Advances in Group Theory and Applications

9 (2020), pp. 71–121
ISSN: 2499-1287
DOI: 10.32037/agta-2020-004

Applying Group Theory Philosophy to Leibniz Algebras: Some New Developments

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(*Received Mar. 30, 2020; Accepted Apr. 15, 2020 — Communicated by F. de Giovanni*)

Abstract

This survey is an attempt of describing of main contours of a recently developing general theory of Leibniz Algebras. This theory based on the employing of methods and approaches which are proved to be exceedingly effective in infinite group theory. The survey addresses a number of natural problems that quite often have analogs in other disciplines, discusses the parts of the theory that have been already developed, and argues which parts of it should be developed further.

Mathematics Subject Classification (2010): 17A32, 17A60

Keywords: Leibniz algebra; Lie algebra; ideal; finitely dimensional Leibniz algebra; locally nilpotent and nilpotent Leibniz algebra

1 Introduction

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *Leibniz algebra* (more precisely a *left Leibniz algebra*) if for all $a, b, c \in L$ it satisfies the Leibniz identity

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]].$$

We will also use another form of this identity:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

Leibniz algebras appeared first in the papers of A.M. Bloh [14, 15, 16], in which they were called the D-*algebras*. However, in that time these works were not in demand. Only after two decades, a real interest to Leibniz algebras rose. It happened thanks to the work of J.-L. Loday [51] (see also [52, Section 10.6]), who "rediscovered" these algebras and used the term *Leibniz algebras* since it was Leibniz who discovered and proved the Leibniz rule for differentiation of functions. The main motivation for the introduction of Leibniz algebras was the study of periodicity phenomena in algebraic K-theory. The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. They found some applications in physics (see, for example, [4, 17, 29, 30]). Nowadays the theory of Leibniz algebras is one of actively developing areas of modern algebra.

Note that the Lie algebras are the partial case of Leibniz algebras. Indeed, if L is a Lie algebra, then

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0.$$

It follows that

$$\begin{aligned}[[a,b],c] &= -[[b,c],a]-[[c,a],b]=[a,[b,c]]+[b,[c,a]]\\ &= [a,[b,c]]-[b,[a,c]],\end{aligned}$$

which shows that every Lie algebra is a Leibniz algebra.

Conversely, suppose that $[a,a]=0$ for each element $a\in L$. Then for arbitrary elements $a,b\in L$ we have

$$\begin{aligned}0=[a+b,a+b] &= [a,a]+[a,b]+[b,a]+[b,b]\\ &= [a,b]+[b,a].\end{aligned}$$

It follows that $[a,b]=-[b,a]$. Then we obtain

$$\begin{aligned}0 &= [[a,b],c]-[a,[b,c]]+[b,[a,c]]\\ &= [[a,b],c]+[[b,c],a]-[[a,c],b]\\ &= [[a,b],c]+[[b,c],a]+[[c,a],b]\end{aligned}$$

for all $a,b,c\in L$. Thus, Lie algebras can be characterized as Leibniz algebras with the identity $[a,a]=0$ for every element $a\in L$. The last equality implies that $[a, b] = -[b, a]$, therefore we can define the Lie algebras as anticommutative Leibniz algebras.

The theory of Leibniz algebras has been developing quite intensively, but this development is uneven and fragmented. This state of theory of Leibniz algebras is fully reflected in a recent book [7]. Most of the results showing the structural features of Leibniz algebras were obtained for finite-dimensional algebras, and many of them over fields of characteristic zero. In some articles, the authors do not even stipulate that they consider finite-dimensional Leibniz algebras, this can only be seen by looking at the proofs. A number of these results are analogues of the corresponding theorems from the theory of Lie algebras. The specifics of Leibniz algebras, the features that distinguish them from Lie algebras, can be seen from the description of Leibniz algebras of small dimensions. But this description also concerns algebras over fields of characteristic zero. And here parallels with the theory of groups are immediately striking, precisely with its period when the theory of finite groups was already quite developed, and the theory of infinite groups only arose, i.e. with the time when the formation of the general theory of groups took place. Therefore, the idea of using this experience naturally arises. It is clear that we cannot talk about some kind of similarity of results, we can talk about approaches and problems, we can talk about application of group theory philosophy. Moreover, any theory has a number of natural problems that arise in the process of its development, and these problems quite often have analogues in other disciplines. In this review, we want to focus on such issues, our goal is to see which parts of the picture involving the general structure of Leibniz algebras have already been drawn, and this will allow us to see which parts of this picture should be developed further, and show significant differences of Leibniz algebras from Lie algebras.

2 Some specific subalgebras of Leibniz algebras

The lack of anticommutativity immediately brings its own specifics. An algebra R over a field F is called *right Leibniz* if it satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c] - [[a, c], b]$$

for all $a, b, c \in R$. Note at once that the classes of left Leibniz algebras and right Leibniz algebras are different. The following simple example justifies it. Let F be an arbitrary field and L be a vector space over F having a basis $\{a, b\}$. Define the operation $[\cdot, \cdot]$ on L by the following rules:

$$[a, a] = [a, b] = b, \ [b, a] = [b, b] = 0.$$

It is not hard to check that L becomes a left Leibniz algebra. But

$$0 = [[a, a], a] \neq [[a, a], a] + [a, [a, a]] = [a, b] = b.$$

Let R be a right Leibniz algebra, then put $[\![a, b]\!] = [b, a]$. Then we have

$$\begin{aligned} [\![[\![a, b]\!], c]\!] &= [c, [b, a]] = [[c, b], a] - [[c, a], b] \\ &= [\![a, [\![b, c]\!]]\!] - [\![b, [\![a, c]\!]]\!]. \end{aligned}$$

Thus, this substitution leads us to a left Leibniz algebra. Similarly, we can make a transfer from a left Leibniz algebra to a right Leibniz algebra.

An algebra L over a field F is called a *symmetric Leibniz algebra* if it is both a left and right Leibniz algebra.

We prefer to work with left Leibniz algebras even though many authors prefer to consider right Leibniz algebras. The choice of left Leibniz algebras is more suitable for us because they have more visible relationships with the differentiation of products (in which the differential operator is written to the right of a differentiable object). Thus, in this article, the term a Leibniz algebra stands for a left Leibniz algebra.

Note the following useful property of the elements of Leibniz algebras. We have

$$\begin{aligned} [a, [b, c]] &= [[a, b], c] + [b, [a, c]], \\ [b, [a, c]] &= [[b, a], c] + [a, [b, c]] \end{aligned}$$

or

$$[a, [b, c]] = [b, [a, c]] - [[b, a], c].$$

It follows that

$$[[a, b], c] + [b, [a, c]] = [b, [a, c]] - [[b, a], c],$$

and hence

$$[[a, b], c] = -[[b, a], c].$$

A Leibniz algebra L is called *abelian* (or *trivial*) if $[a, b] = 0$ for every elements $a, b \in L$. In particular, an abelian Leibniz algebra is a Lie algebra.

Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by all elements $[a, b]$ where $a \in A$, $b \in B$. As usual, a subspace A of L is called a *subalgebra* of L, if $[x, y] \in A$ for every $x, y \in A$. It follows that $[A, A] \leqslant A$.

Let L be a Leibniz algebra over a field F, M be non-empty subset of L, then $\langle M \rangle$ denote the subalgebra of L generated by M.

A subalgebra A is called a *left* (respectively *right*) *ideal* of L, if $[y, x] \in A$ (respectively $[x, y] \in A$) for every $x \in A$, $y \in L$. In other words, if A is a left (respectively right) ideal, then $[L, A] \leqslant A$ (respectively $[A, L] \leqslant A$).

A subalgebra A is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left and right ideal, that is $[x, y], [y, x] \in A$ for every $x \in A$, $y \in L$.

If A is an ideal of L, we can consider the factor-algebra L/A. It is not hard to see that this factor-algebra also is a Leibniz algebra.

Every Leibniz algebra L possesses the following specific ideal. Denote by $\mathrm{Leib}(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. We note that $\mathrm{Leib}(L)$ is an ideal of L. Indeed, for arbitrary elements $a, x \in L$ we have

$$[a, [a, x]] = [[a, a], x] + [a, [a, x]],$$

so $[[a, a], x] = 0$. Furthermore,

$$\begin{aligned}[x + [a, a], x + [a, a]] &= [x, x] + [x, [a, a]] + [[a, a], x] + [[a, a], [a, a]] \\ &= [x, x] + [x, [a, a]].\end{aligned}$$

It follows that $[x, [a, a]] = [x + [a, a], x + [a, a]] - [x, x] \in \mathrm{Leib}(L)$.

Put $K = \mathrm{Leib}(L)$. Then in the factor-algebra L/K we have

$$[a + K, a + K] = [a, a] + K = K$$

for each element $a \in L$. By mentioned above we obtain that L/K is a Lie algebra. Conversely, suppose that H is an ideal of L such that L/H is a Lie algebra. Then $H = [a + H, a + H] = [a, a] + H$, which implies that $[a, a] \in H$ for every element $a \in L$. Then $\mathrm{Leib}(L) \leqslant H$.

The ideal $\mathrm{Leib}(L)$ is called the *Leibniz kernel* of algebra L. We note the following important property of the Leibniz kernel:

$$[[a, a], x] = [a, [a, x]] - [a, [a, x]] = 0.$$

This property shows that $\mathrm{Leib}(L)$ is an abelian subalgebra of L.

Let L be a Leibniz algebra. Define the *lower central series* of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \gamma_\delta(L)$$

by the following rules: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively

$$\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$$

for all ordinals α and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for limit ordinals λ. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L, and we have

$$\gamma_\delta(L) = [L, \gamma_\delta(L)].$$

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots, L] \ldots]]$ is the left normed commutator of k copies of L. Note the following useful properties of subalgebras and ideals.

As usually, we say that a Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote the nilpotency class of L by $\mathrm{ncl}(L)$.

Note some properties of subalgebras and ideals.

Proposition 2.1 *Let L be a Leibniz algebra over a field F.*

(i) *If H is an ideal of L, then $[H, H]$ is an ideal of L.*

(ii) *If H is an ideal of L, then $[L, H]$ is a subalgebra of L.*

(iii) *If H is an ideal of L, then $[H, L]$ is a subalgebra of L.*

(iv) *If H is an ideal of L, then $[L, H] + [H, L]$ is an ideal of L.*

(v) *If H is an ideal of L, then $[\gamma_j(H), \gamma_k(H)] \leqslant \gamma_{j+k}(H)$ for every positive integers j, k.*

(vi) *If H is an ideal of L, then $\gamma_j(H)$ is an ideal of L for each positive integer j. In particular, $\gamma_j(L)$ is an ideal of L for each positive integer j.*

(vii) *If* H *is an ideal of* L*, then* $\gamma_j(\gamma_k(H)) \leqslant \gamma_{jk}(H)$ *for every positive integers* j, k.

We remark that if A, B are ideals of a Leibniz algebra L, then, in general, $[A, B]$ needs not be an ideal. The following example justifies it (see [12]).

Example 2.2 Let L be a vector space over a field F with basis

$$\{e_1, e_2, e_3, e_4, e_5\}.$$

Define the operation on basis vectors by the following rule:

$$\begin{gathered}
[e_j, e_j] = 0, 1 \leqslant j \leqslant 4, \\
[e_1, e_j] = e_j \ \text{ for } j \in \{2, 4, 5\}, [e_1, e_3] = 0, \\
[e_2, e_1] = -e_2, [e_2, e_3] = e_4, [e_2, e_4] = [e_2, e_5] = 0, \\
[e_3, e_j] = 0 \ \text{ for } j \neq 2, [e_3, e_2] = e_5, \\
[e_4, e_j] = 0 \ \text{ for } j \neq 1, [e_4, e_1] = e_5, \\
[e_5, e_j] = 0 \ \text{ for } j \neq 1, [e_5, e_1] = -e_5.
\end{gathered}$$

Let

$$A = Fe_2 + Fe_4 + Fe_5 \quad \text{and} \quad B = Fe_3 + Fe_4 + Fe_5.$$

It is not hard to check that A, B are ideals of L. However, $[A, B] = Fe_4$ is not an ideal.

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{\text{left}}(L) = \{x \in L |\ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{\text{right}}(L) = \{x \in L |\ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $\text{Leib}(L) \leqslant \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is an subalgebra of L, and in general, the left and right centers are different. Moreover, they even may have different dimensions. We will construct now the following examples [41].

Example 2.3 Let F be a field. Put $L = Fe_1 \oplus Fe_2 \oplus Fe_3 \oplus Fe_4$ and define an operation $[\cdot,\cdot]$ by the following rules:

$$\begin{array}{llll} [e_1, e_1] = e_2, & [e_1, e_2] = -e_2 - e_3, & [e_1, e_3] = e_2 + e_3, & [e_1, e_4] = 0, \\ [e_2, e_1] = 0, & [e_3, e_1] = 0, & [e_4, e_1] = e_2 + e_3, & [e_j, e_k] = 0 \end{array}$$

for all $j, k \in \{2, 3, 4\}$. It is possible to check that this operation defines a Leibniz algebra. We can see that $\zeta^{\mathrm{right}}(L) = Fe_4$ and $\zeta^{\mathrm{right}}(L)$ is not an ideal. Furthermore, $\zeta^{\mathrm{left}}(L) = Fe_2 \oplus Fe_3$, so that

$$\zeta^{\mathrm{right}}(L) \cap \zeta^{\mathrm{left}}(L) = \langle 0 \rangle.$$

Moreover,

$$\dim_F(\zeta^{\mathrm{right}}(L)) = 1 < 2 = \dim_F(\zeta^{\mathrm{left}}(L)).$$

Note also that $[L, L] = \mathrm{Leib}(L) = \zeta^{\mathrm{left}}(L)$.

