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Some Topics of Classical Group Theory: the Genesis and Current Stage *

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To Leonid Kurdachenko on his 70^{th} birthday

1 Introduction

This survey was conceived as a brief description of the main contribution of Leonid A. Kurdachenko to group theory. In the process of writing, the authors found that they underestimated the task of describing all original results from more than 250 research papers published by Leonid in many renowned scientific journals. Taking into account the unavoidable size limitations dictated by the nature of a journal article, this task was almost impossible to complete. So, the authors decided to limit their work by imposing the restriction of considering only classical areas of group theory, i.e. the areas that have some significant history of development and lay their roots in the researches done by widely recognized classics of group theory, such as Reinhold Baer, Sergei N. Chernikov, Philip Hall, Bernhard H. Neumann, Issai Shur, Giovanni Zacher and several others. We also decided to carry on this survey in a non-conventional form. Thus we

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split the material into five mini surveys, supporting each of them with appropriate references. In our opinion, it could be useful to the reader to have some short surveys dedicated to specific topics. In addition, such a structure allows us or somebody else to continue the work.

2 Restrictions on conjugacy classes

The starting point of the theory of FC-groups was the following famous result proved by A.P. Dietzmann [Di37].

If M is a finite G–invariant subset of a group G, every element of which has finite order, then the subgroup $\langle M\rangle$ is finite.

As usual, here a subset M of a group G is called G–*invariant* if $a^g \in M$ for all elements a of M and g of G.

This result was one of the stages of research related to the Burnside problem. It drew attention to the structure of groups covered by finite normal subgroups. Such groups were called *locally normal*. It immediately turned out that the main results of the theory of finite groups could be extended to these groups. The first works in this direction were due to R. Baer [Ba40] and P.A. Golberg [Go46].

If G is a locally normal group, then each of its elements admits only finitely many conjugates. Thus groups with this property are a generalization of locally normal groups. The study of these groups was started by Baer [Ba48], who named them FC–*groups*. FC-groups turned out to be very closely related to locally normal ones, and in particular Baer proved that *if G is an FC–group, then the central factor group $G/\zeta(G)$ is periodic*. Furthermore B.H. Neumann [Ne51] proved that *the derived subgroup of any FC–group is locally normal, so that in particular all torsion–free FC–groups are abelian*. These last results imply that every FC–group can be embedded into a direct product $A \times T$, where A is a torsion–free abelian group and T is a locally normal group.

In the papers of M.I. Kargapolov [Ka58], S.N. Chernikov [Ch58] and P. Hall [Ha59], conditions for the embeddability of a periodic FC-group into a direct products of finite groups began to be investigated. This subject, as well as the study of the influence of the class $\mathbf{SD}\mathfrak{F}$ of subgroups of direct products of finite groups on the structure of FC–groups, is an important part of the theory of

FC-groups. Some interesting parallels here arise with the theory of abelian groups. A significant role in this context was played by articles of Yu.M. Gorchakov (see [Go61],[Go62],[Go65],[Go71],[Go74],[Go76]) and M.J. Tomkinson (see [To77],[To78],[To81]). We will not present here these interesting and important results, since they are fully reflected in the monographs [Go78],[To84] and the survey [To96].

However, all these results deal with periodic FC–groups. The theory of non periodic FC–groups began to be developed and was significantly advanced in the works of L.A. Kurdachenko [Ku77],[Ku79],[Ku81],[Ku82],[Ku83],[Ku83a],[Ku86],[Ku86a],[Ku87],[Ku88]. We will not dwell on the results contained in these papers, since they were reflected in sufficient details in the survey paper [OS09]. The research of Kurdachenko in the theory of FC-groups was not restricted only to considering the non-periodic case, as he considered other relevant part of this theory (see the papers [Ku84],[Ku92],[KOT00],[KO03]); for these contributions we refer again to [OS09].

Now we turn to some generalizations of FC-groups, which arise from certain suitable restrictions on classes of conjugate elements.

Let G be a group. If S is a G–invariant subset of G, then its centralizer $C_G(S)$ is a normal subgroup of G, and the factor group

$$\mathrm{Coc}_G(S) = G/C_G(S)$$

is called the *cocentralizer* of S in G.

The influence of cocentralizers of relevant subsets on the structure of the group itself is a subject of study in many branches of group theory. For instance, in the theory of finite groups, many examples playing a significant role for the consideration of the cocentralizers of chief factors were developed. For instance, we can mention formation theory here because local formations are defined by cocentralizers of the chief factors of the groups involved.

Since every conjugacy classes is a G–invariant subset, we can define the FC–groups as the groups in which the cocentralizer of every conjugacy class is finite, and this is a prerequisite for the following general approach.

Let $\mathfrak{X}$ be a class of groups. We say that a group G has *$\mathfrak{X}$-conjugacy classes* (or that G is an *$\mathfrak{X}C$-group*) if $\mathrm{Coc}_G(g^G) \in \mathfrak{X}$ for each $g \in G$, where $g^G = \{g^x \mid x \in G\}$. If $\mathfrak{X} = \mathfrak{I}$ is the class of all trivail groups, then the class $\mathfrak{I}C$-groups is exactly the class $\mathfrak{A}$ of all abelian groups.

Therefore we may consider $\mathfrak{X}C$-groups as a natural generalization of abelian groups. On the other hand, if $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, we obtain that $\mathfrak{F}C$ is the class of FC-groups. Thus the class of FC-groups is a suitable extension of both $\mathfrak{F}$ and $\mathfrak{A}$.

Other generalizations arise from a different approach, which has Dietzmann's Lemma as its source. If $\mathfrak{X}$ is a group class, we will denote by $Cov_n(\mathfrak{X})$ the class of all groups G such that $\langle g \rangle^G$ belongs to $\mathfrak{X}$ for each element g of G, and we shall say that groups in $Cov_n(\mathfrak{X})$ are *covered* by normal $\mathfrak{X}$-subgroups, or that they have a *normal covering* consisting of $\mathfrak{X}$-subgroups. If $\mathfrak{X} = \mathfrak{F}$, then $Cov_n(\mathfrak{F})$ is the class of periodic FC-groups.

A classic natural extension of the class $\mathfrak{F}$ is the class $\mathfrak{C}$ of Chernikov groups, i.e. groups containing an abelian subgroup of finite index which satisfies the minimal condition. Thus the first natural extension of the class of FC-groups is the class $\mathfrak{C}C$ of all groups with *Chernikov conjugacy classes*, or CC*-groups*. Ya.D. Polovitsky [Po64] introduced this class and obtained some initial results on the structure of these groups.

