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# Pronormality in Group Theory *

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Dedicated to Leonid A. Kurdachenko on the occasion of his $70^{\text{th}}$ birthday

**Abstract**

A subgroup $X$ of a group $G$ is said to be *pronormal* if for each element $g$ of $G$ the subgroups $X$ and $X^g$ are conjugate in $\langle X, X^g \rangle$. The aim of this paper is to study pronormality and some close embedding properties, like weak normality and weak pronormality. In particular, it is proved that these properties can be countably detected, and the behaviour of groups which are rich in (generalized) pronormal subgroups is investigated.

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## 1 Introduction

A subgroup $X$ of a group $G$ is said to be *pronormal* if for each element $g$ of $G$ the subgroups $X$ and $X^g$ are conjugate in $\langle X, X^g \rangle$. Of course, normal subgroups and maximal subgroups of arbitrary groups are pronormal, as well as Sylow subgroups of finite groups and Hall subgroups of any finite soluble group. Moreover, it follows

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from the definition that the normalizer of a pronormal subgroup is likewise pronormal.

The concept of a pronormal subgroup was introduced by P. Hall and the first results on this subject appeared in Rose's paper [34]. More recently, several authors have investigated pronormality, mostly dealing with properties of pronormal subgroups of finite groups and with groups which are rich in pronormal subgroups (for more information see the survey paper [20]); for instance, Kuzennyĭ and Subbotin described the structure of infinite locally soluble groups which have only pronormal subgroups (see [25]).

It is easy to show that a subgroup X of a group G is normal if and only if it is both subnormal and pronormal, and actually subnormality can be weakened by requiring that X is *ascendant* in G, i.e. there exists an ascending series

$$X = X_0 \triangleleft X_1 \triangleleft \dots X_\alpha \triangleleft X_{\alpha+1} \triangleleft \dots X_\gamma = G$$

from X to G. In particular, if every subgroup of a group is pronormal, then the group has the so-called T-property. Recall that a group is said to be a T-*group* (or to have the T-*property*) if every subnormal subgroup is normal, or equivalently if normality is a transitive relation; groups with the T-property form an important group class which is local but not subgroup closed. Here, a group class $\mathfrak{X}$ is said to be *local* if it contains every group whose finite subsets lie in an $\mathfrak{X}$-subgroup; clearly, a subgroup closed group class $\mathfrak{X}$ is local if and only if it contains every group whose finitely generated subgroups belong to $\mathfrak{X}$.

A group is called a $\overline{T}$-*group* if all its subgroups have the T-property. Of course, groups admitting only pronormal subgroups have the $\overline{T}$-property. The structure of finite soluble T-groups was described by Gaschütz (see [11]), while Robinson investigated soluble groups with the T-property in the general case (see [29]). It turns out in particular that soluble T-groups are metabelian, and that every finitely generated soluble group with the T-property is either finite or abelian. Moreover, every finite $\overline{T}$-group is supersoluble and all finite soluble T-groups have the $\overline{T}$-property.

Actually, it was proved by Peng that all subgroups of a finite group G are pronormal if and only if G is a soluble T-group (see [28]), and this result was later extended to the class of FC-groups (see [19]). Recall that a group G is called an FC-*group* if every element of G has only finitely many conjugates or, which is the same, if its centalizer has finite index in G; we refer to the monograph [37] for the main properties of groups with the FC-property.

The main purpose of this paper is to give a contribution to the knowledge of pronormal subgroups. In particular, it will be proved that a subgroup of an arbitrary group is pronormal if and only if all its countable subgroups are pronormal, and new information on the structure of groups which are rich in pronormal subgroups will be obtained.

Some further embedding properties, which are closely related to pronormality and depend on the behaviour of the conjugates of a subgroup, will be considered in the second part of the paper. A subgroup $X$ of a group $G$ is said to be *weakly normal* if $X^g = X$ whenever $g$ is an element of $G$ such that $X^g$ is contained in the normalizer $N_G(X)$, while $X$ is called *weakly pronormal* if $X^g = X$ for each element $g$ of $G$ such that $X^g \leqslant X$. Thus every pronormal subgroup is also weakly normal, and all weakly normal subgroups are weakly pronormal; examples will be given to show that pronormality, weak normality and weak pronormality are pairwise different concepts. We must point out that we have chosen to define weak pronormality in this way, although the same terminology was already used elsewhere with different meanings (see [2] and [24]).

Of course, finite subgroups of arbitrary groups are weakly pronormal. Moreover, each subgroup of a periodic group is weakly pronormal, and it is also easy to see that all groups locally satisfying the maximal condition on subgroups have only weakly pronormal subgroups (see for instance [1], Lemma 4.6.3). It follows from the definition that for any set $\pi$ of prime numbers the Sylow $\pi$-subgroups of an arbitrary group are weakly normal, and that a self-normalizing subgroup is weakly normal if and only if it is weakly pronormal; in particular, self-normalizing subgroups of groups locally satisfying the maximal condition on subgroups are weakly normal.

We shall prove that for a subgroup also the properties of being weakly normal or weakly pronormal can be countably detected, and groups with many weakly normal or weakly pronormal subgroups will be investigated.

Our notation is mostly standard and can be found in [31].

## 2 Pronormal subgroups

We start with two elementary results on pronormal subgroups. The first of them is almost obvious and shows in particular that pronormal subgroups of arbitrary groups are weakly normal, while the second describes pronormality in locally nilpotent groups.

**Lemma 2.1** *Let $G$ be a group, and let $X$ be a subgroup of $G$ such that $X^g$ is contained in $N_G(X)$ for some element $g$ of $G$. If $X$ and $X^g$ are conjugate in $\langle X, X^g\rangle$, then $X^g = X$.*

Proof — Let $y$ be an element of $\langle X, X^g\rangle$ such that $X^g = X^y$. As $\langle X, X^g\rangle$ is contained in $N_G(X)$, we have $X^g = X^y = X$. □

**Proposition 2.2** *Let $G$ be a locally nilpotent group. Then every pronormal subgroup of $G$ is normal.*

Proof — Let $X$ be a pronormal subgroup of $G$. If $g$ is any element of $G$, there exists $y$ in $\langle X, X^g\rangle$ such that $X^g = X^y$, and hence $X^{gy^{-1}} = X$. Thus $gy^{-1}$ belongs to $N_G(X)$, and so $g = (gy^{-1})y$ is an element of $\big\langle N_G(X), N_G(X)^g\big\rangle$. It follows that $N_G(X)$ contains a finitely generated subgroup $E$ such that $g$ belongs to $\langle E, E^g\rangle$. On the other hand, the subgroup $Y = \langle E, E^g\rangle$ is nilpotent, and $E^Y = Y$, so that $Y = E$ and $g$ is an element of $E$. Therefore $X^g = X$ and $X$ is normal in $G$. □

Our first main result proves that the pronormality of a subgroup can be detected from the behaviour of its countable subgroups.