Example 2.4 Let F be a field. Put $L = Fe_1 \oplus Fe_2 \oplus Z$ where a subspace Z has a countable basis $\{z_n \mid n \in \mathbb{N}\}$. Put $[z_n, x] = 0$ for every $x \in L$ and

$$[e_1, e_1] = [e_2, e_2] = [e_1, e_2] = [e_2, e_1] = z_1, [e_1, z_1] = [e_2, z_1] = 0.$$

By such definitions, we have

$$0 = [[e_j, e_k], e_m] \text{ and } [e_j, [e_k, e_m]] - [e_k, [e_j, e_m]] = 0 - 0 = 0$$

for all $j, k, m \in \{1, 2\}$. Take into account the equalities

$$\begin{aligned} 0 &= [[e_1, e_2], z] = [e_1, [e_2, z]] - [e_2, [e_1, z]], \\ 0 &= [[e_2, e_1], z] = [e_2, [e_1, z]] - [e_1, [e_2, z]], \end{aligned}$$

we obtain $[e_2, [e_1, z]] - [e_1, [e_2, z]] = 0$ for every $z \in Z$. Now we put

$$[e_1, z_j] = z_j, [e_2, z_j] = z_{j+1}$$

for all $j > 1$. By this definition, we have

$$\begin{aligned} 0 = [[e_j, z], e_k] &\text{ and } [e_j, [z, e_k]] - [z, [e_j, e_k]] = [e_j, 0] - 0 = 0, \\ 0 = [[z, e_j], e_k] &\text{ and } [z, [e_j, e_k]] - [e_j, [z, e_k]] = 0 - [e_j, 0] = 0 \end{aligned}$$

for all $j, k \in \{1, 2\}$ and $z \in Z$. As we have seen above

$$[[e_j, e_k], z] = [e_j, [e_k, z]] - [e_k, [e_j, z]]$$

for all $j, k \in \{1, 2\}$ and $z \in Z$. Hence, L is a Leibniz algebra. By it construction Z is a left center of L, the right center coincides with the center of L and coincides with Fz_1, so that, the left center has finite codimension (and therefore, infinite dimension) and the right center and the center have finite dimension. By the construction, $[L, L] = Z$. Furthermore

$$[e_1 + z_1, e_1 + z_1] = [e_1, e_1] + [z_1, e_1] + [e_1, z_1] + [z_1, z_1] = z_1,$$
$$[e_1 + z_j, e_1 + z_j] = [e_1, e_1] + [z_j, e_1] + [e_1, z_j] + [z_j, z_j] = z_1 + z_j$$

for $j > 1$. It follows that $\mathrm{Leib}(L) = Z$.

The *center* $\zeta(L)$ of L is defined as

$$\zeta(L) = \{x \in L | \ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

3 Leibniz algebras of small dimensions

As usual, we say that a Leibniz algebra L is *finite dimensional*, if the dimension L as a vector space over F is finite. The condition "to be finite dimensional" is very strong. That is why the majority of results on Leibniz algebras were obtained for finite dimensional Leibniz algebras.

If $\dim_F(L) = 1$, then $L = Fa$ for some element $a \in L$. Then $[a, a] = \alpha a$ where $\alpha \in F$. We have $0 = [[a, a], a] = [\alpha a, a] = \alpha[a, a] = \alpha^2 a$. It follows that $\alpha = 0$, that is, $[a, a] = 0$, and L is abelian.

Suppose now that $\dim_F(L) = 2$ and L is not a Lie algebra. It follows that $K = \mathrm{Leib}(L)$ is non-zero. Since K is abelian, $K \neq L$. Hence there exists an element $a \in L$ such that $b = [a, a] \neq 0$. By this choice, $a \notin K$. Then $L = Fa + Fb$, and we have $[b, a] = 0$. The fact that K is an ideal of L implies that $[a, b] = \beta b$ for some $\beta \in F$. Suppose that $\beta \neq 0$ and put $c = \beta^{-1}a$. Then $[c, b] = \beta^{-1}[a, b] = \beta^{-1}\beta b = b$. We have

$$[c, c] = \beta^{-2}[a, a] = \beta^{-2}b = d,$$

and

$$[c, d] = [c, \beta^{-2}b] = \beta^{-2}[c, b] = \beta^{-2}b = d.$$

By this choice, $\{c, d\}$ is a basis of L.

Thus, we obtain the following two non-isomorphic algebras:

$$L_1 = Fa + Fb, [a, a] = b, [b, a] = [a, b] = [b, b] = 0,$$

and

$$L_2 = Fc + Fd, [c, c] = [c, d] = d, [d, c] = [d, d] = 0.$$

We consider now the structure of 3-dimensional Leibniz algebras. The description of Leibniz algebras over a field of complex numbers of dimension 3 was obtained in the papers [6, 19]. The description of right Leibniz algebras over a field of odd characteristic having dimension 3, was obtained in the paper [59]. The description of Leibniz algebras over an arbitrary finite field of dimension 3 one can finds in the paper [68]. The description given in the paper [59] is contained in [7]. Therefore, we present here the results of [68], especially since many of them can be extended to infinite fields.

Further writing $L = A \oplus B$ means that L is a direct sum of the subspaces A and B or the subalgebras A and B. If $L = A + B$ and A is an ideal of L and B is a subalgebra of L, then we will say that L is a *semidirect sum* of A and B and use the following symbol $L = A \dashv B$.

Let now L be a Leibniz algebra of dimension 3 over a finite field F and $\{a, b, c\}$ be a basis of L. The results of paper [68] shows that L is an algebra of one of the following types.

- $S_1 = \langle a \rangle$, $[a, a] = b$, $[a, b] = c$, $[c, a] = [a, c] = [c, b] = [b, c] = [b, b] = [c, c] = 0$, $\mathrm{Leib}(L) = \zeta^{\mathrm{left}}(L) = [L, L] = Fb \oplus Fc$, $\zeta^{\mathrm{right}}(L) = \zeta(L) = \gamma_3(L) = Fc$. In particular, L is a nilpotent cyclic Leibniz algebra.

- $S_2 = A \oplus B$ where A, B are the ideals, $B = Fb$, $[b, b] = 0$, $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a, a] = c$, $[c, a] = [a, c] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = Fc$, $\zeta^{\mathrm{left}}(L) = \zeta^{\mathrm{right}}(L) = \zeta(L) = Fb \oplus Fc$.

- $S_3 = A \dashv B$ where $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a,a] = c$, $[c,a] = [a,c] = 0$, $B = Fb$, $[b,b] = 0$, and $[a,b] = c$, $[b,a] = [b,c] = [c,b] = 0$. Moreover, $\mathrm{Leib}(L) = [L,L] = \zeta^{\mathrm{right}}(L) = \zeta(L) = Fc$, $\zeta^{\mathrm{left}}(L) = Fb \oplus Fc$.

- $S_4 = A \dashv B$ where $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a,a] = c$, $[c,a] = [a,c] = 0$, $B = Fb$, $[b,b] = 0$, and $[a,b] = c$, $[b,a] = \gamma c$, $\gamma \neq 0$, $[b,c] = [c,b] = 0$. Moreover, $\mathrm{Leib}(L) = [L,L] = \zeta^{\mathrm{left}}(L) = \zeta^{\mathrm{right}}(L) = \zeta(L) = Fc$.

- $S_5 = A + B$ where A, B are the ideals, $A = \langle a \rangle$, $B = \langle b \rangle$, $A \cap B = \zeta(L) = Fc$, $[a,a] = [b,b] = c$, $[c,a] = [a,c] = [c,b] = [b,c] = [a,b] = [b,a] = 0$. Moreover, $\mathrm{Leib}(L) = [L,L] = \zeta^{\mathrm{left}}(L) = \zeta^{\mathrm{right}}(L) = \zeta(L) = Fc$, $\mathrm{char}(F) \neq 2$, and equation $x^2 + 1 = 0$ has no solutions in F.

For example, if $F = F_5, F_{13}$, equation $y^2 + 1 = 0$ has a solution, if $F = F_3, F_7, F_{11}$, equation $y^2 + 1 = 0$ has no solutions.

Thus we can see that algebras could have the same defining relation, but different properties.

- $S_6 = A + B$ where A, B are the nilpotent ideals, $A = \langle a \rangle$, $B = \langle b \rangle$, $A \cap B = \zeta(L) = Fc$, $[a,a] = c$, $[b,b] = \rho c$, where ρ is a primitive root of identity of degree $|F| - 1$, $[c,a] = [a,c] = [c,b] = [b,c] = [a,b] = [b,a] = 0$. Moreover, $\mathrm{Leib}(L) = [L,L] = \zeta^{\mathrm{left}}(L) = \zeta^{\mathrm{right}}(L) = \zeta(L) = Fc$, $\mathrm{char}(F) \neq 2$.

- $S_7 = A + B$ where A, B are the ideals, $A = \langle a \rangle$, $B = \langle b \rangle$, $A \cap B = \zeta(L) = Fc$, $[a,a] = c = [a,b]$, $[b,b] = \eta c$, $[c,a] = [a,c] = [c,b] = [b,c] = [b,a] = 0$. Moreover, $\mathrm{Leib}(L) = [L,L] = \zeta^{\mathrm{left}}(L) = \zeta^{\mathrm{right}}(L) = \zeta(L) = Fc$ and polynomial $x^2 + x + \eta$ has no roots in field F.

As we can see, in this last case the properties of the algebra depend on whether the polynomial given above has roots in the field F, and this also depends on the choice of the element η. The difference is already evident even over the fields of the same characteristic. So if $F = F_2$, then for η there is only one value $\eta = 1$. But equation

$$y^2 + y + 1 = 0$$

has no solution in field $F = F_2$. Hence the respectively algebra has no element $d \notin Fc$ such that $[d,d] = 0$. It follows that every subalgebra of L is an ideal. If $F = F_4$ and $\eta_1 = 1$, then equation $y^2 + y + 1 = 0$ has solutions in field $F = F_4$. In this case, L has one dimensional subalgebra, which is no ideal.

In all of the above cases the Leibniz algebras are nilpotent. All of the following Leibniz algebras are not nilpotent.

- $S_8 = A \oplus B$ where A, B are the ideals, $B = Fb$, $[b, b] = 0$, A is a cyclic subalgebra, $A = Fa \oplus Fc$, where $[a, a] = c = [a, c]$, $[c, a] = [c, b] = [b, c] = [a, b] = [b, a] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = Fc$, $\zeta^{\mathrm{left}}(L) = Fb \oplus Fc$, $\zeta^{\mathrm{right}}(L) = \zeta(L) = Fb$.

- $S_9 = A \dashv B$ where $B = Fb$, $[b, b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a, a] = c = [a, c]$, $[a, b] = c$, $[c, a] = [c, b] = [b, c] = [b, a] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = Fc$, $\zeta^{\mathrm{left}}(L) = Fb \oplus Fc$, $\zeta(L) = \zeta^{\mathrm{right}}(L) = \langle 0 \rangle$.

- $S_{10} = A \dashv B$ where $B = Fb$, $[b, b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a, a] = c = [a, c]$, $[b, a] = [b, c] = c$, $[c, a] = [c, b] = [a, b] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = Fc$, $\zeta^{\mathrm{right}}(L) = Fb$, $\zeta(L) = \langle 0 \rangle$.

- $S_{11} = A \dashv B$ where $B = Fb$, $[b, b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a, a] = c = [a, c]$, $[a, b] = a = -[b, a]$, $[b, c] = -2c$, $[c, a] = [c, b] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = Fc$, $\zeta^{\mathrm{right}}(L) = \zeta(L) = \langle 0 \rangle$.

- $S_{12} = A \dashv B$ where $B = Fb$, $[b, b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a, a] = c$, $[a, c] = 0$, $[a, b] = a + \gamma c$, $\gamma \in F$, $[b, a] = -a + \gamma c$, $[b, c] = -2c$, $[c, a] = [c, b] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = Fc$, $\zeta^{\mathrm{right}}(L) = \zeta(L) = \langle 0 \rangle$ whenever $\mathrm{char}(F) \neq 2$ and $\zeta^{\mathrm{right}}(L) = \zeta(L) = Fc$ whenever $\mathrm{char}(F) = 2$.

- $S_{13} = D \dashv A$ where $D = Fd$, $[d, d] = 0$, $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a, a] = c$, $[a, c] = 0$, $[a, d] = \delta d$, $0 \neq \delta \in F$, $[c, a] = [c, d] = [d, c] = [d, a] = 0$, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = Fd \oplus Fc$, $\zeta(L) = \zeta^{\mathrm{right}}(L) = Fc$.