Although CC–groups are not investigated as much as FC–groups, they have been the subject of many recent papers. A satisfactory detailed presentation of the main results of the theory of CC-groups can be found in the survey [DK12].

We note that the class $Cov_n(\mathfrak{C}$ is precisely the class of all periodic CC–groups. Another classical natural extension of the class of finite groups is the class $\mathfrak{P}\mathfrak{F}$ of all polycyclic–by–finite groups. The study of the groups with polycyclic–by–finite conjugacy classes was started in a paper by S. Franciosi, F. de Giovanni and M.J. Tomkinson [FGT90]. In this paper, it was also paper that such class coincides with the class $Cov_n(\mathfrak{P}\mathfrak{F})$.

In his several papers Kurdachenko considered other classes of $\mathfrak{X}C$–groups. A class that naturally combines the classe of Chernikov groups with that of polycyclic–by–finite groups is the class of minimax groups. Recall here that a group is said to be *minimax*, if it has a finite subnormal series whose factors are either polycyclic–by–finite or Chernikov groups. A group G is called an MC*–group* or a group with *minimax conjugacy classes* if $Coc_G(g^G)$ is minimax for each $g \in G$. The study of this class was initiated by L.A. Kurdachenko in the paper [Ku93], where he obtained some basic results on these groups. The study of MC–groups was continued in [KO01].

If G is any group, we denote by $Tor(G)$ the largest periodic normal subgroup of G. Recall that an abelian group A is called an $\mathfrak{A}_0$*–group* if the Sylow p–subgroups of A are Chernikov and the factor group $A/\mathrm{Tor}(A)$ has finite 0–rank. A soluble group is said to be an $\mathfrak{S}_0$*-group* if it has a finite series of normal subgroups whose factors are abelian $\mathfrak{A}_0$–groups. As we can see, the class $\mathfrak{S}_0\mathfrak{F}$ is a significant extension of the class of minimax groups.

Following A.I. Maltsev [Ma48], we say that a group G has *finite special rank* r if every finitely generated subgroup of G has at most r generators and there exists a finitely generated subgroup H of G having exactly r generators. We will denote by $\mathfrak{A}^{\wedge}$ the class of abelian groups of finite special rank and by $\mathfrak{S}^{\wedge}$ the class of soluble groups of finite special rank. In the paper [KS96] one can find a description of $\mathfrak{A}_0C$–groups and $\mathfrak{A}^{\wedge}C$–groups. Morevoer, results concerning the structure of locally nilpotent $\mathfrak{S}_0C$-groups were obtained in [Ka66].

In the paper [Ka66] the following generalization of periodic FC–groups was also considered: periodic FC–groups are the *locally normal groups of category* 0, and if γ is any ordinal number, then a group G is called a *locally normal group of category* γ, if for every finite subset X of G the normal closure $\langle X\rangle^G$ is a locally normal group of category β, for some $\beta < \gamma$. A similar generalization of FC–groups was studied in [GRV02]: FC–groups are the groups in the class FC^0, and if γ is an ordinal, then a group G belong to the class FC^{γ}, if for each element x of G the group $G/C_G(\langle x\rangle^G) \in FC^{\beta}$ for some $\beta < \gamma$.

Let G be a group. If H is a subgroup of G, then we denote by

$$Cl_G(H) = \{H^g \mid g \in G\}$$

the set of all conjugates of H in G. This set is called the *conjugacy class* of H in G. If $Cl_G(H)$ is finite, then the subgroup H is said to be *almost normal* in G. Groups all whose subgroups are almost normal were described in [Ne55], by proving that these groups must be central–by–finite.

Using the same approach as for classes of conjugate elements, one arrives to the concept of $\mathfrak{X}$*–classes of conjugate subgroups*. More precisely, recall that $|Cl_G(H)| = |G : N_G(H)|$ for each subgroup H of G, and put

$$N_G\big(Cl_G(H)\big) = \bigcap_{g\in G} N_G(H^g) = \bigcap_{g\in G} N_G(H)^g.$$

The subgroup $N_G(Cl_G(H))$ is called the *normalizer* of the conjugacy class of H in G. If $\mathfrak{X}$ is a class of groups, a group G is said to have $\mathfrak{X}$-*con*jugacy classes of subgroups if $G/N_G(Cl_G(H)) \in \mathfrak{X}$ for every subgroup H of G (see [KOS04]).

If $\mathfrak{I}$ is the class of all trivial groups, then it is clear that G has $\mathfrak{I}$–conjugacy classes of subgroups if and only if every subgroup of G is normal, i.e. G is a *Dedekind group*. If $\mathfrak{X} = \mathfrak{F}$, then we obtain the groups with finite conjugacy classes of subgroups, which are exactly the groups considered by B.H. Neumann. The next natural step is to consider classes of infinite groups close to that of finite groups and that have been well-studied from different points of view. The first candidates are the class $\mathfrak{C}$ of all Chernikov groups and the class $\mathfrak{PF}$ of all polycyclic–by–finite groups. Ya.D. Polovitsky [Po77] considered groups with Chernikov classes of conjugate subgroups and proved that a periodic group with such property is central–by–Chernikov. By contrast, in the paper [KO05] an example has been constructed, which shows that in general this result is not true. In the same paper, the full description of groups with Chernikov conjugacy classes of subgroups has been obtained. For groups with polycyclic–by–finite conjugacy classes of subgroups the situation is completely analogous to that obtained by B.H. Neumann. Namely, in [KOS2004] it was shown that a group G has the polycyclic–by–finite conjugacy classes of subgroups if and only if the factor group $G/\zeta(G)$ is polycyclic–by–finite.