**Theorem 2.3** *Let $G$ be a group, and let $X$ be a subgroup of $G$. If all countable subgroups of $X$ are pronormal in $G$, then $X$ itself is pronormal in $G$.*

Proof — Assume for a contradiction that $X$ is not pronormal in $G$, so that there exists an element $g$ of $G$ such that $X$ and $X^g$ are not conjugate in $\langle X, X^g\rangle$. Let $a$ be any element of $\langle X, X^g\rangle$. Suppose that $X^a \leqslant X^g$, so that $X^h$ is a proper subgroup of $X$, where $h = ag^{-1}$, and we may consider an element $x$ of $X \setminus X^h$. As the countable subgroup

$$Y = \langle x, x^h, x^{h^2}, \dots, x^{h^n}, x^{h^{n+1}}, \dots\rangle$$

is pronormal in $G$ and $Y^h \leqslant Y$, it follows from Lemma 2.1 that $Y^h = Y$. In particular, $x$ belongs to $Y^h \leqslant X^h$, a contradiction. It follows that $X^a$

is not contained in $X^g$, and hence there exists an element $x(a)$ of $X$ such that $x(a)^a$ does not belong to $X^g$.

Fix now a countable subgroup $Z$ of $X$, and define a chain of countable subgroups

$$Z = Z_1 \leqslant Z_2 \leqslant \ldots \leqslant Z_n \leqslant Z_{n+1} \leqslant \ldots$$

of $X$ by putting

$$Z_{n+1} = \big\langle Z_n, x(a) \mid a \in \langle Z_n, Z_n^g \rangle \big\rangle$$

for each positive integer $n$. As

$$W = \bigcup_{n \in \mathbb{N}} Z_n$$

is a countable subgroup of $X$, it is pronormal in $G$, and hence there exists an element $b$ of $\langle W, W^g \rangle$ such that $W^b = W^g$. Clearly,

$$\langle W, W^g \rangle = \bigcup_{n \in \mathbb{N}} \langle Z_n, Z_n^g \rangle,$$

so that $b$ is in $\langle Z_m, Z_m^g \rangle$ for some positive integer $m$, and hence $x(b)$ is an element of $Z_{m+1} \leqslant W$. Thus $x(b)^b$ belongs to $W^b = W^g \leqslant X^g$, and this last contradiction completes the proof the statement. □

In what follows, we will denote by $\mathfrak{W}_0$ the class of all groups admitting only pronormal subgroups.

**Corollary 2.4** *Let $G$ be a group whose countable subgroups are pronormal. Then $G$ belongs to the class $\mathfrak{W}_0$.*

In contrast to Theorem 2.3, it can be remarked that pronormality cannot be recognizable from the behaviour of finitely generated subgroups, as the following example shows. For each positive integer $n$, let $p_n$ be the $n$-th odd prime number and let

$$G_n = \langle a_n, b_n \mid a_n^2 = b_n^{p_n} = 1, a_n b_n = b_n^{-1} a_n \rangle$$

be the dihedral group of order $2p_n$. In the cartesian product of all these groups consider the subgroup

$$G = \big\langle a, \operatorname*{Dr}_{n \in \mathbb{N}} G_n \big\rangle,$$

where $a = (a_n)_{n\in\mathbb{N}}$. Then $G$ is a locally finite $\overline{T}$-group (see [29] Theorem 6.1.1), so that in particular all its finite subgroups are pronormal. On the other hand, the Sylow 2-subgroup

$$X = \langle a_n b_n \mid n \in \mathbb{N}\rangle = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\langle a_n b_n\rangle$$

is not pronormal in $G$, since $X$ and $X^a$ are not conjugate in $\langle X, X^a\rangle$.

In relation to the above remark, the following interested fact can be observed.

**Proposition 2.5** *Let $G$ be a group whose cyclic subgroups are pronormal. Then $G$ is a $\overline{T}$-group.*

Proof — Let $x$ be any element of $G$. Since the subgroup $\langle x\rangle$ is pronormal in $G$, we have

$$G = N_G(\langle x\rangle)\langle x\rangle^G,$$

and hence

$$\langle x\rangle^G = \langle x\rangle^{N_G(\langle x\rangle)\langle x\rangle^G} = \langle x\rangle^{\langle x\rangle^G}.$$

It follows that $G$ has the $T$-property (see [29], Lemma 2.1.1), and so it is even a $\overline{T}$-group since the hypotheses are inherited by subgroups. □

For our purposes, we need the following result of Robinson concerning the lower central series of a $\overline{T}$-group (see [29]); notice that for any group $G$ we shall denote by $\pi(G)$ the set of all prime numbers that are orders of elements of $G$.

**Lemma 2.6** *Let $G$ be a locally finite $\overline{T}$-group. Then $\gamma_3(G)$ has no elements of order 2 and $\pi\big(\gamma_3(G)\big) \cap \pi\big(G/\gamma_3(G)\big) = \emptyset$.*

Our next result allows in particular to control pronormality from the behaviour of cyclic subgroups in the case of linear groups.

**Theorem 2.7** *For a linear group $G$ the following statements are equivalent:*

(i) *all cyclic subgroups of $G$ are pronormal;*

(ii) *$G$ belongs to the group class $\mathfrak{W}_0$;*

(iii) *$G$ is a soluble $\overline{T}$-group.*

Proof — By Proposition 2.5, it is enough to prove that (ii) and (iii) are equivalent. Suppose now that G is a $\mathfrak{W}_0$, so that in particular it has the $\overline{\mathrm{T}}$-property. Since every finitely generated subgroup of G is residually finite (see [38], Theorem 4.2) and finite $\overline{\mathrm{T}}$-groups are metabelian, we have also that G itself is metabelian.

Assume finally that G is a soluble $\overline{\mathrm{T}}$-group. Clearly, it can be assumed that G is periodic, because soluble non-periodic $\overline{\mathrm{T}}$-groups are abelian. Put $L = \gamma_3(G)$ and $\pi = \pi(L)$. It follows from Lemma 2.6 that L is a Hall normal $\pi$-subgroup of G with no elements of order 2, so that G splits over L and the complements of L in G coincide with the Sylow $\pi'$-subgroups of G (see [38], Theorem 9.18). Thus the characterization of soluble $\mathfrak{W}_0$-groups obtained by Kuzennyĭ and Subbotin proves that G belongs to $\mathfrak{W}_0$ (see [25] or [20]). □

The rest of this section is devoted to the study of groups which are rich in pronormal subgroups, in some reasonable sense. Unfortunately, it is clear that all subgroups of a Tarski group (i.e. an infinite simple group whose proper non-trivial subgroups have prime order) are pronormal, and this remark shows that in general this task can be extremely complicated. One of the most common restrictions adopted in order to avoid Tarski groups and other similar pathologies is the requirement that the group is locally graded: a group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Locally graded groups form a wide class containing for instance all groups which are locally (soluble-by-finite); moreover, any residually finite group is locally graded, and so in particular free groups have this property. Notice in this context the following relevant result, that combines theorems proved in [25] and [33].