- $S_{14} = D \dashv B$ where $B = Fb$, $[b, b] = 0$, $D = Fd \oplus Fc$ is an abelian subalgebra, $[d, d] = [d, c] = [c, d] = [c, c] = 0$, $[b, c] = d$, $[b, d] = \gamma d + \delta d$, $0 \neq \gamma, \delta \in F$, $[c, b] = [d, b] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = Fd \oplus Fc$, $\zeta^{\mathrm{right}}(L) = Fb$, $\zeta(L) = \langle 0 \rangle$.

- $S_{15} = A \dashv D$ where $D = Fd$, $[d, d] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a, a] = c = [a, c]$, $[a, d] = d$, $[c, a] = [c, d] = [d, c] =$

$[d, a] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = Fd \oplus Fc$, $\zeta(L) = \zeta^{\mathrm{right}}(L) = \langle 0 \rangle$.

- $\mathrm{char}(F) \neq 2$, $S_{16} = A \dashv D$ where $D = Fd$, $[d, d] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a, a] = c = [a, c]$, $[a, d] = c + 2d$, $[c, a] = [c, d] = [d, c] = [d, a] = 0$. Moreover, $\mathrm{Leib}(L) = [L, L] = \zeta^{\mathrm{left}}(L) = Fd \oplus Fc$, $\zeta(L) = \zeta^{\mathrm{right}}(L) = \langle 0 \rangle$.

Investigation of Leibniz algebras of dimensions 4 over the fields of characteristic 0 has been conducted in the papers [1, 2, 5, 6, 18, 26, 27, 28, 61, 35]; see also the book [7].

4 The structure of cyclic Leibniz algebras

One of the first questions that naturally arise in the study of any algebraic structure is the question of the structure of its cyclic substructures (that is, substructures generated by one element). In particular, for a Leibniz algebra the question of the structure of its cyclic subalgebras naturally arises. Unlike Lie algebras, associative algebras, groups, etc., cyclic Leibniz algebras is no necessarily abelian.

The structure of cyclic Leibniz algebras has been described in [23]. Now we consider the main results of this paper.

Let L be a Leibniz algebra over a field F and d be an element of L. Put $\ln_1(d) = d, \ln_2(d) = [d, d], \ln_{k+1}(d) = [d, \ln_k(d)], k \in \mathbb{N}$. These elements are called the *left normed commutators of the element* d.

Proposition 4.1 *Let L be a Leibniz algebra over a field F, $d \in L$. Then every non-zero product of k copies of an element d with any bracketing is coincides with $\ln_k(d)$. Hence a cyclic subalgebra $\langle d \rangle$ is generated as a subspace by the elements $\ln_k(d)$, $k \in \mathbb{N}$.*

The following two natural cases appear here.

The elements $d_j = \ln_j(d)$, $j \in \mathbb{N}$ are linearly independent. In this case, a subalgebra $D = \langle d \rangle$ has the lower central series

$$D = \gamma_1(D) \geqslant \gamma_2(D) \geqslant \ldots \geqslant \gamma_j(D) \geqslant \gamma_{j+1}(D) \geqslant \ldots \langle 0 \rangle$$

of length ω, and $\gamma_j(D) = \bigoplus_{t \geqslant j} Fd_t$, $j \in \mathbb{N}$. In this case, we will say that the element d has *infinite depth*.

Consider now the second possibility, namely when the elements $d_j = \ln_j(d)$, $j \in \mathbb{N}$, are not linearly independent. In this case, we have the following result.

Proposition 4.2 *Let* L *be a Leibniz algebra over a field* F, $d \in L$, $D = \langle d \rangle$. *If there exists a positive integer* k *such that*

$$\ln_{k+1}(d) \in F\ln_1(d) + \ldots + F\ln_k(d),$$

then $D = F\ln_1(d) + \ldots + F\ln_k(d)$.

In particular, in this case, the subalgebra $D = \langle d \rangle$ has finite dimension over F, and we will say that an element d has *finite depth*. Let k be the least positive integer such that $\ln_1(d), \ldots, \ln_k(d)$ are linearly independent, but the elements $\ln_1(d), \ldots, \ln_k(d), \ln_{k+1}(d)$ are not linearly independent. Then the subset $\{\ln_1(d), \ldots, \ln_k(d)\}$ is a basis of D and $\dim_F(D) = k$. In this case, we can say that element d has *depth* k.

The case when an element d has finite depth turned out to be much more diverse. The following theorem from paper [23] has described this case.

Theorem 4.3 *Let* L *be a Leibniz algebra over a field* F, $d \in L$, $D = \langle d \rangle$. *Suppose that an element* d *has finite depth. Then* D *is an algebra of one of the following types:*

(i) $D = Fd$ *is abelian,* $[d, d] = 0$;

(ii) *there is a positive integer* k *such that* $\ln_k(d) \neq 0$, *but* $\ln_{k+1}(d) = 0$, *that is* D *is a nilpotent cyclic algebra;*

(iii) $D = V \oplus U$ *where* V *is an abelian ideal,* $V \leqslant \zeta^{\mathrm{left}}(D)$, U *is a nilpotent cyclic subalgebra,* $[D, D] = V \oplus [U, U]$ *is an abelian ideal;*

(iv) $D = \zeta^{\mathrm{left}}(D) \oplus \zeta^{\mathrm{right}}(D)$ *where*

$$[D, D] = \zeta^{\mathrm{left}}(D) = F\ln_2(d) + \ldots + F\ln_k(d),$$

$\zeta^{\mathrm{right}}(D) = Fc$ *for some element* $c \in D$ *and* $[c, y] = [d, y]$ *for each element* $y \in \zeta^{\mathrm{left}}(D)$.

For the case when $F = \mathbb{C}$ is the field of complex numbers, some description of cyclic finite dimensional Leibniz algebras were obtained in the paper [62]. Unlike Theorem 4.3, it does not show the structure of cyclic Leibniz algebras and based on the following. Let an element a has a depth k. Then $\mathrm{ln}_{k+1}(d) = \alpha_2 \mathrm{ln}_2(a) + \ldots + \alpha_k \mathrm{ln}_k(a)$, for some $\alpha_j \in F$, $2 \leqslant j \leqslant k$. In the paper [62], a characterization for the set of coefficients $(\alpha_2, \ldots, \alpha_k)$ was obtained.

Note that the results of this paper have been extended on the case of arbitrary field in [24].

As we already noted above, Lie algebras are a partial type of Leibniz algebras. In this regard, it is interesting to see how the Leibniz algebras that are the minimal non Lie algebras with all proper subalgebras are Lie algebras are organized. A description of such algebras was obtained in [23].

Theorem 4.4 *Let L be a Leibniz algebra over a field F. Suppose that every proper subalgebra of L is a Lie algebra. Then L is an algebra of one of the following types:*

(i) L *is a Lie algebra;*

(ii) *there is a positive integer k such that $\mathrm{ln}_k(a) \neq 0$, but $\mathrm{ln}_{k+1}(a) = 0$, that is L is nilpotent;*

(iii) $L = V \oplus U$ *where V is an abelian ideal, $V \leqslant \zeta^{\mathrm{left}}(D)$, $U = Fu$ and $[u, u] = 0$, $V = Fv + Fv_1$ and $[u, v] = v_1$, $[u, v_1] = 0$;*

Since every abelian Leibniz algebra is a Lie algebra, we obtain the next corollary.

Corollary 4.5 *Let L be a Leibniz algebra over a field F. Suppose that every proper subalgebra of L is abelian. Then L is an algebra of one of the following types:*

(i) L *is a Lie algebra whose proper subalgebras are abelian;*

(ii) *there is a positive integer k such that $\mathrm{ln}_k(a) \neq 0$, but $\mathrm{ln}_{k+1}(a) = 0$, that is L is nilpotent;*

(iii) $L = V \oplus U$ *where V is an abelian ideal, $V \leqslant \zeta^{\mathrm{left}}(D)$, $U = Fu$ and $[u, u] = 0$, $V = Fv + Fv_1$ and $[u, v] = v_1$, $[u, v_1] = 0$.*

This result implies that the description of Leibniz algebras, whose proper subalgebras are abelian, can be deduced to the case of Lie algebras, whose proper subalgebras are abelian. Such Lie algebras are either simple, or solvable. Soluble minimal nonabelian Lie algebras (even soluble minimal non nilpotent Lie algebras) were described in [33, 65, 66]. Simple minimal nonabelian Lie algebras were studied in [31] and [32], but their complete description remains an open problem.

5 Nilpotent Leibniz algebras and their generalizations

Consider now such an important type of Leibniz algebras as the nilpotent Leibniz algebras. The concept of nilpotency arises in many algebraic structures. In most works on the theory of Leibniz algebras, the definition of nilpotency is given through the use of the lower central series. We also gave this standard definition at the beginning of this article. For Lie algebras, for groups and other algebraic structures, the concept of nilpotency can be introduced using upper central series. We show now how this can be done for Leibniz algebras. This was done in [41]. Here are some of its results.

Define the *upper central series*

$$\langle 0\rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \dots \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \dots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rules: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively,

$$\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$$

for all ordinals α, and $\zeta_\lambda(L) = \bigcup_{\mu<\lambda} \zeta_\mu(L)$ for limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L. Denote by $\mathrm{zl}(L)$ the length of the upper central series of L.

It is a well-known that in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length.

Consider the factors $\gamma_k(L)/\gamma_{k+1}(L)$, $k \in \mathbb{N}$. By definition,

$$[L, \gamma_k(L)] = \gamma_{k+1}(L).$$

By Proposition 2.1, $[\gamma_k(L), L] = [\gamma_k(L), \gamma_1(L)] \leqslant \gamma_{k+1}(L)$.

Let

$$\langle 0\rangle = C_0 \leqslant C_1 \leqslant \dots C_\alpha \leqslant C_{\alpha+1} \leqslant \dots C_\gamma = L$$

be an ascending series of ideals of Leibniz algebra L. This series is called *central* if $C_{\alpha+1}/C_\alpha \leqslant \zeta(L/C_\alpha)$ for each ordinal $\alpha < \gamma$. In other words, $[C_{\alpha+1}, L], [L, C_{\alpha+1}] \leqslant C_\alpha$ for each ordinal $\alpha < \gamma$.

Proposition 5.1 *Let L be an Leibniz algebra over a field F, and*

$$\langle 0\rangle = C_0 \leqslant C_1 \leqslant \dots \leqslant C_n = L$$

be a finite central series of L*. Then:*

(i) $\gamma_j(L) \leqslant C_{n-j+1}$, *so that* $\gamma_{n+1}(L) = \langle 0 \rangle$;

(ii) $C_j \leqslant \zeta_j(L)$, *so that* $\zeta_n(L) = L$;

(iii) *if* j, k *are positive integer,* $k \geqslant j$, *then*

$$[\gamma_j(L), \zeta_k(L)], [\zeta_k(L), \gamma_j(L)] \leqslant \zeta_{k-j}(L).$$

Corollary 5.2 *Let* L *be an Leibniz algebra over a field* F *and suppose that* L *has a finite central series*

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant \ldots \leqslant C_n = L.$$

Then L *is nilpotent and* $\mathrm{ncl}(L) \leqslant n$*. Furthermore, the upper central series of* L *is finite,* $\zeta_\infty(L) = L$, $\mathrm{zl}(L) \leqslant n$*. Moreover,* $\mathrm{ncl}(L) = \mathrm{zl}(L)$.

This Corollary shows that a Leibniz algebra L is nilpotent if and only if there is a positive integer k such that $L = \zeta_k(L)$. The least positive integer having this property coincides with nilpotency class of L. So, as in the cases of Lie algebras and groups, the definition of nilpotency can be given here using the notion of the upper central series.

Here it will be appropriate to note the fact that the Leibniz algebra L can be associative. Indeed, if $[L, L] = \gamma_2(L) \leqslant \zeta(L)$, then

$$0 = [[x, y], z] = [x, [y, z]]$$

for all $x, y, z \in L$. Conversely, suppose that L is associative. Then, taking into account the equality $[[x, y], z] = [x, [y, z]]$, from

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

we derive that $[y, [x, z]] = 0$. Since it is true for all

$$x, y, z \in L, \ [L, L] \leqslant \zeta^{\mathrm{right}}(L).$$

Furthermore,

$$0 = [y, [x, z]] = [[y, x], z],$$

which shows that $[L, L] \leqslant \zeta^{\mathrm{left}}(L)$. So we obtain the following proposition.

Proposition 5.3 *Let* L *be a Leibniz algebra over a field* F. *Then* L *is associative if and only if* $[L, L] \leqslant \zeta(L)$.

The concepts of upper and lower central series introduced here immediately leads to the following classes of Leibniz algebras.

A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter.

A Leibniz algebra L is said to be *hypocentral* if its lower hypocenter is trivial.

In the case of finite dimensional algebras, these two concepts coincide, even though, in general, these two classes are very different.

Thus, a cyclic Leibniz algebra $D = \langle d \rangle$, where the element d has infinite depth, is hypocentral and has infinite dimension. At the same time, D has a trivial center.

For finitely generated hypercentral Leibniz algebras we obtained in [48] the following theorem.

Theorem 5.4 *Let* L *be a finitely generated Leibniz algebra over a field* F. *If* L *is hypercentral, then* L *is nilpotent. Moreover,* L *has finite dimension. In particular, a finitely generated nilpotent Leibniz algebra has finite dimension.*

This result is an analog of a similar group theoretical result proved by A.I. Maltsev [54].

