is a typical example of an interior G-algebra over $\mathcal{R}$ with $\gamma : G \to (\mathcal{R}G)^{\times}$ defined by $\gamma(g) = g$ for $g \in G$.

3 Anchor of an irreducible character

In this section, we focus our attention on the characteristics of the anchor of an irreducible character of a finite group G which appeared in [11] by R. Kessar, B. Külshammer, and M. Linckelmann.

Consider $\mathcal{R}G$ to be a G-algebra over $\mathcal{R}$. Let e_B be the block idempotent of the block B of $\mathcal{R}G$. Following Green approach, the defect group D of the block is a unique (up to G-conjugacy) minimal p-subgroup of G such that $e_B \in (\mathcal{R}G)^G_D$ (see the notation in Section 2). We explain the relationship between the anchor of an ordinary irreducible character ψ of G, the defect group of the block of $\mathcal{R}G$ and the vertex group of indecomposable $\mathcal{R}G$-lattice affording ψ.

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of ordinary irreducible characters of G which corresponds to the set of simple kG-modules. Let $\psi \in \mathrm{Irr}(G)$. Then ψ can be uniquely extended to an algebra map $\psi : kG \to k$. We consider the element

$$e_\psi = \frac{\psi(1)}{|G|}\sum_{g\in G}\psi(g^{-1})g,$$

which is the unique central primitive idempotent in kG such that $\psi(e_\psi) \neq 0$. The algebra $\mathcal{R}Ge_\psi$ is a primitive G-interior $\mathcal{R}$-algebra, since $Z(\mathcal{R}Ge_\psi)$ is a subring of $Z(kGe_\psi)$.

Definition 5 Consider G is a finite group and $\psi \in \mathrm{Irr}(G)$. The defect group of the primitive G-interior $\mathcal{R}$-algebra $\mathcal{R}Ge_\psi$ is called an *anchor* of ψ.

Since the anchor is defect group we have the following proposition which is similar to Green theory for defect group.

Proposition 6 *If A_χ is an anchor of an irreducible character χ of G then A_χ is a p-subgroup of G.*

Proof — Let Q be a Sylow p-subgroup of G and $a \in (\mathcal{R}Ge_\chi)^G$. Then

$$\mathrm{tr}_Q^G(a) = |G:Q|a,$$

because p is prime to the index $|G:Q|$. That is, $|G:Q|$ has inverse in F. Therefore, $\mathrm{tr}_Q^G\left(|G:Q|^{-1}a\right) = a$. This implies that $a \in (\mathcal{R}Ge_\chi)_Q^G$. Thus,

$$(\mathcal{R}Ge_\chi)_Q^G = (\mathcal{R}Ge_\chi)^G.$$

We have $e_\chi \in (\mathcal{R}Ge_\chi)^G$, so $e_\chi \in (\mathcal{R}Ge_\chi)_Q^G$. Since A_χ is an anchor of χ, it is the minimal subgroup of G such that $e_\chi \in (\mathcal{R}Ge_\chi)_{A_\chi}^G$. Therefore, $A_\chi \leqslant Q$. So, A_χ is a p-subgroup of G. □

The following proposition is crucial in this work and we extract it from the paper [11] with more details.

Proposition 7 *Suppose that G is a finite group, $\chi \in \mathrm{Irr}(G)$. If A_χ is an anchor of χ then $O_p(G) \leqslant A_\chi$. Where $O_p(G)$ is the largest normal p-subgroup of G.*

Proof — Put $\mathfrak{N} = O_p(G)$. We will use indirect proof. Assume that $\mathfrak{N} \nleq A_\chi$. Then $A_\chi \lneqq A_\chi\mathfrak{N}$. Let $n \in \mathfrak{N}$, we have

$$n - 1 \in J(\mathcal{R}\mathfrak{N}) \subseteq J(\mathcal{R}G).$$

For all $a \in \mathcal{R}Ge_\chi$ and $n \in \mathfrak{N}$ then $na - a$, $an^{-1} - a$ and

$$nan^{-1} - a = nan^{-1} - an^{-1} + an^{-1} - a \subseteq J(\mathcal{R}Ge_\chi).$$