**Lemma 2.8** *Let* G *be a locally graded group in which all subgroups are pronormal. Then* G *is metabelian.*

The previous example also shows that the group class $\mathfrak{W}_0$ is not local. However, this class is at least countably recognizable. Recall here that a group class $\mathfrak{X}$ is said to be *countably recognizable* if it contains every group whose countable subgroups belong to $\mathfrak{X}$. Countably recognizable classes of groups were introduced by Baer in [3], but already in the fifties of last century some relevant group properties were proved to be detectable from the behaviour of countable subgroups. Recently, many new items have been added to the list of the known countably recognizable group classes (see for instance [15],[16]). Of course, any local class is countably recognizable, but for instance it is straightforward to show that nilpotent groups and soluble groups form countably recognizable classes which are not local.

**Corollary 2.9** *The group class $\mathfrak{W}_0$ is a countably recognizable.*

Proof — Let G be a group whose countable subgroups belong to $\mathfrak{W}_0$, and let X be any countable subgroup of G. If g is any element of G, the subgroup $\langle X, g\rangle$ is likewise countable, so that X is pronormal in $\langle X, g\rangle$ and hence X and $X^g$ are conjugate in $\langle X, X^g\rangle$. It follows that each countable subgroup of G is pronormal, and hence all subgroups of G are pronormal by Corollary 2.4. □

The consideration of the locally dihedral 2-group shows that there exist soluble countable groups which are not $\mathfrak{W}_0$-groups, although all its infinite proper subgroups belong to $\mathfrak{W}_0$. Our next result shows that the situation is different in the uncountable case.

**Theorem 2.10** *Let G be an uncountable locally graded group of cardinality $\aleph$. If all proper subgroups of G of cardinality $\aleph$ belong to $\mathfrak{W}_0$ and G has no simple homomorphic images of cardinality $\aleph$, then G is a $\mathfrak{W}_0$-group.*

Proof — Assume for a contradiction that the group G has a simple non-abelian homomorphic image G/N. By hypothesis, G/N has cardinality strictly smaller than $\aleph$, so that N has cardinality $\aleph$ and hence it is metabelian by Lemma 2.8. Clearly, all proper subgroups of the simple group G/N are metabelian, and so G/N is finitely generated. On the other hand, G/N is likewise locally graded (see [26]) and hence it must be finite. As all proper subgroups of G/N have the $\overline{T}$-property, they are supersoluble and hence G/N is soluble (see [21]). This contradiction shows that G has no simple non-abelian homomorphic images.

Since all proper subgroups of G of cardinality $\aleph$ are metabelian by Lemma 2.8, it follows that the group G itself is metabelian (see [14], Theorem D). Thus G is a $\overline{T}$-group (see [13]), and hence all subgroups of $G'$ are normal in G. Clearly, at least one of the abelian groups $G'$ and $G/G'$ has cardinality $\aleph$, and so there exists a normal subgroup K of G of cardinality $\aleph$ such that also G/K has cardinality $\aleph$, and either $K \leqslant G'$ or $G' \leqslant K$ (see [12], Lemma 2.4). If X is any countable subgroup of G, the product XK is a proper subgroup of G of cardinality $\aleph$ and so all its subgroups are pronormal. Therefore all countable subgroups of G have only pronormal subgroups, and hence G belongs to $\mathfrak{W}_0$ by Corollary 2.9. □

Let $\mathfrak{X}$ be a class of groups. A non-trivial group is said to be *minimal non-*$\mathfrak{X}$ (or also an *opponent* of $\mathfrak{X}$) if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. We shall say that a group class $\mathfrak{X}$ is *accessible* if every locally graded group whose proper subgroups belong to $\mathfrak{X}$ is either finite or an $\mathfrak{X}$-group, or equivalently if any locally graded minimal non-$\mathfrak{X}$ group is finite.

It is easy to show that the class $\mathfrak{A}$ of abelian groups is accessible, and $\mathfrak{A}$ shares such a property with other relevant classes of groups, like for instance the class $\mathfrak{N}_c$ of nilpotent groups of class at most c (see [5]) and the class $\mathfrak{S}_d$ of soluble groups of derived length at most d (see [8]). On the other hand, the consideration of the locally dihedral 2-group shows that the class $\mathfrak{N}$ of nilpotent groups is not accessible, while it is still unknown if the class $\mathfrak{S}$ of soluble groups is accessible. For a detailed discussion of accessible group classes and their abstract properties we refer to [17] and [18].

The structure of locally finite groups which are minimal non-T was investigated by Robinson in [30], and it has recently been proved that any locally graded group whose proper subgroups have a transitive normality relation either is finite or a T-group, so that the class of T-groups is accessible (see [13]). In the case of groups with only pronormal subgroups, we can prove the following result.

**Theorem 2.11** *The group class* $\mathfrak{W}_0$ *is accessible.*

Proof — Assume for a contradiction that the statement is false, and let G be an infinite locally graded opponent of the group class $\mathfrak{W}_0$. Then all proper subgroups of G are metabelian by Lemma 2.8, and so G itself is metabelian since $\mathfrak{S}_2$ is an accessible group class. Moreover, all proper subgroups of G have the T-property, so that G is a $\overline{T}$-group and in particular it is periodic.

Let X be a subgroup of G which is not pronormal, and let g be an element of G such that X and $X^g$ are not conjugate in $\langle X, X^g\rangle$, so that $G = \langle X, g\rangle$. As the factor group $G/X_G$ has the $\overline{T}$-property and contains a non-pronormal subgroup, it must be infinite (see [28]), and so it is likewise a counterexample to the statement. Thus we may suppose without loss of generality that X does not contain non-trivial normal subgroups of G. Since all subgroups of $G'$ are normal in G, we have $X \cap G' = \{1\}$, so that X is abelian and hence $X \cap \langle g\rangle = \{1\}$. Put $L = [G', G]$, and let $\pi$ be the set of all prime numbers that are orders of elements of L, so that $G/L$ is a $\pi'$-group by Lemma 2.6. Moreover, the centralizer $C_X(L)$ is normal in XL and XL is obviously normal in G, so that $C_X(L)$ is also normal in G and hence $C_X(L) = \{1\}$. Thus X is isomorphic to a group of power automorphisms of L. In particular, X induces a finite group of automorphisms on each primary component of L, and so it is residually finite (see for instance [29]).