A Leibniz algebra L is said to be *locally nilpotent* if every finite subset of L generates a nilpotent subalgebra.

That is why, hypercentral Leibniz algebras give us examples of locally nilpotent algebras.

In [48] we obtained the following characterization of hypercentral Leibniz algebras.

Theorem 5.5 *Let* L *be a Leibniz algebra over a field* F. *Then* L *is hypercentral if and only if for each* $a \in L$ *and every countable subset* $\{x_n \mid n \in \mathbb{N}\}$ *of elements of* L *there exists a positive integer* k *such that all*

$$[x_1, \ldots, x_j, a, x_{j+1}, \ldots, x_k]$$

are zeros for all j, $0 \leqslant j \leqslant k$.

Corollary 5.6 *Let* L *be a Leibniz algebra over a field* F. *Then* L *is hypercentral if and only if every subalgebra of* L *having finite or countable dimension is hypercentral.*

These results are analogs of the classical results proved for the groups by S.N. Chernikov in [21].

In [12], the following properties of nilpotent ideals of Leibniz algebras have been obtained.

Theorem 5.7 *Let L be a Leibniz algebra over a field F and K_1, K_2 are ideals of L. Suppose that K_1, K_2 are nilpotent and*

$$\mathrm{ncl}(K_1) = c_1, \ \mathrm{ncl}(K_2) = c_2.$$

Then the ideal $K_1 + K_2$ is nilpotent, and $\mathrm{ncl}(K_1 + K_2) = c_1 + c_2$.

Corollary 5.8 *Let L be a Leibniz algebra over a field F. If L has a finite dimension over F, then L has the greatest nilpotent ideal.*

If L is an arbitrary Leibniz algebra, then denote by $\mathrm{Nil}(L)$ the subalgebra, generated by all nilpotent ideals of L. $\mathrm{Nil}(L)$ is called the *nil-radical* of L.

If $L = \mathrm{Nil}(L)$, then L is called a Leibniz *nil-algebra*. Every nilpotent Leibniz algebra is a nil-algebra, but converse is not true even for a Lie algebra. If L is finite dimensional, then Corollary 5.8 shows that $\mathrm{Nil}(L)$ is nilpotent. In the general case, $\mathrm{Nil}(L)$ is locally nilpotent, but converse is not true even for a Lie algebra. Moreover, there exists a Lie nil-algebra, which is not hypercentral. There is a corresponding example in Chapter 6 of the book [3].

In this connection, the following question arises: is an analogous assertion valid for locally nilpotent ideals? For Lie algebras the question has a positive answer, as it was shown by B. Hartley in [38].

Now we show the basic results of [46], which give an affirmative answer on this question.

Theorem 5.9 *Let L be a Leibniz algebra over a field F, and A, B be locally nilpotent ideals of L. Then ideal $A + B$ is locally nilpotent.*

Corollary 5.10 *Let L be a Leibniz algebra over a field F and $\mathfrak{S}$ be a family of locally nilpotent ideals of L. Then the subalgebra generated by $\mathfrak{S}$ is locally nilpotent.*

Corollary 5.11 *Let L be a Leibniz algebra over a field F. Then L has the greatest locally nilpotent ideal.*

Let L be a Leibniz algebra over a field F. The greatest locally nilpotent ideal of L is called the *locally nilpotent radical* of L and will be denoted by $\mathrm{Ln}(L)$.

These results are analogues of those proved for groups by K.A. Hirsch [39] and B.I. Plotkin [57, 58].

Note the following important properties of locally nilpotent Leibniz algebras, which have been obtained in [48].

Theorem 5.12 *Let* L *be a locally nilpotent Leibniz algebra over a field* F.

(i) *If* A, B, $A \leqslant B$ *are ideals of* L *such that factor* B/A *is* L*-chief, then* B/A *is central in* L *(that is* $B/A \leqslant \zeta(L/A)$*). In particular,* $\dim_F(B/A) = 1$.

(ii) *If* A *is a maximal subalgebra of* L, *then* A *is an ideal of* L.

Let L be a Leibniz algebra over a field F, let M be a non-empty subset of L and H be a subalgebra of L. Put

$$\mathrm{Ann}_H^{\mathrm{left}}(M) = \{a \in H \mid [a, M] = \langle 0 \rangle\},$$
$$\mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \mid [M, a] = \langle 0 \rangle\}.$$

The subset $\mathrm{Ann}_H^{\mathrm{left}}(M)$ is called the *left annihilator* or the *left centralizer* of M in subalgebra H. The subset $\mathrm{Ann}_H^{\mathrm{right}}(M)$ is called the *right annihilator* or the *right centralizer* of M in subalgebra H. The intersection

$$\begin{aligned}\mathrm{Ann}_H(M) &= \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M)\\ &= \{a \in H \mid [a, M] = \langle 0 \rangle = [M, a]\}\end{aligned}$$

is called the *annihilator* or the *centralizer* of M in subalgebra H.

It is not hard to see that all of these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then $\mathrm{Ann}_L^{\mathrm{left}}(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathrm{Ann}_L^{\mathrm{left}}(M)$, $b \in M$. Then

$$[[a, x], b] = [a, [x, b]] - [x, [a, b]] = 0 - [x, 0] = 0, \text{ and}$$
$$[[x, a], b] = [x, [a, b]] - [a, [x, b]] = [x, 0] - 0 = 0.$$

If M is an ideal of L, then $\mathrm{Ann}_L(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathrm{Ann}_L(M)$, $b \in M$. Using the above arguments, we obtain that

$$[[a, x], b] = [[x, a], b] = 0.$$

Further,

$$[b,[a,x]] = [[b,a],x] + [a,[b,x]] = [0,x] + 0 = 0, \text{ and}$$
$$[b,[x,a]] = [[b,x],a] + [x,[b,a]] = 0 + [x,0] = 0.$$

Let H be a subalgebra of L. The *left idealizer* or the *left normalizer* of H in L is defined by the following:

$$I_L^{\text{left}}(H) = \{x \in L |\ [x,h] \in H \text{ for all } h \in H\}.$$

Clearly, that the term the *normalizer* arose from group theory analogous.

Similarly, the *right idealizer* of H in L is defined by the following:

$$I_L^{\text{right}}(H) = \{x \in L |\ [h,x] \in H \text{ for all } h \in H\}.$$

The *idealizer* of H in L is defined by the following:

$$I_L(H) = \{x \in L |\ [h,x],[x,h] \in H \text{ for all } h \in H\} = I_L^{\text{left}}(H) \cap I_L^{\text{right}}(H).$$

The left idealizer of H is a subalgebra of L. Indeed, let $x,y \in I_L^{\text{left}}(H)$, $h \in H$, $\alpha \in F$, then

$$[x-y,h] = [x,h] - [y,h] \in H;$$
$$[\alpha x,h] = \alpha[x,h] \in H; \text{ and}$$
$$[[x,y],h] = [x,[y,h]] - [y,[x,h]] \in H.$$

The idealizer of H is also a subalgebra of L. Indeed, let $x,y \in I_L(H)$, $h \in H$, $\alpha \in F$. As above we can show that $x-y$, αx, $[x,y] \in I_L(H)$. Further,

$$[x-y,h] = [x,h] - [y,h] \in H;$$
$$[\alpha x,h] = \alpha[x,h] \in H; \text{ and}$$
$$[h,[x,y]] = [[h,x],y] + [x,[h,y]] \in H.$$

However the right idealizer need not be a subalgebra. This is shown by the following example from [10].

Example 5.13 Let L be a vector space over F, and $\{a,b,c,d\}$ be a basis of L. Define the operation $[\cdot,\cdot]$ by the rules:

$$[a,b]=a,[b,a]=-a+c,[b,b]=d,[a,d]=c,$$
$$[a,a]=[a,c]=0,[b,c]=-c,[d,[d,d]]=0,$$

and

$$[c,x]=[[d,d],x]=0 \text{ for all } x\in L.$$

It is not hard to prove that L is a Leibniz algebra. Let $H=\langle a\rangle$. It follows from $[a,a]=0$ that $\langle a\rangle = Fa$. Clearly,

$$I_L^{\text{right}}(H)=Fb+Fc.$$

However $[c,c]=d\notin I_L^{\text{right}}(H)$, which shows that $I_L^{\text{right}}(H)$ is not a subalgebra of L.

However, if H is a left ideal of L, then its right idealizer is a subalgebra. Indeed, let $x,y\in I_L^{\text{right}}(H)$, $h\in H$, then

$$[h,[x,y]]=[[h,x],y]+[x,[h,y]].$$

By definition, $[h,x],[h,y]\in H$, and $[[h,x],y]\in H$. Since H is a left ideal, $[x,[h,y]]\in H$, which implies that $[x,y]\in I_L^{\text{right}}(H)$.

Let L be a hypercentral Leibniz algebra and let

$$\langle 0\rangle = Z_0\leqslant Z_1\leqslant \dots Z_\alpha\leqslant Z_{\alpha+1}\leqslant\dots Z_\gamma=L$$

be the upper central series of L. Let H be a proper subalgebra of L. Then there exists an ordinal α such that $Z_\alpha\leqslant H$ but H does not include $Z_{\alpha+1}$. Choose an element

$$x\in Z_{\alpha+1}\setminus H.$$

For every element $h\in H$ we have $[x,h],[h,x]\in Z_\alpha$. The inclusion

$$Z_\alpha\leqslant H$$

implies that $[x,h],[h,x]\in H$. This shows that $I_L(H)\neq H$, in particular

$$I_L^{\text{right}}(H)\neq H\neq I_L^{\text{left}}(H),$$

so we obtain the following result.

Proposition 5.14 *Let* L *be a Leibniz algebra over a field* F. *If* L *is hypercentral, then* $I_L(H) \neq H$ *for every proper subalgebra* H *of* L.

Corollary 5.15 *Let* L *be a nilpotent Leibniz algebra over a field* F. *Then* $I_L(H) \neq H$ *for every proper subalgebra* H *of* L.

Let L be a Leibniz algebra over field F. We say that L *satisfies the idealizer condition* if $I_L(A) \neq A$ for every proper subalgebra A of L.

A subalgebra A is called *ascendant* in L if there is an ascending chain of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \dots A_\alpha \leqslant A_{\alpha+1} \leqslant \dots A_\gamma = L$$

such that A_α is an ideal of $A_{\alpha+1}$ for all $\alpha < \gamma$.

It is possible to prove that L satisfies the idealizer condition if and only if every subalgebra of L is ascendant. Next result has been obtained in [48].

Theorem 5.16 *Let* L *be a Leibniz algebra over a field* F. *If* L *satisfies the idealizer condition then,* L *is locally nilpotent.*

This result is analogous to a result proved by B.I. Plotkin for groups in [56].

It should be noted that Leibniz algebras with the idealizer condition form a proper subclass of the class of locally nilpotent Leibniz algebras, since this is the case for Lie algebras. A corresponding example could be found in Chapter 6 of the book [3]. For finitely dimensional Leibniz algebras the following characterization of nilpotent Leibniz algebras is valid.

Theorem 5.17 *Let* L *be a finite dimensional Leibniz algebra over a field* F *of characteristic* 0. *Then the following statements are equivalent.*

(i) L *is nilpotent.*

(ii) *Every proper subalgebra of* L *does not coincide with its idealizer.*

(iii) *Every proper subalgebra of* L *does not coincide with its right idealizer.*

(iv) *Every maximal subalgebra of* L *is an ideal of* L.

(v) *Every maximal subalgebra of* L *is a right ideal of* L.

The most useful of these characterizations were proved in [10].

In [60] the following properties of finite dimensional Leibniz algebras have been obtained.

Theorem 5.18 *Let* L *be a finite dimensional Leibniz algebra over a field* F *and* H *be a nilpotent ideal of* L*. Then* L *is nilpotent if and only if* $L/[H,H]$ *is nilpotent. Moreover, if* $\mathrm{ncl}(H) = c$ *and* $\mathrm{ncl}(L/[H,H]) = d+1$*, then* $\mathrm{ncl}(L) \leqslant \binom{c+1}{2}d - \binom{c}{2}$.

Theorem 5.19 *Let* L *be a finite dimensional Leibniz algebra over a field* F*. If* L *is nilpotent and* H *is a subalgebra of* L *such that* $H + [L,L] = L$*, then* $H = L$*. Conversely, if for every subalgebra* H *such that* $H + [L,L] = L$ *we have* $H = L$*, then* L *is nilpotent.*

Let L be a Leibniz algebra. The intersection of the maximal subalgebras of L is called the *Frattini subalgebra* of L and denoted by $\mathrm{Frat}(L)$. If L does not include maximal subalgebras, then put $L = \mathrm{Frat}(L)$.

The next property of Frattini subalgebra was proved in [12].

Theorem 5.20 *Let* L *be a Leibniz algebra over a field* F *of characteristic* 0*. Then* $\mathrm{Frat}(L)$ *is an ideal of* L*.*

Note that if char(F) is prime, the above statement is not true even for soluble Lie algebras (see [13]).

Combining Theorem 5.20 with Corollary 5.6 of the paper [10], we obtain the following theorem.