We have A_χ is an anchor of χ then there exists $a \in (\mathcal{R}Ge_\chi)^{A_\chi}$ such that $\mathrm{tr}_{A_\chi}^G(a) = e_\chi$. Thus,

$$\mathrm{tr}_{A_\chi}^{A_\chi\mathfrak{N}}(a) - |A_\chi\mathfrak{N} : A_\chi|a \in J(\mathcal{R}Ge_\chi).$$

But p divides $|A_\chi\mathfrak{N} : A_\chi|$, hence, $y = \mathrm{tr}_{A_\chi}^{A_\chi\mathfrak{N}}(a) \in J(\mathcal{R}Ge_\chi)$. Apply $\mathrm{tr}_{A_\chi\mathfrak{N}}^{G}$ to the element y and use the transitivity for the relative trace map, we get

$$e_\chi = \mathrm{tr}_{A_\chi}^{G}(a) = \mathrm{tr}_{A_\chi\mathfrak{N}}^{G}(y) \in J(\mathcal{R}Ge_\chi),$$

which is a contradiction to e_χ being a unit element in $\mathcal{R}Ge_\chi$. Therefore, it must be $O_p(G) \leqslant A_\chi$. □

We remind the reader that the vertex of indecomposable $\mathcal{R}G$-module M is a unique (up to G-conjugacy) minimal p-subgroup V of G such that M is V-projective of G. This is equivalent to that M is a direct summand of the induced G-module $\mathrm{Ind}_V^G(N)$ for some V-module N.

Proposition 8 *Let G be a finite group and $\psi \in \mathrm{Irr}(G)$ which belongs to the block $\mathfrak{B}$ of $\mathcal{R}G$. Suppose that A_ψ is an anchor of ψ. Let $\mathcal{L}$ be an indecomposable $\mathcal{R}G$-lattice affording ψ. The following hold:*

(1) *a defect group of $\mathfrak{B}$ contains A_ψ;*

(2) *a vertex of $\mathcal{L}$ is contained in A_ψ.*

Proof — (1) Consider $\mathcal{D}$ to be a defect group of the block $\mathfrak{B}$. Then by the definition $e_\mathfrak{B} \in \mathrm{tr}_D^G\left((\mathcal{R}G)^\mathcal{D}\right)$. Then there exists $c \in (\mathcal{R}G)^\mathcal{D}$ such that $\mathrm{tr}_\mathcal{D}^G(c) = e_\mathfrak{B}$. Now $ce_\psi \in (\mathcal{R}Ge_\psi)^\mathcal{D}$. Then

$$\mathrm{tr}_\mathcal{D}^G(ce_\psi) = \mathrm{tr}_\mathcal{D}^G(c)e_\psi = e_\mathfrak{B}e_\psi = e_\psi.$$

Thus, $e_\psi \in (\mathcal{R}Ge_\psi)_\mathcal{D}^G$. Since A_ψ is an anchor, it is a minimal p-subgroup such that $e_\psi \in (\mathcal{R}Ge_\psi)_{A_\psi}^G$. Therefore, A_ψ is contained in the defect group $\mathcal{D}$ of $\mathfrak{B}$.

(2) Consider A_ψ is an anchor of ψ, then A_ψ is a defect group of a primitive G-interior $\mathcal{R}$-algebra $\mathcal{R}Ge_\psi$ which implies that

$$e_\psi \in \mathrm{tr}_{A_\psi}^G\left((\mathcal{R}Ge_\psi)^{A_\psi}\right).$$

In particular, there exists $x \in (\mathcal{R}Ge_\psi)^{A_\psi}$ such that $\mathrm{tr}_{A_\psi}^G(x) = e_\psi$. The map $\Gamma : \mathcal{L} \to \mathcal{L}$ which, $\Gamma(l) := xl$ is an element in $\mathrm{End}_{\mathcal{R}A_\psi}(\mathcal{L}) = \mathrm{id}_\mathcal{L}$. By Higman's Criterion (see [13, Theorem 2.2]) $\mathcal{L}$ is A_ψ-projective. Since the vertex V of $\mathcal{L}$ is the minimal p-subgroup such that $\mathcal{L}$ is V-projective, $\mathcal{L}$ is contained in A_ψ. □