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3 Groups with uniform conjugacy classes

B.H. Neumann considered In [Ne54] the important subclass of the class of FC–groups, formed by the so-called BFC-groups. A group G is called a BFC*–group* (or, more precisely, a n–BFC*–group*, where n is a positive integer) if $|g^G| \leqslant n$ for each element g of G. Neumann proved that a group is BFC if and only if it has a finite derived subgroup. In connection with this result, it is natural to ask whether the order of the derived subgroup depends on the size of the conjugacy class. A large series of works has been written in order to obtain estimates for the order of the derived subgroup, and one of the most recent papers on this question was [GM11], where the following bound was obtained:

$$|[G,G]| \leqslant n^{1/2(7+\log n)}$$

The result of Neumann was later generalized in the following natural ways. As we have observed above, Chernikov groups from a class which is one of the closest to that of finite groups, and every Chernikov group G has two natural numerical invariants. Let D be the divisible part of G; then

$$D = C_1 \times \ldots \times C_k,$$

where C_j is a Prüfer subgroup for $j = 1, \ldots, k$, and the non-negative integer k is an invariant of G, that we will denote by $mm(G)$. Another natural invariant of G is the order $o(G)$ of the factor group G/D. The paper [KOS02] investigates the structure of a CC-group G for which there exist positive integers m and b such that $mm(G/C_G(g^G)) \leqslant m$

and $o(G/C_G(g^G)) \leqslant b$ for every element $g \in G$ have been considered, and it was proved that in this case the derived subgroup of G is a Chernikov group. Moreover, some bounding functions were obtained for $mm([G,G])$ and $o([G,G])$.

Another class very close to the class of finite groups is $\mathfrak{PF}$, the class of polycyclic–by–finite groups. Every polycyclic–by–finite group G admits three relevant numerical invariants $is(G)$, $p(G)$ and $h(G)$, that can be defined in the following way. Let S be the largest soluble normal subgroup of G, and put $is(G) = |G/S|$. Of course, the subgroup S has a finite subnormal series with cyclic factors, that may not be the only one; we will denote by $p(G)$ the length of a shortest series of this type. Finally, the invariant $h(G)$ is the *Hirsch number* of G, i.e. the number of infinite cyclic factors in an arbitrary subnormal series of G having cyclic or finite factors. In [KPS01] the authors considered groups G such that the factor group $G/C_G(g^G)$ is polycyclic–by–finite for every element $g \in G$, and there are positive integers i, p and h such that $is\big(G/C_G(g^G)\big) \leqslant i$, $p\big(G/C_G(g^G)\big) \leqslant p$ and $h\big(G/C_G(g^G)\big) \leqslant h$ for all g. Under this assumption, it was proved that, in general, the derived subgroup of the group is not polycyclic–by–finite, but at least it is soluble–by–finite and has finite special rank.

Furthermore, in the paper [FGK95] a broader generalizations of the theorem of B.H. Neumann was considered, by studying the groups G for which there are positive integers d and r such that the factor group $G/C_G(g^G)$ is soluble of derived length at most d and special rank at most r for each element g of G. It was proved that in this case the derived subgroup of G is a soluble subgroup having derived length at most $d+1$ and finite special rank bounded by some function of d and r. Notice that in this paper the condition imposed to the group G is different, requiring that there exists a fixed solvable group Q of finite special rank such that every $G/C_G(g^G)$ is isomorphic to a suitable section of Q. However, the proof uses only the fact that $G/C_G(g^G)$ is soluble of derived length at most d and special rank at most r.

Let p be a prime number. We say that a group G has *finite sectional* p*–rank* r if every elementary abelian p–section of G has order at most p^r and there exists an elementary abelian p–section U/V such that $|U/V| = p^r$. Detailed information about groups of finite sectional p-rank, as well as about other ranks of groups, can be found in the book [DKS17].

The paper [FGK95] studies the structure of groups G for which there are positive integers d and r_p (for every prime p) such that the factor group $G/C_G(g^G)$ is soluble of derived length at most d and sectional p–rank at most r_p for each prime p and each element g G. It was proved that the derived subgroup of such a group is soluble and has finite sectional p–rank, bounded by some function of d and r_p.

In connection with the last two results, one should observe that a different approach produces other information. In [KLMS09] it was proved that if G is a group for which there exists a positive integer r such that the normal closure $\langle g\rangle^G$ has finite special rank at most r for all elements g of G, then the derived subgroup of G has finite special rank bounded by some function of r. Moreover, an analogue of this result for the sectional p-rank was obtained in the paper [KMO12], and in [KMO18] groups G such that $\langle g\rangle^G$ is a boundedly Chernikov group for all $g \in G$ were investigated.

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4 Restrictions on non-abelian subgroups

The presence in a group of a system of abelian subgroups which is large in some sense, or that of only small systems of non-abelian subgroups, significantly affects the structure of the group. The description of finite non–abelian groups all of whose proper subgroups are abelian was given by G. Miller and H. Moreno [MM03], and was one of the first central results in abstract group theory. In the theory of finite groups there is a large number of papers devoted to the consideration of finite groups with various restrictions on non-abelian subgroups. Here, however, we will only talk about infinite groups. Some sophisticated examples of such infinite groups have been developed quite recently by A.Yu. Olshanskii (see the book [Ol91], § 28]), and these examples suggest that it is impossible to expect a complete description of these groups.

In the paper [KP86], the authors considered groups, whose non–normal subgroups have finite index. Moreover, the structure of groups in which the family of all non–abelian subgroups splits into a finite number of isomorphism classes was studied in [KLMS18]; it was shown that under certain natural restrictions such groups are minimax and abelian–by–finite.

Groups in which the set of all non–abelian subgroups satisfies the minimal condition (the groups with $\mathrm{Min}\text{–}(\mathrm{non}\text{–}\mathrm{ab})$) were first discussed by S.N. Chernikov [Ce64]. His results imply that a non–abelian locally soluble group satisfying Min–(non–ab) is a Chernikov group. This result has been extended to the case of locally finite groups by V.P. Shunkov [Sh70]. Notice also that In [KP86] groups whose non–normal subgroups have finite index were considered. By contrast, the class of groups with Max–(non–ab) (i.e. the groups in which the set of all non–abelian subgroups satisfies the maximal condition) does not coincide with the class of groups satisfying Max, even when further stringent hypotheses are added. A simple example here is a group that is the wreath product of a group of prime order by an infinite cyclic group. Groups with Max–(non–ab) were considered by D.I. Zaitsev and L.A. Kurdachenko in [ZK91].