Theorem 5.21 *Let* L *be a Leibniz algebra over a field* F *of characteristic* 0*. If* $\dim_F(L)$ *is finite, then* $\mathrm{Frat}(L)$ *is nilpotent.*

Note the following important property of the Frattini subalgebra.

Proposition 5.22 *Let* L *be a finite dimensional Leibniz algebra over a field* F*. If* M *is a subset of* L *such that* $\langle M, \mathrm{Frat}(L)\rangle = L$*, then* $\langle M\rangle = L$*.*

Indeed, suppose the contrary. Let $\langle M\rangle$ is a proper subalgebra of L. Since $\dim_F(L)$ is finite, there is a maximal subalgebra H such that

$$\langle M\rangle \leqslant H.$$

Being maximal, H includes $\mathrm{Frat}(L)$, so that

$$\langle M, \mathrm{Frat}(L)\rangle \leqslant H \neq L.$$

This contradiction proves that $\langle M\rangle = L$.

Using the Frattini subalgebra, we can obtain the following characterization of nilpotent Leibniz algebras. But first we formulate a slightly generalized statement.

Proposition 5.23 *Let* L *be a finite dimensional Leibniz algebra over a field* F. *Then* $[L, L] = \mathrm{Frat}(L)$ *if and only if every maximal subalgebra of* L *is an ideal.*

Indeed, suppose that each maximal subalgebra of L is an ideal. Let K be an arbitrary maximal subalgebra of L. Then $\langle K, x\rangle = L$ for each element $x \notin K$. Since K is an ideal, L/K is a cyclic algebra. If we suppose that $\mathrm{Leib}(L/K)$ is non-zero, then $\mathrm{Leib}(L/K)$ is a proper subalgebra of L/K, which is impossible. Hence $\mathrm{Leib}(L/K) = \langle 0\rangle$, so that L/K is a cyclic Lie algebra. In particular, it is abelian, which follows that $[L, L] \leqslant K$. It is valid for each maximal subalgebra, therefore their intersection $\mathrm{Frat}(L)$ includes $[L, L]$. On the other hand, factor-algebra $L/[L, L]$ is abelian, so that every its subspace is a subalgebra. Since the intersection of all maximal subspaces of $L/[L, L]$ is zero, then $\mathrm{Frat}(L) = [L, L]$.

Conversely, if $[L, L] = \mathrm{Frat}(L)$, then $\mathrm{Frat}(L)$ is an ideal and the factor-algebra $L/\mathrm{Frat}(L)$ is abelian. It follows that every subalgebra including $\mathrm{Frat}(L)$ is an ideal of L, in particular, every maximal subalgebra of L is an ideal.

Using this result and Theorem 5.12 we obtain the next corollary.

Corollary 5.24 *Let* L *be a finite dimensional Leibniz algebra over a field* F *of characteristic* 0. *Then* L *is nilpotent if and only if* $[L, L] = \mathrm{Frat}(L)$.

Let L be a Leibniz algebra. Define the *lower derived series* of L

$$L = \delta_0(L) \geqslant \delta_1(L) \geqslant \dots \delta_\alpha(L) \geqslant \delta_{\alpha+1}(L) \geqslant \dots \delta_\nu(L)$$

by the following rules: $\delta_0(L) = L$, $\delta_1(L) = [L, L]$, and recursively

$$\delta_{\alpha+1}(L) = [\delta_\alpha(L), \delta_\alpha(L)]$$

for all ordinals α and

$$\delta_\lambda(L) = \bigcap_{\mu<\lambda} \delta_\mu(L)$$

for limit ordinals λ. For the last term $\delta_\nu(L)$ we have

$$\delta_\nu(L) = [\delta_\nu(L), \delta_\nu(L)].$$

The length ν of this series is called the *derived length* of L and denoted by $\mathrm{dl}(L)$.

If $\delta_\nu(L) = \langle 0 \rangle$ for some ordinal ν, then L is called a *hypoabelian* Leibniz algebra. If $\delta_n(L) = \langle 0 \rangle$ for some positive integer n, then we say that L is a *soluble* Leibniz algebra.

If K_1, K_2 are soluble ideals of Leibniz algebra L, then their sum

$$K_1 + K_2$$

is a soluble ideal of L. Therefore if L is a finite dimensional Leibniz algebra, then its subalgebra $\mathrm{Sol}(L)$ generated by all soluble ideals of L is called the *soluble radical* of L. By above remarked, $\mathrm{Sol}(L)$ is a soluble ideal of L, and factor-algebra $L/\mathrm{Sol}(L)$ does not include non-zero soluble ideals.

Note some properties of the nil-radical and the soluble radical obtained in [36].

Theorem 5.25 *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then $[L, \mathrm{Sol}(L)] \leqslant \mathrm{Nil}(L)$.*

Corollary 5.26 *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then $[\mathrm{Sol}(L), \mathrm{Sol}(L)]$ is nilpotent.*

Corollary 5.27 *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then L is soluble if and only if $[L, L]$ is nilpotent.*

The last two corollaries were proved in [5].

The following analogue of the Levi's theorem from Lie algebras is true for a finite dimensional Leibniz algebra; it was proved by D. Barnes [11].

Theorem 5.28 *Let L be a finite dimensional Leibniz algebra over a field F of characteristic 0. Then L includes a subalgebra S (being a semisimple Lie algebra) such that $L = \mathrm{Sol}(L) + S$ and $\mathrm{Sol}(L) \cap S = \langle 0 \rangle$.*

The examples given in [11] show that the subalgebra S is not unique.

6 Anticentrality in dimension of Leibniz algebras

Taking into account the fact that the difference between Leibniz algebras and Lie algebras is in the absence of anticommutativity, we naturally arrive at the following object in Leibniz algebras.

Let L be a Leibniz algebra. Put

$$\alpha(L) = \{z \in L \mid [a, z] = -[z, a] \text{ for every } a \in L\}.$$

This subset is called the *anticenter* of the Leibniz algebra L.

Clearly the anticenter is a subspace of L. It is also a subalgebra of L. Indeed, let $z, y \in \alpha(L)$ and a be an arbitrary element of L. Then

$$\begin{aligned}
[[z, y], a] &= [z, [y, a]] - [y, [z, a]] \\
&= -[z, [a, y]] + [y, [a, z]] \\
&= -[z, [a, y]] - [[a, z], y] \\
&= -([[a, z], y] + [z, [a, y]]) \\
&= -[a, [z, y]].
\end{aligned}$$

Moreover, the anticenter is an ideal of L. In fact, let $z \in \alpha(L)$ and a be an arbitrary element of L. For every element $b \in L$ we have

$$\begin{aligned}
[[z, a], b] &= [z, [a, b]] - [a, [z, b]] = -[[a, b], z] + [a, [b, z]] \\
&= -[[a, b], z] + [[a, b], z] + [b, [a, z]] = -[b, [z, a]]
\end{aligned}$$

and

$$\begin{aligned}
[[a, z], b] &= [a, [z, b]] - [z, [a, b]] = -[a, [b, z]] + [[a, b], z] \\
&= -[a, [b, z]] + [a, [b, z]] - [b, [a, z]] = -[b, [a, z]].
\end{aligned}$$

Note that in [20] it was used the term *Lie-center* of a Leibniz algebra. However, the property of anticommutativity is inherent not only in Lie algebras; therefore, instead of the term Lie-center, it seems to us preferable to use the more general term anticenter. Note also that if $\mathrm{char}(F) = 2$, then the anticenter of the Leibniz algebra coincides with the set

$$\{z \in L \mid [a, z] = [z, a] \text{ for every } a \in L\}.$$

In general, this set is not an ideal. Therefore, it is worthwhile to conduct reviews related to the anticenter over the field F such that $\mathrm{char}(F) \neq 2$, and in this part of the paper we will assume that $\mathrm{char}(F)$ is not 2.

Let L be a Leibniz algebra over a field F, M be a non-empty subset of L and H be a subalgebra of L. Put

$$AC_H(M) = \{a \in H|\ [a, u] = -[u, a] \text{ for all } u \in M\}.$$

The subset $AC_H(M)$ is called the *anticentralizer* of M in the subalgebra H. It is clear that the anticenter of L is the intersection of the anticentralizers of all elements of L. But that's all the good ends. Unlike annihilator, the anticentralizer of subset is not always even a subalgebra, so anticentralizer can no longer be such a good technical tool as centralizer. A corresponding example is constructed in [46]. In Leibniz algebras dual to the center is the derived ideal $[L, L]$ generated by all elements $[x, y]$, $x, y \in L$. Based on our analogy, in which $\alpha(L)$ can be considered as a kind of analogue of the center, and subspace (L, L) generated by all elements $(x, y) = [x, a] + [a, x]$, $x, a \in L$, can be considered as some kind of analog of the derived subalgebra. We note right away that this subspace is an ideal. Moreover, if $x, a \in L$, then

$$[[x, a] + [a, x], y] = 0$$

for every element $y \in L$. Indeed, let $x, y, z \in L$, then

$$[[x, y] + [y, x], z] = [[x, y], z] + [[y, x], z]$$
$$= [x, [y, z]] - [y, [x, z]] + [y, [x, z]] - [x, [y, z]] = 0.$$

Further,

$$[z, [x, y] + [y, x]] = [z, [x, y]] + [z, [y, x]] = [[z, x], y] + [x, [z, y]]$$
$$+[[z, y], x] + [y, [z, x]] = ([[z, x], y] + [y, [z, x]]) + ([x, [z, y]] + [[z, y], x]).$$

On the other hand, $[a, a] + [a, a] = 2[a, a] \in (L, L)$, and $\mathrm{char}(F) \neq 2$ implies that $[a, a] \in (L, L)$, so that $\mathrm{Leib}(L) \leqslant (L, L)$. Since $L/\mathrm{Leib}(L)$ is a Lie algebra, $[x, a] + [a, x] \in \mathrm{Leib}(L)$, so that $\mathrm{Leib}(L) = (L, L)$. Thus, with this approach, the Leibniz kernel will be dual to the anticenter.

Starting from the anticenter, we define the *upper anticentral series*

$$\langle 0 \rangle = \alpha_0(L) \leqslant \alpha_1(L) \leqslant \ldots \alpha_\lambda(L) \leqslant \alpha_{\lambda+1}(L) \leqslant \ldots \alpha_\gamma(L) = \alpha_\infty(L)$$

of a Leibniz algebra L by the following rules: $\alpha_1(L) = \alpha(L)$ is the anticenter of L, and recursively, $\alpha_{\lambda+1}(L)/\alpha_\lambda(L) = \alpha(L/\alpha_\lambda(L))$ for all ordinals λ, and $\alpha_\mu(L) = \bigcup_{\nu<\mu} \alpha_\nu(L)$ for limit ordinals μ. By definition, each term of this series is an ideal of L. The last term $\alpha_\infty(L)$ of this series is called the *upper hyperanticenter* of L. A Leibniz algebra L is said to be *hyperanticentral* if it coincides with the upper hyperanticenter. Denote by $\mathrm{al}(L)$ the length of upper anticentral series of L. If L is hyperanticentral and $\mathrm{al}(L)$ is finite, then L is said to be *antinilpotent*.

Let A, B be the ideals of L such that $B \leqslant A$. The factor A/B is called *anticentral*, if $A/B \leqslant \alpha(L/B)$. By definition, the factor A/B is anticentral if and only if $[x, a] + [a, x] \in B$ for each $a \in A$ and each $x \in L$.

If U, V the ideals of L, then denote by (U, V) a subspace, generated by all elements $[u, v] + [v, u]$, $u \in U$, $v \in V$. As we have seen above,

$$[u, v] + [v, u] \in \zeta^{\mathrm{left}}(L).$$

Using the above arguments, we can show, that (U, V) is an ideal of L.

Note at once, that a factor A/B is anticentral if and only if $(L, A) \leqslant B$.

Now we can introduce an analog of the lower central series. Define the *lower anticentral series* of L

$$L = \kappa_1(L) \geqslant \kappa_2(L) \geqslant \dots \kappa_\alpha(L) \geqslant \kappa_{\alpha+1}(L) \geqslant \dots \kappa_\delta(L)$$

by the following rules: $\kappa_1(L) = L$, $\kappa_2(L) = (L, L)$, and recursively

$$\kappa_{\lambda+1}(L) = (L, \kappa_\lambda(L))$$

for all ordinals λ and

$$\kappa_\mu(L) = \bigcap_{\nu<\mu} \kappa_\nu(L)$$

for limit ordinals μ. The last term $\kappa_\delta(L)$ is called the *lower hypoanticenter* of L. We have $\kappa_\delta(L) = (L, \kappa_\delta(L))$.

As we have seen above

$$\kappa_2(L) = (L, L) = \mathrm{Leib}(L) = K.$$

Furthermore, $\kappa_3(L) = (L, \kappa_2(L))$. If $x \in L$, $a \in K = \kappa_2(L)$, then

$$(x, a) = [x, a] + [a, x] = [x, a],$$

because $\mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$. It follows that

$$\kappa_3(L) = [L, \kappa_2(L)] = [L, \mathrm{Leib}(L)].$$

If A is an ideal of L, then put $\gamma_{L,1}(A) = A$, $\gamma_{L,2}(A) = [L, A]$, and recursively $\gamma_{L,n+1}(A) = [L, \gamma_{L,n}(A)]$ for all positive integers n.