Theorem 9 *Let* G *be a finite group,* $\psi \in \mathrm{Irr}(G)$ *which belongs to the block* $\mathfrak{B}$ *of* $\mathcal{R}G$*. Suppose that* A_ψ *is an anchor of* ψ*. let* $\mathcal{L}$ *be an indecomposable* $\mathcal{R}G$*-lattice affording* ψ*. The following hold:*

(1) *if* ψ *has full defect, then* A_ψ *is a defect group of* $\mathfrak{B}$*;*

(2) *if* $\mathfrak{B}$ *has an abelian defect group* $\mathcal{D}$*, then* $\mathcal{D}$ *is an anchor of* ψ*.*

Proof — (1) Let d be the defect number of ψ. Suppose that ψ has full defect, we know that

$$d = \upsilon_p \left(\frac{|G|}{\psi(1)} \right),$$

where υ_p is a valuation on the field k such that $\upsilon_p(p) = 1$. Therefore $\psi(1)_p = |G : \mathcal{D}|_p$, where $\mathcal{D}$ denotes a defect group of $\mathfrak{B}$. The vertex V of an indecomposable $\mathcal{R}G$-lattice $\mathcal{L}$ is a subgroup of a conjugate of $\mathcal{D}$ and the p-part of the index $|G : V|_p$ divides the $\mathcal{R}$-rank of $\mathcal{L}$ by [3], Theorem 19.26. Hence, it divides $\psi(1)_p = |G : \mathcal{D}|_p$. So, there is a positive integer t such that $|G : \mathcal{D}|_p = t|G : V|_p$. It follows that $|V| = t|D|$. So, it must be $|V| = |D|$ and $V =_G D$. But from Proposition 8, the vertex V of an indecomposable $\mathcal{R}G$-lattice $\mathcal{L}$ is contained in an anchor of ψ. Thus, A_ψ is a defect group of $\mathfrak{B}$.

(2) Suppose that $\mathfrak{B}$ has an abelian defect group $\mathcal{D}$. Then from [12], we have Braure's Height zero Conjecture which states as follows "if block $\mathfrak{B}$ has an abelian defect group then all irreducible characters in $\mathfrak{B}$ have height zero". So, all irreducible characters in the block $\mathfrak{B}$ have full defect. Then the result holds from item (1) in this theorem. □

Corollary 10 *If* $\chi \in \mathrm{Irr}(G)$ *has a degree prime to* p*, then the anchor of* χ *is a Sylow* p*-subgroup of* G*. In particular, a principal irreducible character has a Sylow* p*-subgroup as an anchor.*

Proof — Suppose that $\chi \in \mathrm{Irr}(G)$ has a degree prime to p, then by [5, Corollary 1] a Sylow p-subgroup of G is a vertex of the indecomposable FG-module which affords χ^0. Thus, a Sylow p-subgroup of G is contained in every vertex of indecomposable $\mathcal{R}G$-lattice affording χ. From Proposition 8 (b), the desired is achieved. In particular, since the principal irreducible character has degree one, it is prime to p. Therefore, it has a Sylow p-subgroup of G as an anchor. □

Recall that we say that G is a solvable group if it has a subnormal series whose factor groups are all abelian. We can construct solvable groups by repeated extensions of abelian groups. Solvable group is rich of normal subgroups and hence Clifford theory can be used. For recent book in characters of solvable groups, the reader can see [9].

Let $\chi \in \mathrm{Irr}(G)$. Then χ is said to be p-special if it satisfies: the degree of χ is p-number (a multiple of the prime number p) and if N is a subnormal of G and $\lambda \in \mathrm{Irr}(N)$ such that $\langle \mathrm{Res}_N^G(\chi), \lambda \rangle > 0$, then the determinantal order of λ $O(\lambda) = O\big(\det(\lambda)\big)$ is a p-number (that is, the order of $\det(\lambda)$ as an element of the group of linear characters of G by [8, Proplem 2.3]). The following example appeared in [11].