The next step in this natural classification process is the consideration of groups in which the set of all non–abelian subgroups satisfies the weak minimal condition (the condition Min–∞–(non–ab)). Such groups were studied by Zaitsev in [Za71]. The main result of that paper shows that the situation here is similar to the case of the condition Min; in fact, it was shown that a non–abelian locally almost soluble group satisfies Min–∞–(non–ab) if and only if it is almost soluble and minimax. Groups in which the set of all non–abelian subgroups satisfies the weak maximal condition (the condition Max–∞–(non–ab)) were studied by Kazarin, Kurdachenko and Subbotin in [KKS98]. The situation here is more complicated than in the cases of the condition Max–(non–ab). Groups, whose non–abelian subgroups have finite special rank have been considered in [DES99], and Theorem A of this paper implies that a non–abelian locally (soluble–by–finite) group, whose non-abelian subgroups have finite special rank, likewise has finite special rank.

Another approach has been outlined from the work [RS66], where groups whose non–abelian subgroups are normal were considered. Such groups were called *metahamiltonian* in this article. In the papers [SR68] and [RS70], some properties of metahamiltonian groups were obtained. Notice that any group, all of whose proper subgroups are abelian, is obviously metahamiltonian. Thus the above mentioned very exotic examples constructed by Olshanskii show that a satisfactory study of metahamiltonian groups is possible only if additional restrictions are imposed. One of these natural limitations is the following. A group G is said to be *locally graded* if every non–trivial finitely generated subgroup of G conatins a proper subgroup of finite index. The concept of a locally graded group was introduced by S.N. Chernikov in order to avoid examples like those constructed by Olshanskii.

S.N. Chernikov in [Ce70] proved that the derived subgroup of any locally graded metahamiltonian group is a finite p–group for some prime p. In particular, every locally graded metahamiltonian group is soluble. The structure of finite metahamiltonian p–group, where p is a prime, was clarified by A.A. Mahnev [Ma76]. Moreover, it turns out that every non–abelian subgroup of a locally graded metahamiltonian group G contains the derived subgroup of G (for a proof see [DGM13]).

A complete description of locally graded metahamiltonian groups has been obtained by N.F. Kuzennyi and N.N. Semko in the series of papers [KS83],[KS85],[KS86],[KS87],[KS89],[KS90].

In the paper [DGKM18a] the following characteristic subgroup associated with the family of all non-abelian subgroups of a group was introduced. The *metanorm* of a group G is the subgroup M(G) consisting of all elements g of G such that $X^g = X$ for each non–abelian subgroup X of G. In other words, $M(G)$ is the intersection of the normalizers of all non–abelian subgroups of G. Thus G is metahamiltonian if and only if $G = M(G)$. The structure of the subgroup $M(G)$ and its influence on the structure of the whole group G have been investigated in [DGKM18b].

Note also that in the paper [KSA14] the study of groups whose non–abelian subgroups are subnormal has been initiated, and in particular locally finite groups with such property have been described.

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5 "Schur theorem" and related results

There are two important characteristic subgroups in any group G: the centre and the derived subgroup. The central factor group $G/\zeta(G)$ and the derived subgroup $[G, G]$ indicate how far the given group is far from being abelian (if they are trivial, then G is abelian). The problem of determining the relations between these two objects arises naturally. The starting point of this interesting topic was the following result proved by B.H. Neumann [Ne51]:

If the central factor group $G/\zeta(G)$ *is finite, then the derived subgroup of* G *is also finite.*

Much later, the name *Schur theorem* stuck to this result. The history of the origin of this name is considered in the paper [KS16]. It should also pointed out that at the end of [Ne51], Neumann wrote that he received a letter from Reinhold Baer, who mentioned that this result can be obtained as a corollary of a more general result, that was proved in [Ba45]. In fact, Theorem 3 of this paper proved that if a normal subgroup H of a group G has finite index, then the factor group $([G, G] \cap H)/[H, G]$ is also finite. Nevertheless, Baer stated the above result in its usual form and presented another proof of it in his paper [Ba52].

In [Ne51], B.H. Neumann put the following natural question: *what is the relation between the order of* $G/\zeta(G)$ *and that of the derived subgroup* $[G, G]$*?* He obtained the first bound for the order of $[G, G]$, by reducing the situation to the case of finite groups and using some results of Schur. One year later, R. Baer [Ba52] obtained a bound for the exponent of $[G, G]$. Until recently, the best estimate was the one obtained by J. Wiegold (see [Wi57],[Wi65]). For certain p-groups, the estimate obtained by J. Wiegold was achieved, but for many other types of groups it was very far from being sharp. Recently, the bound obtained by Wiegold was improved (see [We18]); however, also the new estimate may be far from the sharp one.

The above mentioned result of B.H. Neumann suggests the following natural question: *for which classes* $\mathfrak{X}$ *the fact that* $G/\zeta(G) \in \mathfrak{X}$ *always implies that* $[G, G] \in \mathfrak{X}$*?* Neumann's theorem says that the class of all locally finite group has this property, but it follows from the results in [Ad71] that it is no longer true for periodic groups.

For the class of all polycyclic–by–finite groups a positive answer is almost obvious, while for the class of all Chernikov groups a positive answer was obtained in [Sc69]. However, for groups satisfying the minimal or maximal conditions the answer is in general negative. By using Olshanskii's construction, it was exhibited in [KS13] an example of a group G such that $G = [G, G]$, $G/\zeta(G)$ is an infinite p–group whose proper non-trivial subgroups have order p, $\zeta(G)$

is a free abelian group of infinite 0–rank, where p is a very large suitable prime number.

We list here some classes of groups for which the question posed above has a positive answer:

The class of soluble–by–finite minimax groups [Ku93];

The class of locally generalized radical groups of finite special rank [KS13];

The class of locally generalized radical groups of finite sectional p–rank for some prime p [BCKO13];

The class of locally finite groups whose Sylow p–subgroups are Chernikov for a fixed prime p [BCKO13];

The class of locally generalized radical groups of finite sectional p–rank for all primes p [BCKO13];

The class of locally finite groups whose Sylow p–subgroups are Chernikov for all primes p [BCKO13];

The class of locally generalized radical groups of finite 0–rank [DKS18].

A natural generalization of Neumann's result is the following theorem proved by R. Baer in [Ba52]: *if* G *is a group for which the factor group* $G/\zeta_k(G)$ *is finite, then* $\gamma_{k+1}(G)$ *is also finite.* Here $\zeta_k(G)$ and $\gamma_{k+1}(G)$ denote the k-th term of the upper central series and the $(k+1)$-th term of the lower central series of G, respectively.