Thus we obtain

$$\kappa_1(L) = L, \kappa_2(L) = \mathrm{Leib}(L),$$
$$\kappa_3(L) = \gamma_{L,2}(\mathrm{Leib}(L)),$$
$$\kappa_{n+1}(L) = \gamma_{L,n}(\mathrm{Leib}(L))$$

for all positive integers n.

Suppose now that L has finite series of ideals

$$\langle 0 \rangle = A_0 \leqslant A_1 \leqslant A_2 \leqslant \ldots \leqslant A_n = L.$$

This series is said to be *anticentral*, if every factor A_j/A_{j-1} is anticentral, $1 \leqslant j \leqslant n$.

Proposition 6.1 *Let L be an Leibniz algebra over a field F and*

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant \ldots \leqslant C_n = L$$

be a finite anticentral series of L. Then:

(i) $\kappa_j(L) \leqslant C_{n-j+1}$*, so that* $\kappa_{n+1}(L) = \langle 0 \rangle$*;*

(ii) $C_j \leqslant \alpha_j(L)$*, so that* $\alpha_n(L) = L$*.*

For right Leibniz algebras this statements were proved in [20], for left Leibniz algebras the proof is similar.

Corollary 6.2 *Let L be an antinilpotent Leibniz algebra. Then the length of the lower anticentral series coincides with the length of the upper anticentral series. Moreover, the length of these two series is the smallest among the lengths of all anticentral series of L.*

The length of the upper anticentral series (or lower anticentral series) is called the *class of antinilpotency* of a Leibniz algebra L, and denote it by $\mathrm{ancl}(L)$. In [20] it was called a Lie-nilpotent algebra and class of Lie-nilpotency. However, the concept of Lie-nilpotency arose much earlier in the theory of associative rings, so to avoid confusion it is better to use a different term. In addition, as we have already noted, the property of anticommutativity is inherent not only in Lie algebras, so it is better to focus on it. In general, the results of [20] and [46] show that this approach does not yet seem very effective. The above properties show a certain analogy between nilpotent and antinilpotent Leibniz algebras. However, this analogy is very shallow. Thus, every chief central factor of Leibniz algebra L has dimension 1. On the other hand, every chief factor of Lie algebra is anticentral, but it can have even infinite dimension. Further, we have showed above that finitely generated nilpotent Leibniz algebra has finite dimension. On the other hand, there are finitely generated Lie algebras, which have infinite dimension.

7 Almost hypercentral Leibniz algebras

As we can see Corollary 5.2 states that the fact $\gamma_{c+1}(L) = \langle 0 \rangle$ is equivalent to the fact that $\zeta_c(L) = L$, i.e. the lower and the upper central series in nilpotent Leibniz algebras have the same length. The next natural step is the consideration of the case when the upper (respectively lower) central series has finite length. For this case the question about the relationships between $L/\zeta_k(L)$ and $\gamma_{k+1}(L)$ naturally appears.

If L is a Lie algebra such that $L/\zeta_k(L)$ is finitely dimensional, then $\gamma_{k+1}(L)$ is also finitely dimensional. It follows from Theorem 5.2 of paper [64] by I. Stewart. A corresponding result for groups has been obtained early by R. Baer [9]. In paper [41] the following analog of these theorems has been obtained.

Theorem 7.1 *Let* L *be a Leibniz algebra over a field* F. *Suppose that codimension* $\mathrm{codim}_F(\zeta_k(L)) = d$ *is finite. Then* $\gamma_{k+1}(L)$ *has finite dimension. Moreover* $\dim_F(\gamma_{k+1}(L)) \leqslant 2^{k-1}d^{k+1}$, $k \geqslant 1$.

As a corollary we obtained a bound for the dimension of $\gamma_{k+1}(L)$ in a Lie algebra L.

Corollary 7.2 *Let* L *be a Lie algebra over a field* F. *Suppose that codimension* $\mathrm{codim}_F(\zeta_k(L)) = d$ *is finite. Then* $\gamma_{k+1}(L)$ *has finite dimension. Moreover* $\dim_F(\gamma_{k+1}(L)) \leqslant d^k(d+1)/2$.

An important specific case here is the case when the centre of a Leibniz algebra L has finite codimension. For Lie algebras the following result is well known (see, for example, [67]).

Theorem 7.3 *Let* L *be a Lie algebra over a field* F. *If a factor-algebra* $L/\zeta(L)$ *has finite dimension* d, *then the derived subalgebra* $[L, L]$ *also has finite dimension, and, in addition,* $\dim_F([L, L]) \leqslant d(d+1)/2$.

A corresponding result for groups was proved much earlier.

Theorem 7.4 *Let* G *be a group,* C *a subgroup of the center* $\zeta(G)$ *such that* G/C *is finite. Then the derived subgroup* $[G, G]$ *is finite.*

In this formulation this statement first appears in the paper of B.H. Neumann [55]. This theorem was obtained by R. Baer [9] also.

For Leibniz algebras the following analog of these results has been obtained in [41].

Theorem 7.5 *Let* L *be a Leibniz algebra over a field* F. *Suppose that codimensions* $\mathrm{codim}_F(\zeta^{\mathrm{left}}(L)) = d$ *and* $\mathrm{codim}_F(\zeta^{\mathrm{right}}(L)) = r$ *are finite. Then* $[L, L]$ *has finite dimension, moreover* $\dim_F([L, L]) \leqslant d(d+r)$.

In this connection the following question appears. Suppose that only $\mathrm{codim}_F(\zeta^{\mathrm{left}}(L))$ is finite. Is $\dim_F([L, L])$ finite? The above constructed Example 2.4 gives a negative answer on this question.

Corollary 7.6 *Let* L *be a Leibniz algebra over a field* F. *Suppose that codimension* $\mathrm{codim}_F(\zeta(L)) = d$ *is finite. Then* $[L, L]$ *has finite dimension. Moreover* $\dim_F([L, L]) \leqslant d^2$.

Corollary 7.7 *Let* L *be a Leibniz algebra over a field* F. *Suppose that codimension* $\mathrm{codim}_F(\zeta(L)) = d$ *is finite. Then the Leibniz kernel of* L *has finite dimension at most* $d(d-1)/2$.

The paper [44] considered Lie algebras L whose upper hypercenter have finite codimension. Such algebra includes a finite dimensional ideal K such that the factor-algebra L/K is hypercentral. In addition, some restrictions were obtained for the dimension of this finite dimensional ideal. A similar result for groups was obtained in [25] and [42]. These results relate to a rather extensive topic linked to the study of the relationships between the upper and lower central series in various algebraic structures, see the survey [47].

In the paper [43] the following extension of Theorem 7.1 has been obtained.

Theorem 7.8 *Let* L *be a Leibniz algebra over a field* F. *Suppose that the upper hypercenter of* L *has finite codimension, say* d. *Then* L *includes a finite dimensional ideal* E *such that the factor-algebra* L/E *is hypercentral. Moreover,* $\dim_F(E) \leqslant d(d+1)$.

In this regard, it will not be out of place to cite the following results proved in [46]. We are talking about the inversion of Theorem 7.8. The inversion of this theorem is wrong for both groups and Leibniz algebras. However, if a derived subgroup of a group G is finite, then the second hypercenter of G has finite index [37]. The same situation holds for Leibniz algebras, as the following results proved in [46] show.

Theorem 7.9 *Let L be a Leibniz algebra over a field F. Suppose that the derived ideal of L has finite dimension d. Then the second hypercenter of L has finite codimension at most $2d^2(1+2d)$.*

Theorem 7.10 *Let L be a Leibniz algebra over a field F. If the anticenter of L has finite codimension d, then the Leibniz kernel of L has finite dimension at most d^2.*

8 Restrictions on subalgebras

Another natural question concerns the relationship of the subalgebras and ideals. In particular, what is a structure of Leibniz algebras, all of whose subalgebras are ideals? It is not hard to prove that a Lie algebra, all of whose subalgebras are ideals, is abelian. For groups the situation is different. There exists a non-abelian groups, all of whose subgroups are normal. Such groups have been described in [8]. In the case of associative algebras, the situation is much more complicated. And for Leibniz algebras the situation is quite diverse. At once it is possible to specify a simple example of non-abelian Leibniz algebra, all of whose subalgebras are ideals.

Let L be a vector space over a field F, having dimension 2, $\{a, b\}$ be a basis of L. Define the operation $[\cdot,\cdot]$ by the following rule: $[a, a] = b$, $[b, b] = [b, a] = [a, b] = 0$. A direct check justifies that L becomes a Leibniz algebra. If $\lambda a + \mu b$ is an arbitrary element of L and $\lambda \neq 0$, then

$$[\lambda a + \mu b, \lambda a + \mu b] = \lambda^2 b.$$

Since $\lambda^2 \neq 0$, we obtain that the subalgebra generated by $\lambda a + \mu b$ includes Fb. Since L/Fb is abelian, $\langle \lambda a + \mu b\rangle$ is an ideal. Hence every cyclic subalgebra of L is an ideal. It follows that every subalgebra of L is an ideal.

As we shall see later, any non-abelian Leibniz algebra, whose subalgebras are ideals, is constructed from such algebras as from bricks. Here are more details.

A Leibniz algebra L is called an *extraspecial* algebra if it satisfies the following condition: $\zeta(L)$ is non-trivial and has dimension 1, $L/\zeta(L)$ is abelian.

It is important to observe that there are extraspecial Leibniz algebras in which not every subalgebra is an ideal. The following example of an extraspecial Leibniz algebra from [45] shows this. Moreover, the existence of subalgebras that are not ideals depends on the choice of the field.

Example 8.1 Let F be a field, put $L = Fa \oplus Fb \oplus Fc$. Define on L an operation $[\cdot,\cdot]$ by the following rules: $c = [a,a] = [b,b] = [a,b]$, $[c,c] = [c,a] = [c,b] = [a,c] = [b,c] = [b,a] = 0$. From this definition it follows that $[L,L] \leqslant Fc$, $c \in \zeta(L)$, $\langle c\rangle = Fc$. The equality

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]]$$

occurs automatically, because $[x,y],[y,z],[x,z] \in \zeta(L)$. Thus L is a Leibniz algebra. Let x be an arbitrary element of L, then

$$x = \lambda a + \mu b + \nu c$$

for some $\lambda, \mu, \nu \in F$. We have

$$\begin{aligned}
[x,x] &= [\lambda a + \mu b + \nu c, \lambda a + \mu b + \nu c] \\
&= \lambda^2[a,a] + \lambda\mu[a,b] + \lambda\nu[a,c] + \lambda\mu[b,a] + \mu^2[b,b] \\
&\quad + \mu\nu[b,c] + \lambda\nu[c,a] + \mu\nu[c,b] + \nu^2[c,c] \\
&= \lambda^2 c + \lambda\mu c + \mu^2 c = (\lambda^2 + \lambda\mu + \mu^2)c.
\end{aligned}$$

Let $F = F_2$. If $(\lambda,\mu) \neq (0,0)$, then $\lambda^2 + \lambda\mu + \mu^2 = 1$, that is, $[x,x] = c$ whenever $x \notin Fc$. It follows that $\zeta(L) = Fc$ and $\langle x\rangle = Fx \oplus Fc$. It follows that $\langle x\rangle$ is an ideal of L. Since Fc is an ideal, we obtain that every subalgebra of L is an ideal.

Let $F = F_5$. Suppose that $\lambda^2 + \lambda\mu + \mu^2 = 0$. It follows that

$$\Big(\lambda + \frac{1}{2}\mu\Big)^2 = \mu^2\Big(\frac{1}{4} - 1\Big).$$

In F_5 the solution of the equation $4x = 1$ is 4, so that $\frac{1}{4} - 1 = 3$.

But the equation $x^2 = 3$ has no solutions in F_5. This shows that the equality

$$\lambda^2 + \lambda\mu + \mu^2 = 0$$

is true only when $\lambda = \mu = 0$. Thus if $(\lambda, \mu) \neq (0, 0)$, then $[x, x] \neq 0$ and $[x, x] \in Fc$. Hence, in this case, every subalgebra of L is an ideal.

If $F = \mathbb{Q}$, then using the similar arguments we obtain again that every subalgebra of L is an ideal and the center of L is Fc.

Consider now the case when $F = F_3$. For element $x = a + b$ we have

$$[a + b, a + b] = 3c = 0.$$

It follows that $\langle x \rangle = Fx$. But $[x, a] = [a + b, a] = c \notin Fx$, which shows that a cyclic subalgebra $\langle x \rangle$ is not an ideal.

The following theorem concerned with Leibniz algebras whose every subalgebra is an ideal was proved in [45].