Example 11 Let $p = 2$, $G = GL(2,3)$ which has a Sylow 2−subgroup semidihedral group

$$SD_{16} = \langle a, b | a^8 = b^2 = 1, bab^{-1} = a^3 \rangle.$$

Let $H = SL(2,3)$ which has a Sylow 2−subgroup Q_8 and C_3 is a complement of Q_8 in H. Consider $\psi \in \mathrm{Irr}(Q_8)$ as follows:

	[1]	$[a^2]$	[a]	[b]	[ab]
ψ	2	-2	0	0	0

Where $\phi \in \mathrm{Irr}(H)$ is the unique extension of ψ with determinantal or-

	(I)	[(Z)]	[(b)]	[(c)]	[(d)]	[(Zc)]	[(Zd)]
ϕ	2	-2	0	-1	-1	1	1

der $O(\phi)$ is a power of 2. Since we have $Q_8 \trianglelefteq SL(2,3)$ and $SL(2,3)/Q_8 \simeq C_3$ solvable group. We have $\det(\psi) = 1_{Q_8}$ from Exercise 2.4 in [8]. The order of ψ is $O(\psi) = O(1_{Q_8}) = 1$, then we have

$$\big(|SL(2,3) : Q_8|, O(\psi)\psi(1)\big) = (3,2) = 1.$$

Hence, from [8, Corollary 6. 28], $\psi \in \mathrm{Irr}(Q_8)$ has a unique extension $\phi \in \mathrm{Irr}\,\big(SL(2,3)\big)$ with

$$\big(|SL(2,3) : Q_8|, O(\psi)\big) = 1 \quad \text{and} \quad O(\phi) = O(\psi).$$

Take $\chi \in \mathrm{Irr}(G)$ as the extension ϕ to G as follows:

Then χ is 2-special. Furthermore, $\chi^0 \in \mathrm{IBr}\,\big(GL(2,3)\big)$ as follows:

	(I)	2A	2B	3	4	6	8A	8B
χ	2	-2	0	-1	0	1	$-\sqrt{-2}$	$\sqrt{-2}$

	[1]	[(3)]
χ^0	2	-1

Note that $\chi^0 = \mathrm{Ind}_H^G(\eta^0)$, where η^0 is a linear Brauer character of H.

	(I)	[(c)]	[(d)]
η^0	1	ω^2	ω

It is clear from the computations above that the restriction $\mathrm{Res}_{C_3}^G \chi^0$ is equal to the aggregate of two different irreducible Brauer characters of C_3. Thus, Q_8 is a vertex of the unique indecomposable $\mathcal{F}G$-module which affords χ^0 and Q_8 is a subgroup of each vertex of indecomposable $\mathcal{R}G$-lattice affording χ. From the previous details, χ is not found by induced any character of H. Thus, the indecomposable $\mathcal{R}G$-lattice affording χ is not H-projective. Hence, Q_8 is a proper subgroup of a vertex of $\mathcal{R}G$-lattice affording χ. Therefore, a vertex of indecomposable $\mathcal{R}G$-lattice affording χ is a Sylow 2-subgroup of G. Let A_χ be an anchor of χ. We have SD_{16} is a vertex of the indecomposable $\mathcal{R}G$-lattice affording χ. From Proposition 8, we obtain A_χ contains SD_{16} and a defect group of the block of $\mathcal{R}G$ which contains χ contains A_χ. Thus, the anchor of χ is the defect group of the block of $\mathcal{R}G$ which containing χ. It is equivalent to a Sylow 2-subgroup of G.

4 Main Results

In this section, let G_i be a finite group for $i = 1, 2$. We prove that the direct product of the anchors χ_i, where $\chi_i \in \mathrm{Irr}(G_i)$ for $i = 1, 2$, is the anchor of the irreducible character $\chi_1 \otimes \chi_2$. We prove that if G is p-group then the anchor of the irreducible character of G is G itself. Let A_χ be the anchor of irreducible character χ of G. We show that the degree of any irreducible character divides the index $[G : Z(A_\chi)]$ in a p-group G. We generalize the result to nilpotent groups and give a suitable example. We study and explain the relationship between the anchor of an irreducible character of a subgroup of G which does not contain a Sylow p-subgroup of G and the anchor of an irreducible character of G.