The following question arises naturally from Baer's theorem: *for which group classes* $\mathfrak{X}$ *the fact that* $G/\zeta_k(G) \in \mathfrak{X}$ *always implies that* $\gamma_{k+1}(G) \in \mathfrak{X}$*?*

Also in this case we present a list of group classes for which the question posed above has a positive answer:

The class of polycyclic–by–finite groups;

The class of Chernikov groups [OP92];

The class of locally generalized radical groups of finite special rank [KO13];

The class of locally generalized radical groups of finite sectional p–rank for some prime p [DKO15];

The class of locally finite groups whose Sylow p–subgroups are Chernikov for some prime p [DKO15];

The class of locally generalized radical groups of finite sectional p–rank for all primes p [DKO15];

the class of locally finite groups whose Sylow p–subgroups are Chernikov for all primes p [DKO15];

The class of locally generalized radical groups of finite 0–rank [DKS18].

Baer's theorem was also generalized in [DGMS11], where it was proved that if the factor group $G/\zeta_\infty(G)$ is finite, then G contains a finite normal subgroup S such that G/S is hypercentral. Here $\zeta_\infty(G)$ denotes the *upper hypercentre* of G, i.e. the last term of its upper central series. This result raised the following natural question: *are the orders of* $G/\zeta_\infty(G)$ *and* S *realed in some way? In particular, is the order of* S *bounded?* The answer to this question was obtained in [KOS13]. A further generalization of the above theorem has been considered in [DKO15],[DKS18], where it was proved in particular that if $G/\zeta_\infty(G)$ is Chernikov group, then G contains a Chernikov normal subgroup S such that G/S is hypercentral. In this connection note also that if $G/\zeta_\infty(G)$ is polycyclic, then G need not contain a polycyclic normal subgroup S such that G/S is hypercentral; an example of this kind can be found in [DKO15].

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6 The influence of the arrangement of subgroups

The investigation of groups satisfying certain arrangements of subgroups allows algebraists to introduce and describe many important classes of groups. Most of the conditions related to the arrangement of subgroups in a group are based on the fundamental notions of normality and of different kinds of subgroup chains.

If G is a group and H is a subgroup of G, we can define the *upper normalizer chain* of H in G as

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_\alpha \leqslant H_{\alpha+1} \leqslant \ldots \leqslant H_\gamma \leqslant H_{\gamma+1} = G,$$

where $H_{\beta+1} = N_G(H_\beta)$ for $\beta < \gamma$,

$$H_\lambda = \bigcup_{\beta<\lambda} H_\beta$$

if $\lambda \leqslant \gamma$ is a limit ordinal and $N_G(H_\gamma) = H_\gamma$. We can consider here the following two extreme cases: If $H_\gamma = G$, then the subgroup H must be ascendant in G, while we have $H_\gamma = H$ precisely when the subgroup H is self-normalizing in G. Thus we can associate to every subgroup of G an ascendant and a self-normalizing subgroup in a natural way. The structure of groups, whose subgroups are either ascendant or selfnormalizing, was considered by Kurdachenko, Otal, Russo and Vincenzi in [KORV11]. They proved that *if* G *is a locally* *finite non-(locally nilpotent) group whose subgroups are either ascendant or selfnormalizing, then* $G = P \ltimes A$*, where* $P = \langle x \rangle$ *is a cyclic* p*-subgroup for some prime* p*,* $A = [G, G]$ *is a nilpotent normal* p'*-subgroup,* $C_P(A) = \langle x^p \rangle$ *and* $C_G(P) = P$*. Conversely, if* G *is a group having this structure, then every subgroup of* G *either is ascendant or self-normalizing.* Moreover, the authors proved that if G is a hyperabelian group whose subgroups are either ascendant or self-normalizing, then all subgroups of G are ascendant, provided that G is either locally nilpotent or non-periodic. Recall here that if $\mathfrak{X}$ is a class of groups, a group is said to be a *hyper-*$\mathfrak{X}$*-group* if it has an ascending series of normal subgroups whose factors belong to $\mathfrak{X}$.

Consider again the upper normalizer chain of a subgroup H. If K is a subgroup of G containing H_γ, then H_γ is self-normalizing in K; however, the subgroup K itself need not be self-normalizing. This remark leads to the definition of the following important type of subgroups.

Let G be a group. A subgroup H is said to be *weakly abnormal* in G if every subgroup of G containing H (and H itself, in particular) is self-normalizing. Using the concept of normal closure, the following characteristization of weakly abnormal subgroups was obtained in [BB88]: *a subgroup* H *of a group* G *is weakly abnormal if and only if* $x \in H^{\langle x \rangle}$ *for each element* x *of* G.

If H is a maximal subgroup of G which is not normal, then obviously $H = N_G(H)$. Thus, if $x \notin H$, we have $G = \langle H, H^x \rangle$ and in particular, $x \in \langle H, H^x \rangle$, and we come to the following type of subgroups: a subgroup H of a group G is called *abnormal* if $g \in \langle H, H^g \rangle$ for each element g of G. Abnormal subgroups were considered by Philip Hall in the paper [Ha37], and the term *abnormal* is also due to Hall, even though it appeared first in the article [Ca61] of R. Carter. Since $H^{\langle x \rangle}$ contains $\langle H, H^x \rangle$, every abnormal subgroup is weakly abnormal. The converse is not true, and a corresponding counterexample can be found in [BB88].

By their nature, abnormality is an antagonist of normality: a subgroup H of a group G is both normal and abnormal only if it coincides with the whole group. As we saw earlier, every maximal non-normal subgroup is abnormal. More interesting is an example of J. Tits: the subgroup $T(n, F)$ of all triangular matrices is abnormal in the general linear group $GL(n, F)$ over an arbitrary field F. Finite (soluble) groups provide us with many assorted examples of abnormal subgroups. Among them, we note Carter subgroups (that is

nilpotent self-normalizing subgroups), introduced in [Ca61], where Carter also obtained the following characterization of abnormal subgroups.
A subgroup H *of a group* G *is abnormal if and only if the following two conditions hold:* (i) *every subgroup of* G *containing* H *is self-normalizing;* (ii) *If* K *and* L *are two conjugate subgroups of* G *containing* H, *then* $K = L$.

Condition (ii) could be omitted In the case of soluble groups. For finite groups, this fact was noted in the book of B. Huppert (see [Hu67], p.733, Theorem 11.17]. For infinite groups, the most general expansion of this result was obtained in [KPS10].