Theorem 8.2 *Let L be a Leibniz algebra over a field F, all of whose subalgebras are ideals. If L is non-abelian, then $L = E \oplus Z$ where $Z \leqslant \zeta(L)$, and E is an extraspecial subalgebra such that $[a, a] \neq 0$ for every element $a \notin \zeta(E)$.*

To any extraspecial algebra we can link a bilinear form in the following way. Let $Z = \zeta(L)$, $V = L/Z$, and c be a fixed non-zero element of Z. Define the mapping

$$\Phi : V \times V \to F$$

by the following rule: if $x, y \in L$, then $[x, y] \in Z$, so that $[x, y] = \alpha c$ for some element $\alpha \in F$. Put

$$\Phi(x + Z, y + Z) = \alpha.$$

This definition is correct. Indeed, let x_1, y_1 be elements of L such that $x_1 + Z = x + Z$, $y_1 + Z = y + Z$. This means that $x_1 = x + c_1$, $y_1 = y + c_2$ for some elements $c_1, c_2 \in Z$. Then

$$[x_1, y_1] = [x + c_1, y + c_2] = [x, y] + [x, c_2] + [c_1, y] + [c_1, c_2] = [x, y].$$

The mapping Φ is bilinear. In fact, let $x, y, u \notin Z$, $[x, u] = \lambda c$, $[y, u] = \mu c$. Then

$$[x + y, u] = [x, u] + [y, u] = \lambda c + \mu c = (\lambda + \mu)c,$$

so that

$$\Phi(x+Z+y+Z, u+Z) = \Phi(x+y+Z, u+Z)$$
$$= \lambda+\mu = \Phi(x+Z, u+Z) + \Phi(y+Z, u+Z).$$

Similarly, we can show that

$$\Phi(x+Z, y+Z+u+Z) = \Phi(x+Z, y+Z) + \Phi(x+Z, u+Z).$$

Let $\beta \in F$, then $[\beta x, y] = \beta[x, y] = \beta(\alpha c) = (\beta\alpha)c$. Thus

$$\Phi(\beta(x+Z), y+Z) = \Phi(\beta x+Z, y+Z) = \beta\alpha = \beta\Phi(x+Z, y+Z).$$

Likewise we can show that $\Phi(x+Z, \beta(y+Z)) = \beta\Phi(x+Z, y+Z)$.

By the definition of an extraspecial algebra we obtain that a bilinear form Φ is non-degenerate. Moreover, Theorem 8.2 shows that $\Phi(x, x) \neq 0$ for every non-zero element x.

Conversely, let V be a vector space over a field F and Φ be a bilinear form on V such that $\Phi(x, x) \neq 0$ for every non-zero element $x \in V$. Put $L = V \oplus F$. Define the operation $[\cdot, \cdot]$ on L by the following rule: if $a, b \in V$, $\alpha, \beta \in F$, then $[(a, \alpha), (b, \beta)] = (0, \Phi(a, b))$.

Put $C = \{(0, \alpha) |\ \alpha \in F\}$. Then $\dim_F(C) = 1$. By this definition,

$$[L, L] = [L, C] = [C, L] = [C, C] = \langle 0 \rangle.$$

It follows from here that the constructed algebra a Leibniz algebra. Furthermore, $C \leqslant \zeta(L)$. Moreover, $C = \zeta(L)$. Indeed, let $(z, \gamma) \in \zeta(L)$ and suppose that $z \neq 0$. Then

$$[(z, \gamma), (a, \alpha)] = [(a, \alpha), (z, \gamma)] = (0, 0),$$

in particular, $[(z, \gamma), (z, \gamma)] = (0, 0)$. But

$$[(z, \gamma), (z, \gamma)] = (0, \Phi(z, z)).$$

Since $z \neq 0$, $\Phi(z, z) \neq 0$, and we obtain a contradiction. This contradiction proves the equality $C = \zeta(L)$. The properties of this bilinear form were considered in details in the survey [40]. Here is one of the corollaries.

Corollary 8.3 *Let L be an extraspecial Leibniz algebra over a field F, having countable dimension. If $[a, a] \neq 0$ for every element $a \notin \zeta(L)$, then L*

has a basis $\{c, e_n | n \in \mathbb{N}\}$ *such that* $[c, e_n] = [e_n, c] = 0$, $0 \neq [e_n, e_n] \in Fc$ *for all* $n \in \mathbb{N}$, $[e_j, e_k] = 0$ *whenever* $j > k$ *and* $[e_j, e_k] = 0$ *whenever* $k > j + 3$, $j, k \in \mathbb{N}$.

Let us now consider some other natural questions of the general theory of Leibniz algebras.

Note that the relation "to be a subalgebra of a Leibniz algebra" is transitive. However, the relation "to be an ideal" is not transitive even for Lie algebras.

Therefore it is natural to ask the question about the structure of Leibniz algebras, in which the relation "to be an ideal" is transitive.

In this context, the following important type of subalgebras naturally arises. A subalgebra A is called a *left* (respectively *right*) *subideal* of L if there is a finite series of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$$

such that A_{j-1} is a left (respectively right) ideal of A_j, $1 \leqslant j \leqslant n$.

Similarly, a subalgebra A is called a *subideal* of L, if there is a finite series of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$$

such that A_{j-1} is an ideal of A_j, $1 \leqslant j \leqslant n$.

We note the following property of nilpotent Leibniz algebras.

Proposition 8.4 *Let* L *be a nilpotent Leibniz algebra over a field* F. *Then every subalgebra of* L *is a subideal of* L.

A Leibniz algebra L is called a T-*algebra*, if the relation "to be an ideal" is transitive. In other words, if A is an ideal of L and B is an ideal of A, then B is an ideal of L. It follows that in a Leibniz T-algebra every subideal is an ideal.

Lie algebras, in which relation "to be an ideal" is transitive have been studied by I. Stewart [63] and A.G. Gejn and Yu.N. Mukhin [34]. In particular, soluble T-algebras and finite dimensional T-algebras over a field of characteristic 0 have been described.

As in the mentioned above cases, the situation in Leibniz algebras is much more complex and diverse than it was in Lie algebras. Here are few simple examples illustrating this point. Let F be an arbitrary field, L be a vector space over F with a basis $\{a, c\}$. Define the operation $[\cdot, \cdot]$ on L by the following rule: $[a, a] = c$, $[c, a] = [a, c] = [c, c] = 0$.

Then L is a cyclic Leibniz algebra, Fc is an unique its non-zero subalgebra. Moreover, Fc is the center of L, in particular, Fc is an ideal of L. Thus every subalgebra of L is an ideal.

Let now $F = F_2$ and L be the Leibniz algebra constructed above. Put

$$A = L \oplus Fv$$

and let $[v, v] = [v, c] = [c, v] = 0, [v, a] = [a, v] = a$. It is not hard to check that A is a Leibniz algebra and L is an ideal of A. Moreover, if B is a non-zero ideal of A and L does not include B, then $B = A$. As we have seen above, Fc is an unique non-zero ideal of L. But $Fc = \zeta(L)$, thus Fc is an ideal of A. Thus A is a Leibniz T-algebra.

Let $F = F_2$ and $D = L \oplus Fu$. Put now $[u, u] = [u, c] = [c, u] = 0$, $[u, a] = [a, u] = a + c$. It is not hard to check that D is a Leibniz algebra and L is an ideal of D. As we did above, we can check that D is a Leibniz T-algebra.

As we will see further, these examples are typical in some sense.

The subalgebra $\mathrm{Ba}(L)$ generated by all nilpotent subideals of L is called the *Baer radical* of L. It is possible to show that $\mathrm{Ba}(L)$ is an ideal of L and $\mathrm{Nil}(L) \leqslant \mathrm{Ba}(L)$. If $L = \mathrm{Ba}(L)$, then L is called a Leibniz *Baer algebra*. Every nil-algebra is a Baer algebra, but the converse is not true even for a Lie algebra (see, for example, [3, Theorem 6.4.5]).

As in the cases mentioned above, the situation for the Leibniz algebra is much more complex and diverse than it was for Lie algebras. Here are few simple examples illustrating this point.

Let F be an arbitrary field, L be a vector space over F with a basis $\{a, c\}$. Define the operation $[\cdot, \cdot]$ on L by the following rules: $[a, a] = c$, $[c, a] = [a, c] = [c, c] = 0$. Then L is a cyclic Leibniz algebra, Fc is its unique non-zero subalgebra. Moreover, Fc is the center of L, in particular, Fc is an ideal of L. Thus every subalgebra of L is an ideal.

Let now $F = F_2$ and L be the Leibniz algebra constructed above. Put

$$A = L \oplus Fv$$

and let $[v, v] = [v, c] = [c, v] = 0$, $[v, a] = [a, v] = a$. It is not hard to check that A is a Leibniz algebra and L is an ideal of A. Moreover, if B is a non-zero ideal of A and L does not include B, then $B = A$. As we have seen above, Fc is an unique non-zero ideal of L. But $Fc = \zeta(L)$, thus Fc is an ideal of A. Thus A is a Leibniz T-algebra.

Let $F = F_2$ and $D = L \oplus Fu$. Put now

$$[u,u] = [u,c] = [c,u] = 0, [u,a] = [a,u] = a + c.$$

It is not hard to check that D is a Leibniz algebra and L is an ideal of D. As above, we can check that D is a Leibniz T-algebra.

As we will see, these examples are typical in some sense. The description of Leibniz T-algebras has been obtained in the paper [49]. Here are the main results of this paper.

Theorem 8.5 *Let* L *be a Leibniz* T*-algebra over a field* F*. If* L *is a Baer algebra, then either* L *is abelian, or* $L = E \oplus Z$ *where* $Z \leqslant \zeta(L)$ *and* E *is an extraspecial subalgebra such that* $[a,a] \neq 0$ *for every element* $a \notin \zeta(E)$*.*

A Leibniz algebra L is called *hyperabelian* if it has an ascending series

$$\langle 0 \rangle = L_0 \leqslant L_1 \leqslant \ldots L_\alpha \leqslant L_{\alpha+1} \leqslant \ldots L_\gamma = L$$

of ideals whose factors $L_{\alpha+1}/L_\alpha$ are abelian for all $\alpha < \gamma$. If this series is finite, then we obtain a soluble Leibniz algebra.

The structure of a Leibniz T-algebra essentially depends on the structure of its nil-radical.

Theorem 8.6 *Let* L *be a hyperabelian Leibniz* T*-algebra over a field* F*. If* L *is non-nilpotent and* $\mathrm{Nil}(L) = D$ *is abelian, then* $L = D \oplus V$ *where* $V = Fv$*,* $[v,v] = 0$*,* $[v,d] = d = -[d,v]$ *for every element* $d \in \mathrm{Nil}(L)$*. In particular,* L *is a Lie algebra.*

Theorem 8.7 *Let* L *be a hyperabelian Leibniz* T*-algebra over a field* F*. If* $\mathrm{char}(F) \neq 2$*, then* $\mathrm{Nil}(L)$ *is abelian.*

In other words, if $\mathrm{char}(F) \neq 2$, then every Leibniz T-algebra is a Lie algebra. Thus we can see that the case when $\mathrm{char}(F) = 2$ is very specific here. We will consider this case with the following additional restriction.

We say that a field F is 2-*closed*, if the equation $x^2 = a$ has a solution in F for every element $a \neq 0$. We note that every locally finite (in particular, finite) field of characteristic 2 is 2-closed.

Theorem 8.8 *Let* L *be a hyperabelian Leibniz* T*-algebra over a field* F*. Suppose that* L *is non-nilpotent and* $\mathrm{Nil}(L)$ *is non-abelian. If a field* F *is 2-closed and* $\mathrm{char}(F) = 2$*, then* $L = (Fe \oplus Fc) \oplus Fv$ *where*

$$[e,e] = c, [c,e] = [e,c] = [c,v] = [v,c] = 0,$$

$$[v,v] = 0, [v,e] = e + \gamma c = [e,v], \gamma \in F.$$

Two ideals are naturally associated with each subalgebra A of a Leibniz algebra L: the ideal A^L which is the intersection of all ideals including A (that is an ideal, generated by A); and the ideal $\mathrm{Core}_L(A)$ which is the sum of all ideals that are contained in A.

A subalgebra A of L is called a *contraideal* of L if $A^L = L$.

From the definition it follows that the contraideals are natural antipodes to the concepts of ideals. Therefore, the study of Leibniz algebras whose subalgebras are either ideals or contraideals seems to us very natural. The description of such Leibniz algebras has been obtained in paper [50].

A Leibniz algebra L is called *quasisimple* if the central factor-algebra $L/\zeta(L)$ is simple and $L = [L, L]$.

Let L be a quasisimple Leibniz algebra and A be a non-trivial subalgebra of L. If $\zeta(L)$ does not include A, then $(A + \zeta(L))/\zeta(L)$ is non-trivial. The fact that the factor-algebra $L/\zeta(L)$ is simple implies that

$$\begin{aligned}(A + \zeta(L)/\zeta(L))^{L/\zeta(L)} &= (A + \zeta(L))^L/\zeta(L)\\ &= (A^L + \zeta(L))/\zeta(L) = L/\zeta(L),\end{aligned}$$

that is $A^L + \zeta(L) = L$. If we suppose that $A^L \neq L$, then the isomorphism

$$L/A^L = (A^L + \zeta(L))/A^L \simeq \zeta(L)/(A^L \cap \zeta(L))$$

shows that L/A^L is abelian, which is impossible. Hence $A^L = L$. Thus, every subalgebra of a quasisimple Leibniz algebra is either an ideal or a contraideal.