As $G = \langle X, g\rangle$ is infinite, also the subgroup X must be infinite, and hence also the set $\pi(L)$ is infinite. Moreover, G is periodic and $X \cap L = \{1\}$, so that the intersection $X_0 = X \cap L\langle g\rangle$ is finite, and X contains a proper subgroup Y such that the index $|X : Y|$ is finite and larger than the order of $X_0$. It follows now from the Dedekind modular law that the product $YL\langle g\rangle$ is a proper subgroup of G, so that all its subgroups are pronormal and in particular Y and $Y^g$ are conjugate in $\langle Y, Y^g\rangle$. Let $z$ be an element of $\langle Y, Y^g\rangle$ such that $Y^z = Y^g$, so that $v = gz^{-1}$ belongs to the normalizer $N_G(Y)$. Suppose that $\langle X, v\rangle$ is a proper subgroup of G. Then X is pronormal in $\langle X, v\rangle$, so that there exists $w$ in $\langle X, X^v\rangle$ such that $X^w = X^v$ and hence $X^{wz} = X^g$. On the other hand, $\langle X, X^v\rangle \leqslant \langle X, X^g\rangle$, so that $wz$ belongs to $\langle X, X^g\rangle$, against the assumption. Therefore $\langle X, v\rangle = G$, so that Y is a normal subgroup of G, which is impossible as X does not contain non-trivial normal subgroups of G. This last contradiction completes the proof of the theorem. □

Uncountable groups in which all uncountable subgroups have a given embedding property, like normality or subnormality, have recently been studied (see for instance [6] and [7]). We point out here that a soluble uncountable group in which all uncountable subgroups are pronormal need not have only pronormal subgroups. In fact, Ehrenfeucht and Faber constructed a nilpotent uncountable group G of class 2 in which all abelian subgroups are countable and all uncountable subgroups contain $G'$ (see [9]); in particular, all uncountable subgroups of G are normal, while obviously every non-normal subgroup of G is not even pronormal.

## 3 Weakly normal subgroups

Weak normality was introduced by Müller in [27], under the influence of Reinhold Baer and with the additional requirement that the subgroup has only finitely many conjugates. Actually, almost all known results on weakly normal subgroups have been proved in the finite case.

Of course, every weakly normal subgroup is weakly pronormal, but any non-normal subgroup of order 2 of the dihedral group $D_8$ is not weakly normal, although it is (trivially) weakly pronormal. Moreover, every pronormal subgroup is weakly normal, but there exist weakly normal subgroups that are not pronormal. To see this, consider the natural permutational wreath product $G = H \wr K$, where $H$ has order 3 and $K$ is isomorphic to the symmetric group $Sym(3)$. Then $|N_G(K) : K| = 3$ and $K$ is the only subgroup of $N_G(K)$ isomorphic to $Sym(3)$, so that $K$ is a weakly normal subgroup of $G$; on the other hand, $K$ cannot be pronormal in $G$ (see for instance [23], Corollary to Theorem 1).

Our next three statements show that some basic facts concerning pronormality can be extended to weakly normal subgroups.

**Lemma 3.1** *Let $G$ be a group, and let $X$ be a weakly normal subgroup of $G$. Then also the normalizer $N_G(X)$ is weakly normal in $G$.*

Proof — If $g$ is an arbitrary element of $G$ normalizing $N_G(X)$, we have

$$X^g \leqslant N_G(X)^g = N_G(X),$$

so that $X^g = X$ and $g$ belongs to $N_G(X)$. Thus $N_G\big(N_G(X)\big) = N_G(X)$. Let $h$ be any element of $G$ such that $N_G(X)^h \leqslant N_G\big(N_G(X)\big)$. Then

$$X^h \leqslant N_G(X)^h \leqslant N_G(X),$$

so that $X^h = X$ and $N_G(X)^h = N_G(X)$. Therefore the subgroup $N_G(X)$ is weakly normal in $G$. □

**Lemma 3.2** *Let $G$ be a group, and let $X$ be a subgroup of $G$. Then $X$ is normal in $G$ if and only if it is ascendant and weakly normal.*

Proof — Obviously, it is enough to prove that $X$ is normal in $G$, whenever it is weakly normal and there exists an ascending series

$$X = X_0 \lhd X_1 \lhd \ldots X_\alpha \lhd X_{\alpha+1} \lhd \ldots X_\gamma = G.$$

Assume for a contradiction that $X$ is not normal in $G$, and let $\mu \leqslant \gamma$ be the smallest ordinal such that $X$ is not normal in $X_\mu$. Clearly, $\mu > 1$ and $\mu$ is not a limit ordinal, so that we may consider $X_{\mu-1}$ and $X \lhd X_{\mu-1} \lhd X_\mu$. If $g$ is any element of $X_\mu$, we have $X^g \leqslant X_{\mu-1} \leqslant N_G(X)$, so that $X^g = X$ and hence $X$ is normal in $X_\mu$. This contradiction completes the proof. □

**Corollary 3.3** *Let $G$ be a group in which all subgroups are weakly normal. Then $G$ is a $\overline{T}$-group.*

It follows from Lemma 3.2 that every weakly normal subgroup of a hypercentral group is normal. On the other hand, all pronormal subgroups of a locally nilpotent group are normal by Proposition 2.2, but the following easy example shows that a weakly normal subgroup of a locally nilpotent group need not be normal. Let $p$ be a prime number, and let $G = H \wr K$ be the standard wreath product of a group $H$ of order $p$ by an infinite abelian group $K$ of exponent $p$; of course, $G$ is locally nilpotent and $N_G(K) = K$, so that $K$ is a weakly normal subgroup of $G$ which is not normal (and of course not even pronormal).

It is also of interest to remark that a weakly normal subgroup $X$ of a group $G$ may contain a characteristic subgroup $Y$ which is not weakly normal in $G$. In fact, any cyclic subgroup of order 6 of the symmetric group $Sym(5)$ is weakly normal, while its 2-component cannot be weakly normal.

It was proved in [35] that any locally graded non-periodic group admitting only weakly normal subgroups is abelian. As finite $\overline{T}$-groups are soluble, we have that every periodic locally graded $\overline{T}$-group is metabelian, and it follows that all groups with only weakly normal subgroups are metabelian. Moreover, all subgroups of a soluble $\overline{T}$-group are weakly normal (see [35]), and hence the following result can be stated.

**Theorem 3.4** *A locally graded group has only weakly normal subgroups if and only if it is soluble and has the $\overline{T}$-property.*

It is not clear whether the requirement that all subgroups are weakly normal characterizes groups with the $\overline{T}$-property, at least within the universe of locally graded groups. By the above quoted results in [35], a positive answer to this question would be equivalent to the conjecture that every non-periodic locally graded $\overline{T}$-group is abelian (see [22], Question 14.36).