Recall that a group G is called an $\tilde{N}$-*group* if M is normal in L, whenever M and L are subgroups of G and M is maximal in L. Notice that the property of being an $\tilde{N}$-group is local (see [KC47], §8). In particular, every locally nilpotent group is an $\tilde{N}$-group, but converse is not true [Wi77]. It was proved in [KPS10] that in a hyper-$\tilde{N}$-group G a subgroup H is abnormal if and only if every subgroup containing H is self-normalizing; in other words, in hyper-$\tilde{N}$-groups every weakly abnormal subgroup is abnormal. Of course, this result holds in particular for radical, hyperabelian, and soluble groups.

We consider now another natural series associated to any subgroup H of a group G, which is in some sense dual to the upper normalizer chain. This is the descending series

$$G = H_0 \geqslant H_1 = H^G \geqslant \ldots \geqslant H_\alpha \geqslant H_{\alpha+1} \geqslant \ldots H_\gamma$$

defined by the rules $H_{\beta+1} = H^{H_\beta}$ for every ordinal $\beta < \gamma$ and

$$H_\lambda = \bigcap_{\beta<\lambda} H_\beta$$

if $\lambda \leqslant \gamma$ is a limit ordinal. This series is called the *series of normal closures* of H in G. The term H_α of the series is called the *α-th normal closure* of H in G and will be denoted by $H^{G,\alpha}$. The last term H_γ of this series is called the *lower normal closure* of H in G and will be denoted by $H^{G,\infty}$. If $H = H^{G,\infty}$, then the subgroup H is said to be *descendant* in G. It is possible to prove that if every subgroup of a locally (soluble-by-finite) group is descendant, then the group is locally nilpotent.

An important special case of descendant subgroups are *subnormal subgroups*, which are precisely those descendant subgroups admitting a finite series of normal closures. Subnormal subgroups strongly affect the structure of a group. For instance, If every subgroup of a group G is subnormal, then a remarkable result due to W. Möhres [Mo90] proves that G is soluble. Subnormal subgroups have been extensively studied for quite a long period of time. We will not delve into this subject, since it is widely reflected in the book of J.C. Lennox and S.E. Stonehewer [LS87] and the survey of C. Casolo [Cs08].

Another extreme case leads us to the following type of subgroups. A subgroup H of a group G is said to be *contranormal* in G if $H^G = G$. The term *contranormal* was introduced by J.S. Rose in [Ro68]. Note at once that every subgroup is contranormal in its lower normal closure. If K is a subgroup of G such that $H \leqslant K \leqslant H^{G,\infty}$, then K is certainly contranormal in $H^{G,\infty}$, but the subgroup H need not contranormal in K. Thus we naturally arrive at the following important type of subgroups: a subgroup H of a group G is said to be *nearly abnormal* if H is contranormal in every subgroup K such that $H \leqslant K \leqslant G$ (see [KPS10]). If H is an abnormal subgroup of a group G, and K is a subgroup containing H, then $x \in \langle H, H^x\rangle$ for every element $x \in K$. Since obviously embedding $\langle H, H^x\rangle \leqslant H^{\langle x\rangle}$, we obtain that $H^K = K$. Therefore, every abnormal subgroup is nearly abnormal; in particular, every abnormal subgroup of a group is contranormal. However, not every contranormal subgroup is abnormal, as the following easy example shows.

Let P be a quasicyclic 2-group, and consider the semidirect product $G = \langle d\rangle \ltimes P$, where $|d| = 2$ and $d^{-1}ad = a^{-1}$ for all $a \in P$. Then G is hypercentral, and so it satisfies the normalizer condition. It follows that G has no proper abnormal subgroups, while $\langle d\rangle$ is a proper contranormal subgroup of G.

Let H be a nearly abnormal subgroup of a group G, and consider an arbitrary subgroup K of G containing H and such that

$$L = N_G(K) \neq K.$$

Then K is normal in L and so $H^L \neq L$, so that H is not contranormal in L. This contradiction shows that $N_G(K) = K$; in other words, every nearly abnormal subgroup of G is weakly abnormal. The above quoted result from [KPS10] implies that *in a hyper-$\tilde{N}$-group a subgroup is nearly abnormal if and only if it is is abnormal*.

Conversely, let H be a weakly abnormal subgroup of a group G, and let K be an arbitrary subgroup of G containing H. Suppose that $K \neq H^K$. As the subgroup H^K is normal in K, we have

$$N_K(H^K) = K \neq H^K,$$

so that $N_G(H^K) \neq H^K$, and H^K is not self-normalizing. This contradiction shows that $H^K = K$. Therefore all weakly abnormal subgroups are nearly abnormal.

Our next considerations deal with subgroups for which the upper normalizer series is very short. A subgroup H of a group G is said to be *transitively normal* if H is normal in every subgroup K of G such that $H \leqslant K$ and H is subnormal in K (see [KS06]). These subgroups have been considered under a different name in [My02]; namely, a subgroup H of a group G is said to satisfy the *subnormalizer condition* in G, if $N_G(K) \leqslant N_G(H)$ whenever H is normal in the subgroup K of G.

Some properties of transitively normal subgroups (again under another name) in FC-groups have been studied in [GV03], where in particular the authors proved that in an FC-group all transitively normal p-subgroups are pronormal. Recall that a subgroup H of a group G is said to be *pronormal* if for every $g \in G$ the subgroups H and H^g are conjugates in the subgroup $\langle H, H^g \rangle$. From this definition, it clearly follows that every abnormal subgroup is pronormal. On the other hand, every normal subgroup is also pronormal. Thus pronormal subgroups have managed to combine these two antagonistic types of subgroups. Very important subgroups of finite soluble groups as Sylow subgroups, Hall subgroups, system normalizers, and Carter subgroups are pronormal.

Pronormality is tightly connected to the well know T-*property*, i.e. to the property of transitivity of normality. The groups having this property are called T-*groups*, so that a group is a T-group if and only if all its subnormal subgroups are normal. A group G is called a $\overline{T}$-*group* if every subgroup of G is a T-group. The structure of finite soluble T-groups has been described by W. Gaschütz [Ga57]. In particular, he proved that every finite soluble T-group is a $\overline{T}$-group. Observe that every finite $\overline{T}$-group is metabelian. Infinite soluble groups with the $\overline{T}$-property have been invetsigated by D.J.S. Robinson [Ro64]. The main results of this study can be summarized as follows.