Theorem 8.9 *Let L be a Leibniz algebra, whose subalgebras are either ideals or contraideals. If L is not soluble, then L is a simple Lie algebra or a quasisimple Leibniz algebra.*

Theorem 8.10 *Let L be a soluble Leibniz algebra, whose subalgebras are either ideals or contraideals. Then L is an algebra of one of the following types:*

(i) L *is abelian;*

(ii) $L = E \oplus Z$ *where E is an extraspecial subalgebra such that $[e, e] \neq 0$ for each element $e \notin \zeta(E)$ and $Z \leqslant \zeta(L)$;*

(iii) $L = D \oplus Fb$ *where* $[y, y] = 0 = [b, b]$, $[b, y] = y = -[y, b]$ *for every $y \in D$, in particular, L is a Lie algebra;*

(iv) $L = D \oplus Fb$ *where* $[y, y] = [y, b] = [b, b] = 0$, $[b, y] = y$ *for every* $y \in D$, *in particular,* $D = [L, L] = \mathrm{Leib}(L)$;

(v) $L = B \oplus A$ *where* $A = Fa_1 \oplus Fc_1$, $[a_1, a_1] = c_1$, $[c_1, a_1] = 0$, $[a_1, c_1] = c_1$ *and* $[b, b] = [b, a_1] = [b, c_1] = [c_1, b] = 0$, $[a_1, b] = b$ *for every* $b \in B$, *in particular,* $B \oplus Fc_1 = [L, L] = \mathrm{Leib}(L)$;

(vi) $\mathrm{char}(F) = 2$, $L = D \oplus Fa$ *where* D *has a basis* $\{z, b_\lambda |\ \lambda \in \Lambda\}$ *such that* $[a, a] = \alpha z$, $[a, b_\lambda] = b_\lambda = [b_\lambda, a]$, $[a, z] = [z, a] = 0$, $[z, b_\lambda] = [b_\lambda, z] = 0$ *and* $0 \neq [b_\lambda, b_\lambda] \in Fz$, $\lambda \in \Lambda$, $[b_\lambda, b_\mu] = 0$ *for all* $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, *in particular,* $D = [L, L]$, $Fz = \mathrm{Leib}(L)$.

Corollary 8.11 *Let* L *be a Lie algebra, whose subalgebras are either ideals or contraideals. Then* L *is an algebra of one of the following types:*

(i) L *is simple;*

(ii) L *is quasisimple;*

(iii) L *is abelian;*

(iv) $L = D \oplus Fb$ *where* $[y, y] = 0 = [b, b]$, $[b, y] = y = -[y, b]$ *for every* $y \in D$.

A subalgebra A of L is called *core-free* in L if $\mathrm{Core}_L(A) = \langle 0 \rangle$. From the definition it follows that the core-free subalgebras are natural antipodes to the concepts of ideals. Therefore, the study of Leibniz algebras whose subalgebras are either core-free or ideals seems to us a very natural task. The description of such Leibniz algebras has been obtained in [22].

First example of such algebras are Leibniz algebras whose subalgebras are ideals.

Note also that a Lie algebra, whose subalgebras are ideals, are abelian.

On the other hand, if L is a simple Leibniz algebra, then every its proper subalgebra is core-free. We note that in this case L is a Lie algebra.

We would like to show another example, which is typical in some sense.

Let L be a cyclic nilpotent Leibniz algebra of dimension 3. That is

$$L = Fa \oplus Fb \oplus Fc$$

where $b = [a, a]$, $c = [a, b]$. Here $\mathrm{Leib}(L) = Fb \oplus Fc$. If A is a subalgebra of L such that $\mathrm{Leib}(L)$ does not include A, then $A = L$. If $A \leqslant \mathrm{Leib}(L)$ and $Fc \leqslant A$, then A is an ideal of L. If A does not include Fc, then A is not an ideal, $\dim_F(A) = 1$, therefore $\mathrm{Core}_L(A) = \langle 0 \rangle$.

Let L be a Leibniz algebra. The intersection of all non-zero ideals $\mathrm{Mon}(L)$ of L is called the *monolith* of the Leibniz algebra L. If $\mathrm{Mon}(L) \neq \langle 0 \rangle$, then the Leibniz algebra L is called *monolithic*, and, in this case, $\mathrm{Mon}(L)$ is the least non-zero ideal of L.

Thus we can see that the following natural cases appear here: L is a non-monolithic Leibniz algebra; L is a monolithic Leibniz algebra.

As the following results show, the second case is essential.

Theorem 8.12 *Let L be a non-monolithic Leibniz algebra. If every subalgebra of L, which is not an ideal, is core-free, then every subalgebra of L is an ideal.*

Corollary 8.13 *Let L be a non-monolithic Lie algebra. If every subalgebra of L, which is not an ideal, is core-free, then L is abelian.*

The monolithic case splits naturally into two subcases: the Leibniz algebra L has a non-zero center; the Leibniz algebra L has a zero center.

Theorem 8.14 *Let L be a Leibniz algebra. Suppose that L includes a subalgebra, which is not an ideal and that every subalgebra of L, which is not an ideal, is core-free. If the center of L is non-zero, then L satisfies the following conditions:*

(i) L *is monolithic and* $\mathrm{Mon}(L) = \zeta(L) = \gamma_3(L)$, *in particular,* $\dim_F(\zeta(L)) = 1$;

(ii) $\gamma_2(L) = [L, L] \leqslant \zeta_2(L)$ *and* $\gamma_2(L)$ *has dimension 2;*

(iii) *every subalgebra of L, which is not an ideal, is abelian;*

(iv) *every subalgebra of $L/\zeta(L)$ is an ideal.*

Conversely, if L is a Leibniz algebra, satisfying the above conditions, then every subalgebra of L either is core-free or an ideal.

Corollary 8.15 *Let L be a monolithic non-abelian Lie algebra, having non-trivial center. Then every subalgebra of L, which is not an ideal, is core-free if and only if L is an extraspecial algebra.*

The situation when L has a non-central monolith was considered in the following theorem.

Theorem 8.16 *Let L be a monolithic Leibniz algebra whose center is zero. Suppose that every subalgebra of L, which is not an ideal, is core-free. If L is a not Lie algebra, then the following conditions hold:*

(i) $\mathrm{Mon}(L)$ *is a minimal ideal of* L;

(ii) $\mathrm{Mon}(L)$ *is a maximal abelian ideal of* L;

(iii) $L = \mathrm{Mon}(L) \oplus A$ *for some abelian subalgebra* A;

(iv) $\mathrm{Ann}_L(\mathrm{Mon}(L)) = \mathrm{Ann}_L^{\mathrm{left}}(\mathrm{Mon}(L)) = \mathrm{Mon}(L)$.

Conversely, if L is a Leibniz algebra satisfying the above conditions, then every subalgebra of L either is core-free or an ideal.

We note also that in this case a core-free subalgebra can be non-abelian, as the following example shows it.

Example 8.17 Let F be an arbitrary field, L be a vector space over F with a basis $\{a, b, a_1, a_2\}$. Define the operation $[\cdot, \cdot]$ on L in the following way:

$$[a, a] = a_1, [a, a_1] = a_2, [a, a_2] = -a_1 - a_2, [a, b] = 0,$$
$$[b, a] = a_1 + a_2, [b, b] = 0, [b, a_1] = -a_1, [b, a_2] = -a_2,$$
$$[a_1, a] = 0, [a_1, b] = 0, [a_2, a] = 0, [a_2, b] = 0,$$
$$[a_1, a_1] = 0, [a_1, a_2] = 0, [a_2, a_1] = 0, [a_2, a_2] = 0.$$

It is possible to check that L is a Leibniz algebra, $\mathrm{Leib}(L) = Fa_1 + Fa_2$, $\mathrm{Leib}(L) = \mathrm{Mon}(L)$, factor-algebra $L/\mathrm{Mon}(L)$ is abelian. We note that every subalgebra of L, which is not an ideal, is core-free. But the subalgebra $\langle b, a_1\rangle$ is a not ideal, is not abelian and is core-free.

For Lie algebras we obtained the following result.

Proposition 8.18 *Let L be a monolithic Lie algebra whose center is zero. Suppose that every subalgebra of L, which is not an ideal, is core-free. Then $\mathrm{Mon}(L)$ is a minimal ideal of L such that $\mathrm{Ann}_L(\mathrm{Mon}(L)) = \mathrm{Mon}(L)$ and the factor-algebra $L/\mathrm{Mon}(L)$ is abelian. Moreover, every core-free subalgebra of L is abelian.*

If $\mathrm{Mon}(L)$ is abelian, the description is more comprehensive.

Let L be a Leibniz algebra and a be a fixed element of L. Consider the mapping

$$r_a : L \to L$$

defined by the rule $r_a(x) = [x, a]$, $x \in L$. It is not hard to see that r_a is a linear mapping, $\beta\, r_a = r_{\beta\, a}$ and $r_a + r_b = r_{a+b}$ for all $a, b \in L$ and $\beta \in F$. Put $c_a(x) = x + [x, a]$, $x \in L$, that is $c_a(x) = i + r_a(x)$ where i is an identity permutation of L. Clearly c_a is also linear mapping.

Theorem 8.19 *Let L be a monolithic Lie algebra whose center is zero. Suppose that every subalgebra of L, which is not an ideal, is core-free. If the monolith of L is abelian, then the following conditions hold:*

(i) $\mathrm{Mon}(L)$ *is a minimal ideal of* L*;*

(ii) $\mathrm{Mon}(L)$ *is a maximal abelian ideal of* L*;*

(iii) $\mathrm{Ann}_L(\mathrm{Mon}(L)) = \mathrm{Mon}(L)$*;*

(iv) $L = \mathrm{Mon}(L) \oplus A$ *for some abelian subalgebra* A*;*

(v) *if* $L = \mathrm{Mon}(L) \oplus C$ *for some subalgebra* C*, then there exists an element* $v \in A$ *such that* $C = c_v(A)$*; moreover,* c_v *is an automorphism of algebra* L.

Like Lie algebras, Leibniz algebras are also associated with associative algebras, but this connection is a little more complicated.

Let A be an associative algebra over a field F and let $f : A \to A$ be an endomorphism of A such that $f^2 = f$. Define the binary operation $[\cdot, \cdot]$ on A by the following rule: $[a, b] = f(a)b - bf(a)$ for all $a, b \in A$. We have

$$\begin{aligned}
[[a, b], c] &= [f(a)b - bf(a), c] \\
&= f(f(a)b - bf(a))c - cf(f(a)b - bf(a)) \\
&= f(f(a)b)c - f(bf(a))c - cf(f(a)b) + cf(bf(a)) \\
&= f(a)f(b)c - f(b)f(a)c - cf(a)f(b) + cf(b)f(a);
\end{aligned}$$

$$\begin{aligned}
[a, [b, c]] &= [a, f(b)c - cf(b)] \\
&= f(a)(f(b)c - cf(b)) - (f(b)c - cf(b))f(a) \\
&= f(a)f(b)c - f(a)cf(b) - f(b)cf(a) + cf(b)f(a);
\end{aligned}$$

$$\begin{aligned}[b,[a,c]] &= [b, f(a)c - cf(a)]\\ &= f(b)(f(a)c - cf(a)) - (f(a)c - cf(a))f(b)\\ &= f(b)f(a)c - f(b)cf(a) - f(a)cf(b) + cf(a)f(b).\end{aligned}$$

Then

$$\begin{aligned}[a,[b,c]] - [b,[a,c]] &= f(a)f(b)c - f(a)cf(b) - f(b)cf(a) + cf(b)f(a)\\ &\quad -(f(b)f(a)c - f(b)cf(a) - f(a)cf(b) + cf(a)f(b))\\ &= f(a)f(b)c - f(a)cf(b) - f(b)cf(a) + cf(b)f(a)\\ &\quad -f(b)f(a)c + f(b)cf(a) + f(a)cf(b) - cf(a)f(b))\\ &= f(a)f(b)c + cf(b)f(a) - f(b)f(a)c - cf(a)f(b)\\ &= [[a,b],c].\end{aligned}$$

Thus, with respect to the operations $+$ and $[\cdot,\cdot]$ A becomes a Leibniz algebra. Note that if f is the identity permutation of A, then we obtain a standard transition from associative algebras to Lie algebras.

We did not talk about the links of Leibniz algebras with other algebraic structures. However, in conclusion, we would like to note one such link of Leibniz algebras with not very ordinary but interesting algebraic structures that were introduced by J.-L. Loday (see [53]).

Let D be a vector space over a field F. Then D is called a *dialgebra* if two associative binary operation $\vdash$ and $\dashv$ are defined on D and they satisfy the following conditions:

(D1) $x \vdash (y \dashv z) = (x \vdash y) \dashv z$;

(D2) $x \dashv (y \vdash z) = x \dashv (y \dashv z)$;

(D3) $(x \dashv y) \vdash z = (x \vdash y) \vdash z$ for all $x, y, z \in D$.

Note that our use of $\vdash$ and $\dashv$ in this bracket is the opposite of that of J.-L. Loday. This convention matches our preference for left Leibniz algebras instead of right Leibniz algebras.

For a given a dialgebra D we define the operation $[\cdot,\cdot]$ by the rule

$$[x,y] = x \vdash y - y \dashv x, \ x, y \in D.$$

One can check that D becomes a Leibniz algebra relatively the operations $+$ and $[\cdot,\cdot]$. This algebra is called a *Leibniz algebra associated with dialgebra* D. And conversely, J.-L. Loday proved that for any Leibniz algebra L there exists a dialgebra D(L) such that a Leibniz algebra associated with D(L) includes a subalgebra, which is isomorphic to L.

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