**Lemma 3.5** *Let $G$ be a group, and let $X$ be a subgroup of $G$ such that $X^g$ is contained in $N_G(X)$ for some element $g$ of $G$. If $Y$ is any countable subgroup of $X$, there exists a countable subgroup $Z$ of $X$ such that $Y \leqslant Z$ and $Z^g \leqslant N_G(Z)$.*

Proof — Put $Z_1 = Y$, and if a countable subgroup $Z_n$ of $X$ containing $Y$ has been chosen for some positive integer $n$, let $Z_{n+1}$ be the subgroup generated by $Z_n$ and by the set $Z_n^g \setminus N_G(Z_n)$. Then also

$$Z = \bigcup_{n \in \mathbb{N}} Z_n$$

is a countable subgroup of $X$. Assume for a contradiction that $Z^g$ is not contained in $N_G(Z)$, so that there exist elements $u = z^g$ of $Z^g$ and $w$ of $Z$ such that $w^u$ does not belong to $Z$. Let $m$ be a positive integer such that both $z$ and $w$ are in $Z_m$. As $w^u$ does not belong to $Z_m$, the element $z^g$ is not in $N_G(Z_m)$ and hence it belongs to $Z_{m+1}$, which is of course impossible. Therefore the subgroup $Z^g$ is contained in $N_G(Z)$. □

We can now prove that also weak normality, like pronormality, may be detected from the behaviour of countable subgroups.

**Theorem 3.6** *Let $G$ be a group, and let $X$ be a subgroup of $G$. If all countable subgroups of $X$ are weakly normal in $G$, then $X$ itself is weakly normal in $G$.*

Proof — Let $g$ be any element of $G$ such that $X^g \leqslant N_G(X)$. It follows from Lemma 3.5 that for each element $x$ of $X$ there exists a countable subgroup $Y_x$ of $X$ such that $x$ belongs to $Y_x$ and $Y_x^g \leqslant N_G(Y_x)$. Then $Y_x^g = Y_x$ since every $Y_x$ is weakly normal in $G$, and so

$$X^g = \bigcup_{x \in X} Y_x^g = \bigcup_{x \in X} Y_x = X.$$

Therefore the subgroup $X$ is weakly normal in $G$. □

We shall denote by $\mathfrak{W}_1$ the class of all groups with only weakly normal subgroups, which properly contains the class $\mathfrak{W}_0$, since there exist locally finite $\overline{T}$-groups that are not in $\mathfrak{W}_0$.

**Corollary 3.7** *Let $G$ be a group whose countable subgroups are weakly normal. Then $G$ belongs to the group class $\mathfrak{W}_1$.*

Next result shows that the hypothesis of Corollary 3.7 can be weakened in the case of a locally (soluble-by-finite) group, by imposing the weak normality only to finitely generated subgroups.

**Theorem 3.8** *Let* $G$ *be a locally (soluble-by-finite) group in which all finitely generated subgroups are weakly normal. Then* $G$ *belongs to* $\mathfrak{W}_1$*.*

Proof — Let $E$ be any finitely generated subgroup of $G$. Clearly, all subgroups of finite index of $E$ are finitely generated, and so they are weakly normal in $G$. It follows that each finite homomorphic image of $E$ has only weakly normal subgroups, and so it is a $\overline{T}$-group. Then $E$ is soluble, and so it has the T-property (see [32], Theorem 2). As the T-property is local, we obtain that $G$ itself is a soluble $\overline{T}$-group, and hence all its subgroups are weakly normal by Theorem 3.4. □

It follows from Theorem 3.4 that the class of locally graded $\mathfrak{W}_1$-groups coincides with that of soluble groups with the $\overline{T}$-property, and so it is a local class. Although it is an open question whether $\mathfrak{W}_1$ itself is a local class, we can prove that $\mathfrak{W}_1$ can be at least countably detected.

**Corollary 3.9** *The class* $\mathfrak{W}_1$ *is countably recognizable.*

Proof — Let $G$ be a group whose countable subgroups belong to the class $\mathfrak{W}_1$, and let $X$ be any countable subgroup of $G$. If $g$ is an element of $G$ such that $X^g$ is contained in $N_G(X)$, the subgroup $Y = \langle X, g\rangle$ is countable and $X^g \leqslant N_Y(X)$, so that $X^g = X$ because $X$ is weakly normal in $Y$. Therefore all countable subgroups of $G$ is weakly normal, and hence $G$ is a $\mathfrak{W}_1$-group by Corollary 3.7. □

A result of Rose states that if $X$ and $Y$ are pronormal subgroups of a group $G$ such that $X^Y = X$, then also the product $XY$ is pronormal in $G$ (see [34]). The following example shows that a corresponding result does not hold for weak normality.

Consider the standard wreath product $G = H \wr K$, where

$$H = \langle a, b \mid a^3 = b^2 = 1, ab = ba^{-1}\rangle$$

is isomorphic to the symmetric group $Sym(3)$ and $K = \langle c\rangle$ has order 2, and put $a' = a^c$ and $b' = b^c$. The subgroup $X = \langle aa', c\rangle$ is cyclic of order 6 and it is the Fitting subgroup of its normalizer

$$N_G(X) = \langle aa', bb', c\rangle.$$

If $g$ is any element of $G$ such that $X^g \leqslant N_G(X)$, then $|N_G(X) : X^g| = 2$ so that $X^g$ is a nilpotent normal subgroup of $N_G(X)$ and hence $X^g = X$. Therefore the subgroup $X$ is weakly normal in $G$. On the other hand, the product of X and the normal subgroup $N = \langle a, a' \rangle$ is not weakly normal, because it is subnormal but not normal in G. Notice also that the same example proves that homomorphic images of weakly normal subgroups need not be weakly normal.

In contrast to the above example, Sementovskiĭ proved that the subgroup generated by any collection of pronormal subgroups is weakly normal, although it is not always pronormal (see [4] or [36]). Our next lemma provides a condition for a subgroup generated by weakly normal subgroups to be weakly normal.

**Lemma 3.10** *Let* G *be a group, and let* X *be a subgroup of* G. *If* X *is generated by a set* $\mathcal{H}$ *of subgroups which are normal in* X *and weakly normal in* G, *then also* X *is weakly normal in* G.

Proof — Let H be any element of $\mathcal{H}$. Then H is normal in the normalizer $N_G(X)$, because it is normal in X and weakly normal in G, and hence $N_G(X) \leqslant N_G(H)$. If g is any element of G such that $X^g$ is contained in $N_G(X)$, for each element H of $\mathcal{H}$ we have

$$H^g \leqslant X^g \leqslant N_G(X) \leqslant N_G(H)$$

and so $H^g = H$. Since X is generated by the subgroups in $\mathcal{H}$, it follows that $X^g = X$ and hence X is weakly normal in G. □

It is interesting to notice that, when G is a finite group, in the above statement it is enough to require the normality only of the elements of $\mathcal{H}$ which do not have prime-power order (see [27]). It follows in particular that if X is a subgroup of a finite group G and all Sylow subgroups of X are weakly normal in G, then X itself is weakly normal in G. This fact can be alternatively obtained as a consequence of Sementovskiĭ's result, since every primary weakly normal subgroup of a finite group is pronormal.

**Lemma 3.11** *Let* G *be a group, and let* X *be a finite weakly normal* p*-subgroup of* G, *where* p *is a prime number. If the normal closure* $X^G$ *is locally finite, then* X *is pronormal in* G.