Let G *be a locally soluble* $\overline{T}$-*group.* (i) *If* G *is not periodic, then it is abelian.* (ii) *If* G *is periodic and* L *is the locally nilpotent residual of* G*, then:* (a) G/L

is a Dedekind group; (b) $\pi(L) \cap \pi(G/L) = \emptyset$*;* (c) $2 \notin \pi(L)$ (d) *every subgroup of* L *is normal in* G*.*

T.A. Peng [Pe71] gave the first characterization of finite $\overline{T}$-groups in terms of pronormal subgroups. He proved that *all subgroups of a finite group* G *are pronormal if and only if* G *is a* $\overline{T}$*-group*; moreover, in order to have the $\overline{T}$-property for a finite group, it is enough to require that every cyclic subgroup is pronormal.

This result was generalized to infinite groups in [KS87], where it is proved that *for a group* G *which is either locally soluble group or periodic and locally graded the following conditions are equivalent:* (i) *every cyclic subgroup of* G *is pronormal;* (ii) G *is a soluble* $\overline{T}$*-group. Moreover, If* G *is a group whose subgroups are pronormal and* L *is the locally nilpotent residual of* G*, we have:* (iii) *if* G *is periodic and locally graded, then* L *complements every Sylow* $\pi(G/L)$*-subgroup of* G*;* (iv) *If* G *is locally soluble and non-periodic and locally soluble, then it is abelian.*

Notice that assertion (iv) above has been extended to the case of non-periodic locally graded groups in [RRV07].

N.F. Kuzennyi and I.Ya. Subbotin completely described periodic also locally graded groups in which all primary subgroups are pronormal [KS89] and infinite locally soluble groups in which all infinite subgroups are pronormal [KS88]. Actually, they proved that *the class of infinite groups whose subgroups are pronormal is a proper subclass of the class of infinite groups for which normality is transitive, and it is also a proper subclass of the class of infinite groups in which all primary subgroups are pronormal.* However, the pronormality condition for all subgroups can be weakened to the pronormality only for abelian subgroups (see [KS89]).

We mentioned above some relevant results on transitivity of normality. Transitivity of such important subgroup properties as pronormality, abnormality and other related properties have been studied by L.A. Kurdachenko, I.Ya. Subbotin and J.Otal (see [KS02] and [KOS05]). Groups in which pronormality is a transitive relation are called TP*-groups* and those in which every subgroup is a TP-group are called $\overline{TP}$*-groups*. It was proved in [KS02] that a locally souble group is a $\overline{TP}$-group if and only if it has the $\overline{T}$-property, and for TP-groups the following description was obtained.

(i) *A periodic soluble group* G *has the* TP*-property if and only if* $G = (B \times P) \ltimes A$*, where* A *and* B *are abelian* $2'$*-subgroups,* P *is a 2-subgroup* *with the* T*-property,* $\pi(A) \cap \pi(B) = \emptyset$*,* $[G, G] = A \times [P, P]$*, every subgroup of* $[G, G]$ *is normal in* G *and* A *is a complement to every Sylow* $\pi(B \times P)$*-subgroup of* G*.*

(ii) *If* G *is a soluble group and* $C_G([G, G])$ *is not periodic, then* G *is a* TP*-group if and only if it is a* T*-group.*

(iii) *If* G *is a soluble non-periodic group and* $C_G([G, G])$ *is periodic, then* G *is a* TP*-group if and only if it is a hypercentral* T*-group.*

The complete list of all types of periodic soluble TP-groups can be found in [KS02].

The property of being an abnormal subgroup is not transitive. However, P. Hall noted that if H is a normal subgroup of G, D is an abnormal subgroup of DH and DH is abnormal in G, then D is abnormal in G. The following relevant result was proved in [KS05]: *if the group* G *contains a normal subgroup* H *satisfying the normalizer condition and such that the factor group* G/H *has no proper abnormal subgroups, then abnormality is transitive in* G.

J.S. Rose introduced in [Ro65] the notion of a *balanced chain* connecting a subgroup H to the group G; this is a chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_{n-1} \leqslant H_n = G$$

such that for each non-negative integer $j \leqslant n-1$ the subgroup H_j is either normal or abnormal in H_{j+1}. In a finite group, every subgroup can be connected to the group by some balanced chain. It is natural to consider the case when these balanced chains can be chosen short, i.e. in such a way that their lengths are bounded by a fixed small number k. In particular, if $k = 1$ every subgroup of the group is either normal or abnormal, and the finite groups with such property were studied by Fattahi [Fa74]. Infinite groups of this type and some of their generalizations were described in [KS03] and [DFKS98]. In this context, groups whose subgroups are either abnormal or subnormal have been considered; we mention also that L.A. Kurdachenko and H. Smith [KSm05] considered groups whose subgroups are either subnormal or self-normalizing. Observe that if in a group the normalizer of any subgroup is abnormal, then the mentioned balanced chains can be chosen of lengths at most 2, and a similar conclusion holds for groups in which every subgroup is abnormal in its normal closure.

If G is a soluble $\overline{T}$-group, then every subgroup of G is abnormal in its normal closure. Notice also that the normalizer of any pronormal subgroup is an abnormal subgroup, so that the property for a subgroup to have an abnormal normalizer is a generalization of pronormality. Kurdachenko, Russo, Subbotin and Vincenzi [KRSV08] proved that (i) *a radical group* G *is a* $\overline{T}$*-group if and only if every cyclic subgroup of* G *is abnormal in its normal closure*, and that (ii) *a periodic soluble group* G *is a* $\overline{T}$*-group if and only if its locally nilpotent residual is abelian and the normalizer of each cyclic subgroup of* G *is abnormal in* G.

In the non-periodic case, there exist non-abelian groups in which the normalizers of all subgroups are abnormal. On the other hand, it is known that non-periodic locally soluble groups in which every subgroup is pronormal are abelian (see [KS87]).

In this context, the following result proved in [KRSV08] is of interest. *Let* G *be a non-periodic group whose locally nilpotent residual* L *is abelian. If the normalizer of every cyclic subgroup of* G *is abnormal and the Sylow* p*-subgroup of* L *have finite exponent for each prime* p*, then* G *is abelian*.

J. Otal, N. Semko and N. Semko Jr. in [OSS13] obtained interesting results concerning groups whose transitively normal subgroups are either normal or self-normalizing. They showed in particular that such groups are almost locally nilpotent ((locally nilpotent)-by-finite), and gave a detailed description of their structure.