Theorem 12 *Let* $\chi_i \in \mathrm{Irr}(G_i)$ *with an anchor* A_{χ_i}*, for* $i = 1, 2$*. Then*

$$A_{\chi_1} \times A_{\chi_2} = A_{\chi_1 \otimes \chi_2}.$$

Proof — Then $\chi_1 \otimes \chi_2 \in \mathrm{Irr}(G_1 \times G_2)$ from [8, Theorem 4.21]. We have that A_{χ_i} is an anchor of χ_i, for $i = 1, 2$. Hence A_{χ_i} is a minimal p-subgroup of G_i with respect to the condition $e_{\chi_i} \in (\mathcal{R}G_i e_{\chi_i})^{G_i}_{A_{\chi_i}}$ for $i = 1, 2$. Then there is $y_i \in (\mathcal{R}G_i e_{\chi_i})^{A_{\chi_i}}$ such that $\mathrm{tr}^{G_i}_{A_{\chi_i}}(y_i) = e_{\chi_i}$ for $i = 1, 2$. From Lemma 3, we have

$$(A_1 \otimes_{\mathcal{R}} A_2)^{G_1 \times G_2}_{A_{\chi_1} \times A_{\chi_2}} \simeq (A_1)^{G_1}_{A_{\chi_1}} \otimes_{\mathcal{R}} (A_2)^{G_2}_{A_{\chi_2}}.$$

We get

$$\begin{aligned} e_{\chi_1} \otimes e_{\chi_2} &= \mathrm{tr}^{G_1}_{A_{\chi_1}}(y_1) \otimes \mathrm{tr}^{G_2}_{A_{\chi_2}}(y_2), \\ &= \mathrm{tr}^{G_1 \times G_2}_{A_{\chi_1} \times A_{\chi_2}}(y_1 \otimes y_2) \in (\mathcal{R}G_1 e_{\chi_1} \otimes_{\mathcal{R}} \mathcal{R}G_2 e_{\chi_2})^{G_1 \times G_2}_{A_{\chi_1} \times A_{\chi_2}}. \end{aligned}$$

Hence, $A_{\chi_1 \otimes \chi_2} \leqslant A_{\chi_1} \times A_{\chi_2}$. Conversely assume $A_{\chi_1} \times A_{\chi_2} \nleq A_{\chi_1 \otimes \chi_2}$. Then from Lemma 2, we have

$$\mathrm{Br}_{A_{\chi_1 \otimes \chi_2}}(e_{\chi_1} \otimes e_{\chi_2}) = 0.$$

Hence, it follows from Proposition 4 that $\mathrm{Br}_{A_{\chi_1}}(e_{\chi_1}) \otimes_{\mathcal{R}} \mathrm{Br}_{A_{\chi_2}}(e_{\chi_2})$ belongs to

$$I_{A_{\chi_1}}(\mathcal{R}G_1 e_{\chi_1}) \otimes_{\mathcal{R}} (\mathcal{R}G_2 e_{\chi_2})^{A_{\chi_2}} + (\mathcal{R}G_1 e_{\chi_1})^{A_{\chi_1}} \otimes_{\mathcal{R}} I_{A_{\chi_2}}(\mathcal{R}G_2 e_{\chi_2}).$$

Thus, either $\mathrm{Br}_{A_{\chi_1}}(e_{\chi_1}) \in I_{A_{\chi_1}}(\mathcal{R}G_1 e_{\chi_1})$ or $\mathrm{Br}_{A_{\chi_2}}(e_{\chi_2}) \in I_{A_{\chi_2}}(\mathcal{R}G_2 e_{\chi_2})$, this implies that $\mathrm{Br}_{A_{\chi_1}}(e_{\chi_1}) = 0$ or $\mathrm{Br}_{A_{\chi_2}}(e_{\chi_2}) = 0$. This means that e_{χ_1} and e_{χ_2} are not local points, which contradict with A_{χ_i} is a defect group of $\mathcal{R}G_i e_{\chi_i}$ for $i = 1, 2$. Thus,

$$A_{\chi_1} \times A_{\chi_2} = A_{\chi_1 \otimes \chi_2}$$

and we are done. □

Proposition 13 *If* G *is a* p*-group then the anchor of any irreducible character of* G *is* G *itself.*

Proof — Plesken [14] proved that if G is a p-group and $\psi \in \mathrm{Irr}(G)$, then there is an $\mathcal{R}G$-indecomposable lattice which affords ψ with vertex G. Then Proposition 8 (2) implies the result. □

We present our main result for a p-group G the degree of any irreducible character χ of G divides the index of the center of its anchor.