Proof — Let g be any element of G, and let P be a Sylow p-subgroup of $Y = \langle X, X^g \rangle$ containing X. As Y is finite, there is an element y of Y such that $X^g \leqslant P^y$, and of course the weakly normal subgroup $X^g$ is normal in $P^y$ by Lemma 3.2. Then

$$(X^g)^{g^{-1}y} = X^y \leqslant P^y \leqslant N_G(X^g),$$

and hence $X^y = (X^g)^{g^{-1}y} = X^g$. Therefore the subgroup $X$ is pronormal in $G$. □

**Corollary 3.12** *Let $G$ be a locally finite group whose cyclic subgroups are weakly normal. Then all cyclic subgroups of $G$ are pronormal.*

Proof — If $x$ is any element of $G$, it follows from Lemma 3.11 that each primary component of $\langle x\rangle$ is a pronormal subgroup of $G$, so that $\langle x\rangle$ itself is pronormal in $G$ (see [34]). □

It is known that if all cyclic subgroups of an FC-group are pronormal, then the group belongs to $\mathfrak{W}_0$ (see [19], Theorem 3.9), and so by Corollary 3.12 we have that every periodic FC-group whose cyclic subgroups are weakly normal is a $\mathfrak{W}_0$-group. With a little more work, it can be proved that this is actually true for arbitrary FC-groups.

**Corollary 3.13** *Let $G$ be an FC-group whose cyclic subgroups are weakly normal. Then $G$ belongs to $\mathfrak{W}_0$.*

Proof — Let $x$ be any element of $G$. As the factor group $G/Z(G)$ is periodic (see [37]), in order to prove that $\langle x\rangle$ is pronormal in $G$, we may replace $G$ by the factor group $G/\langle x\rangle \cap Z(G)$. Then $x$ has finite order and so the normal closure $\langle x\rangle^G$ is finite by Dietzmann's Lemma. As in the proof of Corollary 3.12 we have now that all cyclic subgroups of $G$ are pronormal, and hence $G$ belongs to $\mathfrak{W}_0$. □

Notice that the above statement shows in particular that within the universe of FC-groups the classes $\mathfrak{W}_0$ and $\mathfrak{W}_1$ coincide.

It is well known that any periodic linear group is locally finite, and so Corollary 3.12 and Theorem 2.7 yield that $\mathfrak{W}_0$ contains all periodic linear groups whose cyclic subgroups are weakly normal. On the other hand, the periodicity cannot be omitted in this case: in fact, every free group is linear and our next result shows that in a free group all cyclic subgroups are weakly normal, although by Theorem 3.4 a free non-abelian group must contain subgroups that are not weakly normal.

**Lemma 3.14** *Let $G$ be a free group. Then all cyclic subgroups of $G$ are weakly normal.*

Proof — Let $X$ be any cyclic non-trivial subgroup of $G$, and let $g$ be an element of $G$ such that $X^g \leqslant N_G(X)$. Then $N_G(X^g) = N_G(X)$, because the normalizer $N_G(X)$ is a maximal cyclic subgroup of G, so that $|N_G(X) : X| = |N_G(X) : X^g|$ and hence $X^g = X$. Therefore all cyclic subgroups of G are weakly normal. □

A direct combination of Theorem 3.4 and Theorem 2.7 gives the following interesting information.

**Corollary 3.15** *A linear group* G *is a* $\mathfrak{W}_0$*-group if and only if it belongs to the group class* $\mathfrak{W}_1$.

Lemma 3.10 can be applied to drop out the condition that the group G is locally (soluble-by-finite) from the statement of Theorem 3.8, at least when the subgroup X belongs to a suitable group class. Our next two corollaries illustrate this fact.

Recall first that a group G is said to be a PC-*group* if the factor group $G/C_G(\langle g\rangle^G)$ is polycyclic-by-finite for each element g of G. Groups with the PC-property were introduced in [10] as a natural generalization of FC-groups, and they are precisely those groups that can be covered by polycyclic-by-finite normal subgroups.

**Corollary 3.16** *Let* G *be a group, and let* X *be a subgroup of* G *whose finitely generated subgroups are weakly normal in* G*. If* X *is a* PC*-group, then it is weakly normal in* G.

Proof — Since X is covered by its polycyclic-by-finite normal subgroups, and each of them is weakly normal in G, it follows from Lemma 3.10 that X is weakly normal in G. □

**Corollary 3.17** *Let* G *be a group, and let* X *be a locally nilpotent subgroup of* G*. If all finitely generated subgroups of* X *are weakly normal in* G*, then* X *itself is weakly normal in* G.

Proof — Let E be any finitely generated subgroup of X. Then the subgroup $\langle E, x\rangle$ is nilpotent for each x in X, so that E is normal in $\langle E, x\rangle$ by Lemma 3.2, and hence even in X. It follows now from Lemma 3.10 that the subgroup X is weakly normal in G. □

Another direct consequence of Lemma 3.10 is the following result, that was already proved in [27] by a different argument; it should be seen in relation to the fact that homomorphic images of weakly normal subgroups are not in general weakly normal.

**Corollary 3.18** *Let* G *be a group, and let* X *be a weakly normal subgroup of* G*. If* N *is a normal subgroup of* G *such that* $X^N = X$*, then also the product* XN *is weakly normal in* G*.*

Our next theorem shows that groups with only weakly normal subgroups form an accessible class.

**Theorem 3.19** *The group class* $\mathfrak{W}_1$ *is accessible.*

Proof — Let G be an infinite locally graded group whose proper subgroups belong to $\mathfrak{W}_1$. Then all proper subgroups of G are metabelian, and so G itself is a metabelian group (see [8]). As the class of T-groups is accessible (see [13]), the group G has the $\overline{T}$-property, and hence all its subgroups are weakly normal by Theorem 3.4. □

The last result of this section deals with uncountable groups with many weakly normal subgroups.

**Theorem 3.20** *Let* G *be an uncountable locally graded group of cardinality* $\aleph$ *which has no simple homomorphic images of cardinality* $\aleph$*. If all proper subgroups of* G *of cardinality* $\aleph$ *belong to* $\mathfrak{W}_1$*, then* G *itself is a* $\mathfrak{W}_1$*-group.*

Proof — Every proper subgroup of G of cardinality $\aleph$ is a soluble $\overline{T}$-group by Theorem 3.4, and so it is metabelian. Assume for a contradiction that G has a simple non-abelian homomorphic image G/N. Then G/N has cardinality strictly smaller than $\aleph$, and so N has cardinality $\aleph$. In particular, the subgroup N is metabelian and hence G/N is locally graded (see [26]). Moreover, every proper subgroup of G/N has only weakly normal subgroups, so that all subgroups of G/N are weakly normal by Theorem 3.19 and hence G/N is soluble. This contradiction shows that G has no simple non-abelian homomorphic images, and so G is metabelian (see [14], Theorem D). Then G has the $\overline{T}$-property (see [13]), and a further application of Theorem 3.4 yields that all subgroups of G are weakly normal. □

The consideration of the locally dihedral 2-group shows that a statement similar to Theorem 3.20 does not hold in the countable case. Notice that the example by Ehrenfeucht and Faber mentioned in Section 2 proves also that there exists an uncountable nilpotent group of class 2 in which all uncountable subgroups are normal but every non-normal subgroup is not even weakly normal.