For convenience, we will call normal and abnormal subgroups U-*normal*, and groups in which all subgroups are U-normal will be called U-*groups*. Moreover, groups for which U-normality is transitive will be called TU-*groups*, and those in which all subgroups have the TU-property will be said $\overline{TU}$-*groups*. The study of such groups was initiated in [Su92] and was continued in [KS03]. Later, in [KS07] the authors described some soluble groups in which U-subgroups form a lattice.

A subgroup H of a group G is said to be *permutable* (or *quasinormal*) in G, if $HK = KH$ for every subgroup K of G. The study of properties of permutable subgroups is presented in the book [Sc94]. According to a well known theorem of S.E. Stonehewer, permutable subgroups are always ascendant. Thus it is natural to consider the opposite case, to study groups whose ascendant subgroups are permutable. A group G is said to be an AP-*group* if every ascendant subgroup of G is permutable. These groups are very close to those in which the property of being a permutable subgroup is transitive, i.e. groups G such that whenever H is a permutable subgroup of G and K is a permutable subgroup of H, then K is permutable in G. These groups are called PT-*groups*. A description of finite soluble PT-groups has been given by G. Zacher [Za61], which looks close to the description of finite soluble T-groups due to Gaschütz. Namely, if G is a finite soluble group and L is its nilpotent residual, then G has the PT-property if and only if every subgroup of G/L is permutable, $\pi(L) \cap \pi(G/L) = \emptyset$, $2 \notin \pi(L)$ and all subgroups of L are normal in G. Moreover, a finite group G is a PT-group if and only if all its subnormal subgroups are permutable. The structure of infinite soluble PT-groups has been described by F. Menegazzo (see [Me68],[Me69]).

The important paper [BKOP08] is dedicated to the study of infinite AP-groups. In particular, the authors obtained there a detailed description of radical hyperfinite AP-groups.

Let G be a group. A subgroup H of G is called an NE-*subgroup* if $N_G(H) \cap H^G = H$ (see [Li98]). The following characterization of finite $\overline{T}$-groups is interesting: *if every (primary) subgroup of a finite group* G *is an* NE-*subgroup, then* G *is a* $\overline{T}$-*group* (see [Li06]).

Recall also that a subgroup H of a group G is said to be an H-*subgroup* if $N_G(H) \cap H^g \leqslant H$ for all elements $g \in G$. This concept was introduced in [BMHV00], where the authors proved that if every (primary) subgroup of a finite group G is an H-subgroup, then G is a $\overline{T}$-group.

Recall, that a subgroup H of a group G is called *weakly normal* if for each element g of G such that $H^g \leqslant N_G(H)$ we have $g \in N_G(H)$ (see [Mu66]). It turns out that every pronormal subgroup is weakly normal. Moroever, it has been proved that *if all (primary) subgroups of a finite group* G *are weakly normal, then* G *is a* $\overline{T}$-*group* (see [BE03]).

The following result is tightly related to this theme: *if* G *is a finite group whose cyclic subgroup of order* 4 *or a prime are* H-*subgroups, then* G *is supersoluble* (see [CH04]). Notice also that Yangming Li [Li06] proved that the same conclusion holds when all subgroups have the NE-property.

For infinite groups it has been proved that if G is a locally finite group in which every cyclic subgroup of order 4 or a prime is transitively normal, then G is hypercyclic. Moreover, if L is the locally nilpotent residual of G, then L is an abelian Hall subgroup of G

and all subgroups of L are normal in G (see [KO13]). Using this theorem, Kurdachenko and Otal obtained the following more general characterization of $\overline{T}$-groups: *if* G *is a locally finite group whose primary cyclic subgroup are transitively normal, then* G *is a* $\overline{T}$*-group*.

We point out that this result cannot be extended to arbitrary periodic groups, since in any Tarski group all subgroups are transitively normal. The situation is similar in the non-periodic case, since Olshanskii has constructed another sophisticated example of an infinite torsion-free group G whose proper subgroups are cyclic, and clearly every subgroup of G is transitively normal. Moreover, it seems very plausible that every pronormal subgroup of the Grigorchuk's group is normal, while in the group constructed by Olshanskii all pronormal subgroups are abnormal.

Recall that a group G is called *generalized radical* if it has an ascending series whose factors are either locally nilpotent or locally finite. Kurdachenko and Otal [KO13] proved that *if* G *is a non-periodic locally generalized radical group in which every cyclic subgroup of is transitively normal, then either* G *is abelian or* $G = R\langle b\rangle$, *where* R *is abelian,* $b^2 \in R$ *and* $a^b = a^{-1}$ *for each element* $a \in R$. *Moreover, in the second case, the following conditions hold*: (i) *if* $b^2 = 1$, *then the Sylow* 2*-subgroup* D *of* R *is elementary abelian*; (ii) *if* $b^2 \neq 1$, *then either* D *is elementary abelian or* $D = E \times \langle v\rangle$ *where* E *is elementary abelian and* $\langle b, v\rangle$ *is a quaternion group. Conversely, if a group has the above structure, then all its cyclic subgroups are transitively normal*.

We shall say that a subgroup H of a group G is *weakly pronormal* (or that H has the *Frattini property*), if for all subgroups K and L such that $H \leqslant K \leqslant L$ and K is normal in L, we have $L = N_L(H)K$. The inclusion $\langle H, H^x\rangle \leqslant H^{\langle x\rangle}$ shows that every pronormal subgroup is weakly pronormal, so that in particular pronormal subgroups have the Frattini property.

Weakly pronormal subgroups have been characterized in [BB88] in the following way: *a subgroup* H *of a group* G *is weakly pronormal if and only if the* H *and* H^x *conjugate in* $H^{\langle x\rangle}$ *for each element* $x \in G$.

Recall that a group G is called an N-*group* or a group with the *normalizer condition*, if $N_G(H) \neq H$ for every subgroup H of G. Thus a group is an N-group if and only if all its subgroups are ascendant. The following result is a generalization of the one proved in [KOS05] and mentioned above. *Let* G *be a hyper-*N*-group and let* D *be a subgroup of* G. *Then* D *is pronormal in* G *if and only if it has has Frattini property*.

Notice here that this theorem is also a generalization of a similar result proved in [GV01] for hyperabelian groups.

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