Theorem 14 *If* G *is a* p*-group and* $\chi \in \mathrm{Irr}(G)$ *has anchor* A_χ *then* $\chi(1)$ *divides the index* $|G : Z(A_\chi)|$.

Proof — First we need to show that: If ψ is a character of $Z(G)$ then

$$\sum_{x \in Z(G)} |\psi(x)|^2 \geqslant |Z(G)|\psi(1).$$

Suppose that

$$\psi = \sum\nolimits_{\psi_i \in \mathrm{Irr}\left(Z(G)\right)} n_i \psi_i \qquad (n_i \geqslant 0, n_i \in \mathbb{Z}).$$

We have $\sum n_i^2 \geqslant \sum n_i$ for all $\psi_i \in \mathrm{Irr}(Z(G))$ for all i. Now,

$$\begin{aligned} \langle \psi, \psi \rangle &= \left\langle \sum_{\psi_i \in \mathrm{Irr}\left(Z(G)\right)} n_i \psi_i, \sum_{\psi_i \in \mathrm{Irr}\left(Z(G)\right)} n_i \psi_i \right\rangle \\ &= \sum n_i^2 \langle \psi_i, \psi_i \rangle \geqslant \sum n_i = \psi(1). \end{aligned} \tag{$\star$}$$

From the First Orthogonality Relation, see [8, Corollary 2.14] we have

$$\langle \psi, \psi \rangle = \frac{1}{|Z(G)|} \sum_{x \in Z(G)} \psi(x)\overline{\psi(x)} = \frac{1}{|Z(G)|} \sum_{x \in Z(G)} |\psi(x)|^2.$$

Substituting by $\langle \psi, \psi \rangle$ in ($\star$) we obtain

$$\begin{gathered} \frac{1}{|Z(G)|} \sum_{x \in Z(G)} |\psi(x)|^2 \geqslant \psi(1) \quad \text{and} \\ \sum_{x \in Z(G)} |\psi(x)|^2 \geqslant |Z(G)|\psi(1). \end{gathered} \tag{$\star\star$}$$

If $\chi \in \mathrm{Irr}(G)$, then $\chi_{Z(G)}$ is a character of $Z(G)$ and $\star\star$ implies

$$\chi_{Z(G)}(1)\,|Z(G)| \leqslant \sum_{x\in Z(G)} |\chi_{Z(G)}(x)|^2 \leqslant \sum_{x\in G} |\chi(x)|^2 = |G|.$$

Since $\chi(1) = \chi_{Z(G)}(1)$, therefore, $\chi(1) \leqslant |G : Z(G)|$. From [8, Theorem 3.11], the degree of the irreducible character of G is dividing the order of G. Further, if G is a p-group, $\chi(1)$ must divide the index $|G : Z(G)|$. Indeed, since G is a p-group, by Proposition 13 we conclude that $\chi(1)$ divides the index $|G : Z(A_\chi)|$. □

Corollary 15 *If* G *is a finite nilpotent group and* $\chi \in \mathrm{Irr}(G)$ *has anchor* A_χ*, then* $\chi(1)$ *divides the index* $|G : Z(A_\chi)|$ *for all* $\chi \in \mathrm{Irr}(G)$.

Proof — Suppose G is a finite nilpotent group of order $\prod_{i=1}^{m} p_i^{n_i}$, where $p_1, p_2, \ldots, p_m$ are distinct prime numbers. Then G is the direct product of its Sylow groups from [7, Proposition 7.5], that is, $G = \prod_{i=1}^{m} S_i$, where S_i is a Sylow p_i-subgroup of G. If $\chi_{S_i} \in \mathrm{Irr}(S_i)$, then

$$\chi_{S_i}(1) \,|\, |S_i : Z(A_{\chi_{S_i}})|$$

for $i = 1, \ldots, m$. So, there is $t_i \in \mathbb{Z}^+$ such that $|S_i : Z(A_{\chi_{S_i}})| = t_i \chi_{S_i}(1)$ for $i = 1, \ldots, m$. By [7, Corollary 8. 11] we have

$$\prod_{i=1}^{m} S_i / \prod_{i=1}^{m} Z(A_{\chi_{S_i}}) \simeq \prod_{i=1}^{m} S_i / Z(A_{\chi_{S_i}}).$$