# 4 Weakly pronormal subgroups

The first statement of this section shows that, like pronormality and weak normality, also the property of being weakly pronormal can be countably recognized.

**Theorem 4.1** *Let* $G$ *be a group, and let* $X$ *be a subgroup of* $G$. *If all countable subgroups of* $X$ *are weakly pronormal in* $G$, *then* $X$ *itself is weakly pronormal in* $G$.

PROOF — Assume for a contradiction that $X^g$ is properly contained in $X$ for some element $g$ of $G$, and let $x$ be an element of $X \setminus X^g$. Clearly, $X^{g^n} \leqslant X$ for each non-negative integer $n$, so that

$$Y = \langle x, x^g, x^{g^2}, \dots, x^{g^n}, x^{g^{n+1}}, \dots \rangle$$

is a countable subgroup of $X$ and hence it is weakly pronormal in $G$. As $Y^g \leqslant Y$, it follows that $Y^g = Y$, which is impossible because $x$ belongs to $Y \setminus Y^g$. This contradiction proves the statement. □

The following example shows that the hypotheses of Theorem 4.1 cannot be weakened by assuming that all finitely generated subgroups of $X$ are weakly pronormal.

Let $p$ be any prime number, and let $G = \langle a \rangle \wr \langle x \rangle$ be the standard wreath product of a group $\langle a \rangle$ of order $p$ and an infinite cyclic group $\langle x \rangle$. The base group $B$ of $G$ can be written as a direct product

$$B = \operatorname*{Dr}_{n \in \mathbb{Z}} \langle a_n \rangle,$$

where $a_0 = a$ and $a_n^x = a_{n+1}$ for all $n$. Clearly, the subgroup

$$C = \langle a_n \mid n \geqslant 1 \rangle$$

properly contains $C^x$, and so it is not weakly pronormal in $G$, while all its finitely generated subgroups are finite and so trivially weakly pronormal. Of course, this example should be seen in relation to the similar situation pertaining pronormal subgroups.

Notice also that the same example can be used to show that the subgroup generated by two weakly pronormal subgroups $H$ and $K$ need not be weakly pronormal, even if $[H, K] = \{1\}$, in contrast to the behaviour of pronormal subgroups (see for instance [34], 1.8). To see this, put $F = \{n! \mid n \in \mathbb{N}\}$ and in the wreath product G constructed above consider the subgroups

$$H = \langle a_h \mid h \in F \rangle \quad \text{and} \quad K = \langle a_k \mid k \in \mathbb{N} \setminus F \rangle.$$

For each positive integer $m$, we have $a_1^{x^m} = a_{m+1}$ and $a_2^{x^m} = a_{m+2}$, and at least one of the positive integers $m+1$ and $m+2$ does not belong to F, so that $H^{x^m}$ cannot be contained in H. Moreover, if $n > 2$ is any integer such that $m < n! - (n-1)!$, we have

$$(n-1)! < n! - m < n!,$$

so that $k = n! - m$ is not in the set F and $a_k^{x^m} = a_{n!}$ does not belong to K. It follows that both subgroups H and K are weakly pronormal in G, altough their product $C = H \times K$ is not.

A useful extension property of weak pronormality is described by the following result.

**Lemma 4.2** *Let* G *be a group, and let* N *be a normal subgroup of* G*. If* X *is a subgroup of* G *such that* $X \cap N$ *and* XN *are weakly pronormal in* G*, then also* X *is weakly pronormal in* G*.*

Proof — Let $g$ be any element of G such that $X^g$ is contained in X. Since

$$(X \cap N)^g = X^g \cap N \leqslant X \cap N \quad \text{and} \quad (XN)^g = X^g N \leqslant XN,$$

it follows that $X^g \cap N = X \cap N$ and $X^g N = XN$. Therefore

$$X = X \cap X^g N = X^g (X \cap N) = X^g$$

and hence X is weakly pronormal in G. □

As an example of application of Lemma 4.2, we have the following sufficient condition for a subgroup to be weakly pronormal; it shows for instance that a subgroup intersecting trivially some term of the lower central series of the group is always weakly pronormal.

**Corollary 4.3** *Let* G *be a group, and let* X *be a subgroup of* G *such that* $X \cap \gamma_n(G)$ *is weakly pronormal in* G *for some positive integer* $n$*. Then* X *is weakly pronormal in* G*.*

Proof — Obviously, the factor group $G/\gamma_n(G)$ locally satisfies the maximal condition on subgroups, and so all its subgroups are weakly pronormal. In particular, $X\gamma_n(G)$ is a weakly pronormal subgroup of $G$, and hence $X$ is weakly pronormal in $G$ by Lemma 4.2. □

Notice that a result corresponding to Lemma 4.2 cannot be proved for pronormality or for weak normality. In fact, it is enough to observe that if $G$ is any soluble $T$-group for which the $\overline{T}$-property does not hold, it follows from Corollary 3.3 that $G$ contains a subgroup $X$ which is not weakly normal (and so neither pronormal), but $X \cap G'$ and $XG'$ are obviously normal in $G$.

Our next lemma shows in particular that if a subgroup $X$ of a group $G$ is weakly pronormal in some normal subgroup of finite index of $G$, then $X$ is also weakly pronormal in $G$.

**Lemma 4.4** *Let $G$ be a group, and let $N$ be a normal subgroup of finite index of $G$. If $X$ is a subgroup of $G$ such that $X \cap N$ is weakly pronormal in $N$, then $X$ is weakly pronormal in $G$.*

Proof — Put $Y = X \cap N$, and let $g$ be any element of $G$ such that $Y^g$ is contained in $Y$. As the index $|G : N|$ is finite, there exists a positive integer $m$ such that $g^m$ belongs to $N$, and obviously $Y^{g^m}$ is contained in $Y$. But $Y$ is weakly pronormal in $N$, so that $Y = Y^{g^m} \leqslant Y^g$ and hence $Y = Y^g$. Thus $Y$ is weakly pronormal in $G$. Moreover, the product $XN$ is a weakly pronormal subgroup of $G$ since the factor group $G/N$ is finite, and so $X$ is weakly pronormal in $G$ by Lemma 4.2. □

It was previously remarked that the normalizer of any pronormal subgroup is pronormal, and similarly all weakly normal subgroups have a weakly normal normalizer. The situation is different in the case of weakly pronormal subgroups, as the following example shows.