It follows that

$$\left| \prod_{i=1}^{m} S_i : \prod_{i=1}^{m} Z(A_{\chi_{S_i}}) \right| = \prod_{i=1}^{m} t_i \chi_{S_i}(1).$$

The irreducible character χ of $G = \prod_{i=1}^{m} S_i$ is of the form

$$\chi = \chi_{S_1} \otimes \chi_{S_2} \otimes \ldots \otimes \chi_{S_m},$$

where $\chi_{S_i} \in \mathrm{Irr}(S_i),\ i = 1, \ldots, m$, from [8, Theorem 4.21]. Hence

$$\chi(1) = \prod_{i=1}^{m} \chi_{S_i}(1).$$

From Theorem 12, $A_\chi = A_{\chi_{S_1}} \times \ldots \times A_{\chi_{S_m}}$. From Proposition 13 we know that the anchor of any irreducible character of a Sylow p-subgroup:= S is S. Thus, if "we deal with $p = p_j$", then the anchor of the irreducible character χ_{S_j} of S_j is S_j while the anchor of the other irreducible characters χ_{S_i} of S_i is the trivial group for all $i \neq j$. So,

$$A_\chi = \{1_G\} \times \ldots \times \{1_G\} \times S_j \times \{1_G\} \times \ldots \times \{1_G\} \simeq S_j.$$

We have that the center of a direct products of groups is equal to the product of the centers of the groups by Exercise 5.1.1 in [4],

$$Z(A_\chi) = Z(A_{\chi_{S_1}}) \times \ldots \times Z(A_{\chi_{S_m}}).$$

Thus, $|G : Z(A_\chi)| = t.\chi(1)$, where $t = \prod_{i=1}^{m} t_i$. Thus, we get the desired claim. □

Example 16 Consider the extraspecial p-group $G = C_{p^2} : C_p$, of order p^3. The anchor of any irreducible character χ of G is G itself, that is, $A_\chi = G$ for all $\chi \in \mathrm{Irr}(G)$. The center $Z(G) \simeq C_p$. The degree of

$$\mathrm{Irr}(G) = \underbrace{1, 1, \ldots, 1}_{p^2 \text{ times}}, \underbrace{p, \ldots, p}_{(p-1) \text{ times}}.$$

For all $\chi \in \mathrm{Irr}(G)$, $\chi(1) \mid |G : Z(A_\chi)| = p^2$.

Theorem 17 *Let* H *be a subgroup of* G *which does not contain a Sylow* p*-subgroup of* G. *Let* $\chi \in \mathrm{Irr}(G)$ *and* $\theta \in \mathrm{Irr}(H)$ *such that* χ *lies over* θ, *that is* $\langle \mathrm{Res}^G_H \chi, \theta \rangle \neq 0$. *If* H *contains the anchor of* χ. *Then the anchor of* θ *is contained in the anchor of* χ.

Proof — Let A_χ be the anchor of χ and A_θ be the anchor of θ. Assume the result does not hold. From Definition 1, A_χ is the minimal p-subgroup of G such that $e_\chi \in (\mathcal{R}Ge_\chi)^G_{A_\chi}$. Then there exists $c \in (\mathcal{R}Ge_\chi)^{A_\chi}$ such that $\mathrm{tr}^G_{A_\chi}(c) = e_\chi$. Since e_θ is the unique primitive idempotent in $Z(kH)$ satisfying $0 \neq \theta(e_\theta)$. So a primitive H-algebra $\mathcal{R}He_\theta$ contains e_θ as the unique identity such that $\langle \mathrm{Res}^G_S \chi, \theta \rangle \neq 0$, then $e_\chi e_\theta = e_\theta$. We have $\mathcal{R}He_\theta \subseteq \mathcal{R}Ge_\chi$, so $ce_\theta \in (\mathcal{R}Ge_\chi)^{A_\chi}$. Now

$$\mathrm{tr}^G_{A_\chi}(ce_\theta) = \mathrm{tr}^G_{A_\chi}(c)e_\theta = e_\chi e_\theta = e_\theta.$$

Thus, $e_\theta \in (\mathcal{R}Ge_\chi)^G_{A_\chi}$. Without loss of generality, we may assume $e_\theta \in (\mathcal{R}He_\theta)^G_{A_\chi}$. By [15, Proposition 11.4 (a)], we have $A_\chi \leqslant H \leqslant G$

then

$$\mathrm{tr}_H^G\left((\mathcal{R}He_\theta)_{A_\chi}^H\right) = (\mathcal{R}He_\theta)_{A_\chi}^G.$$

It follows from [15, p.89] that $(\mathcal{R}He_\theta)_{A_\chi}^H$ is an ideal of $(\mathcal{R}He_\theta)^H$ and that $(\mathcal{R}He_\theta)_{A_\chi}^G$ is an ideal $(\mathcal{R}He_\theta)^G$. Therefore,

$$(\mathcal{R}He_\theta)_{A_\chi}^G \subseteq (\mathcal{R}He_\theta)_{A_\chi}^H,$$

since $(\mathcal{R}He_\theta)^G \leqslant (\mathcal{R}He_\theta)^H$. Thus,

$$e_\theta \in (\mathcal{R}He_\theta)_{A_\chi}^G \leqslant (\mathcal{R}He_\theta)_{A_\chi}^H.$$

From [10, Corollary 1.4 (i)], we obtain

$$(\mathcal{R}He_\theta)_{A_\chi}^H \subseteq (\mathcal{R}He_\theta)_{A_\theta}^H.$$

This is a contradiction since A_θ is the minimal p-subgroup of H such that $e_\theta \in (\mathcal{R}He_\theta)_{A_\theta}^H$. □

Example 18 Let $p = 2$, $G = S_5$ be the symmetric group of degree five, and $H = A_5 \leqslant G$ is the alternating group of degree five. Consider $\chi \in \mathrm{Irr}(G)$ of degree four, such that $\theta = \mathrm{Res}_H^G \chi \in \mathrm{Irr}(H)$. The anchor A_χ of χ is isomorphic to C_2. Which is the cyclic subgroup of order two. That is because χ belongs to the block B_1 which is of defect one. It follows that the defect group of the block B_1 is of order 2^1 which is isomorphic to C_2. The anchor of θ is the trivial group as θ is of defect zero.

In Theorem 17 if H is a Sylow p-subgroup of G, then from Proposition 13 the anchor of any irreducible character of H is H itself. Since the anchor of any irreducible character of G is a p-subgroup of G, it is contained in the anchor of any irreducible character of H.

Example 19 (1) Let $p = 2$, $G = S_4$ be the symmetric group of degree four, with Sylow 2-subgroup the dihedral group D_8. Consider $\chi \in \mathrm{Irr}(S_4)$ as follows:

	[(1)]	[(12)]	[(123)]	[(12)(34)]	[(1234)]
χ	2	0	-1	2	0

by restriction to D_8 we obtain:

	[(1)]	[(12)]	[(1324)]	[(12)(34)]	[(14)(23)]
$\mathrm{Res}_{D_8}^{S_4}\chi$	2	0	0	2	2

Note that $\mathrm{Res}_{D_8}^{S_4}\chi = \theta_1 + \theta_4$, where $\theta_1,\ \theta_4$ are linear irreducible characters of D_8. The anchor A_χ is the Klein four subgroup V_4 from [11, Example 5.8 (2)]. Now all elements θ_i of $\mathrm{Irr}(D_8)$ have anchor D_8 itself, from Proposition 13. Thus,

$$A_\chi = V_4 \leqslant A_{\theta_i} = D_8, \text{ for } i = 1, 4.$$

(2) Let $p = 2$, $G = S_5$ and a Hall-subgroup $H = S_4$ of G which all have a Sylow 2-subgroup isomorphic to the dihedral group D_8. Consider $\chi \in \mathrm{Irr}(G)$ of degree four. The anchor A_χ of χ is isomorphic to C_2, the cyclic group of order two. Note that $\mathrm{Res}_H^G\chi = \theta_1 + \theta_3$, where $\theta_1, \theta_3 \in \mathrm{Irr}(H)$ such that θ_1 the principal character and θ_3 is of degree 3. By Corollary 10 their anchors are Sylow 2-subgroups of H. Thus, $A_\chi \leqslant A_{\theta_i}$ for $i = 1, 3$.

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