Let $\mathbb{Z}$ be the set of all integers in their natural order, and let $g$ be the order automorphism of $\mathbb{Z}$ mapping each $n$ to $n+1$. Consider now a group $A$ of prime order $p$, and the wreath power $W = \mathrm{Wr}A^{\mathbb{Z}}$. Of course, $g$ determines an automorphism of $A$ of infinite order, and we put $G = \langle g \rangle \ltimes A$. If $I_1$ is the set of all negative integers and $I_2$ that of positive integers, we have that $(I_1, \{0\}, I_2)$ is a segmentation of $\mathbb{Z}$, and hence

$$W = (W_1 \wr A_0) \wr W_2,$$

where

$$W_1 = WrA^{I_1}, \quad A_0 \simeq A \quad \text{and} \quad W_2 = WrA^{I_2}$$

(for more details on wreath powers and segmentations of ordered sets see for instance [31] Part 2, Chapter 6). Then $B_0 = A_0^{W_2}$ is a weakly pronormal subgroup of G, while its normalizer

$$N_G(B_0) = B_0 W_2$$

properly contains $(B_0W_2)^g$ and so it is not weakly pronormal in G.

We shall denote by $\mathfrak{W}_2$ the class of all groups in which all subgroups are weakly pronormal. Thus $\mathfrak{W}_2$ contains all periodic groups and all groups locally satisfying the maximal condition on subgroups; in particular, $\mathfrak{W}_1$ is a proper subclass of $\mathfrak{W}_2$.

It turns out that $\mathfrak{W}_2$ coincides with the class of the so-called HNN-*free* groups, i.e. groups which do not contain HNN-extensions (see [33]). Recall here that, given an isomorphism $\varphi : A \to B$ between two different subgroups A and B of a group G, and an infinite cyclic group $\langle t\rangle$, the factor group of the free product $G * \langle t\rangle$ by the normal closure of the subset

$$\left\{t^{-1}at\big(\varphi(a)\big)^{-1} \mid a \in A\right\}$$

is called the HNN-*extension* of G relative to A,B and $\varphi$. This concept plays a relevant role in several parts of group theory. It is obvious that the class of HNN-free groups is subgroup closed, and it has been proved in [33] that such class is local and so also countably recognizable. Our next result shows that the class of locally soluble $\mathfrak{W}_2$-groups coincides with that of locally polycyclic groups.

**Lemma 4.5** *A locally (soluble-by-finite) group* G *belongs to the class* $\mathfrak{W}_2$ *if and only if it is locally (polycyclic-by-finite).*

Proof — Since $\mathfrak{W}_2$ is subgroup closed, it can be assumed without loss of generality that G is finitely generated and so soluble-by-finite. Then every normal subgroup of G is finitely generated (see [33], Lemma 2), and hence G is polycyclic-by-finite. □

As a consequence of the above statement, any finitely generated metabelian group which is not polycyclic must contain a subgroup which is not weakly pronormal, and hence the class $\mathfrak{W}_2$ is not extension closed. On the other hand, it follows from Lemma 4.4 that $\mathfrak{W}_2$ is at least closed with respect to finite extensions.

**Corollary 4.6** *Let* G *be a group containing a* $\mathfrak{W}_2$*-subgroup of finite index. Then* G *belongs to the group class* $\mathfrak{W}_2$*.*

Proof — Of course, the group G contains a normal subgroup N which belongs to $\mathfrak{W}_2$. If X is any subgroup of G, the intersection $X \cap N$ is weakly pronormal in N and hence X is weakly pronormal in G by Lemma 4.4. □

We can now prove that the class $\mathfrak{W}_2$ has no opponents. It is interesting to notice that in this case the restriction to the universe of locally graded groups is unnecessary, a phenomenon that rarely occurs.

**Theorem 4.7** *Let* G *be a group whose proper subgroups belong to* $\mathfrak{W}_2$*. Then* G *is a* $\mathfrak{W}_2$*-group.*

Proof — Assume for a contradiction that the statement is false, so that G must be finitely generated since the group class $\mathfrak{W}_2$ is local. Let X be a subgroup of G which is not weakly pronormal, and let g be an element of G such that $X^g$ is properly contained in X. Then

$$\ldots < X^{g^n} < \ldots < X^g < X < X^{g^{-1}} < \ldots < X^{g^{-n}} < \ldots$$

and so

$$Y = \bigcup_{n \in \mathbb{Z}} X^{g^n}$$

is a proper subgroup of G which is normalized by g. Clearly, X is not weakly pronormal in $\langle X, g \rangle$, so that $\langle X, g \rangle = G$ and hence $G = \langle Y, g \rangle$. It follows that Y is normal in G, and so G has a finite non-trivial homomorphic image. Therefore all subgroups of G are weakly pronormal by Corollary 4.6, and this contradiction completes the proof. □

**Corollary 4.8** *The group class* $\mathfrak{W}_2$ *is accessible.*

If G is a countable group whose (countably) infinite proper subgroups belong to $\mathfrak{W}_2$, we have obviously that all proper subgroups of G are in $\mathfrak{W}_2$ and hence all subgroups of G are weakly pronormal by Theorem 4.7. A similar result holds for uncountable groups, provided that the group has no large simple homomorphic images.

**Theorem 4.9** *Let* G *be an uncountable group of cardinality* $\aleph$ *which has no simple homomorphic images of cardinality* $\aleph$*. If all proper subgroups of* G *of cardinality* $\aleph$ *belong to* $\mathfrak{W}_2$*, then* G *itself is a* $\mathfrak{W}_2$*-group.*

Proof — By Corollary 4.6 it can be assumed that G has no proper subgroups of finite index. Suppose first that G contains a proper normal subgroup N of cardinality $\aleph$, and let X be any subgroup of G. Put $Y = X \cap N$ and assume for a contradiction that $Y^g$ is properly contained in Y for some element g of G. Clearly, Y is not weakly pronormal in $\langle N, g \rangle$, and hence $\langle N, g \rangle = G$, which is impossible as G has no proper subgroups of finite index. This contradiction shows that Y is weakly pronormal in G. On the other hand, every proper subgroup of G/N belongs to $\mathfrak{W}_2$, and hence all subgroups of G/N are weakly pronormal by Theorem 4.7. In particular, the product XN is a weakly pronormal subgroup of G, and so X is weakly pronormal in G by Lemma 4.2.

Suppose now that all proper normal subgroups of G have cardinality strictly smaller than $\aleph$ so that, since G has no simple homomorphic images of cardinality $\aleph$, each of them is contained in a proper subgroup of cardinality $\aleph$ (see [12], Corollary 2.6). Moreover, G cannot have maximal normal subgroups and so it follows from Zorn's Lemma that it is the union of a chain of proper normal subgroups. Thus every finitely generated subgroup E of G is contained in a proper normal subgroup of G, and so also in a proper subgroup of cardinality $\aleph$; in particular, E has only weakly pronormal subgroups. As the class $\mathfrak{W}_2$ is local, it follows that G itself belongs to $\mathfrak{W}_2$. □

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