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The History of the Divisibility Conjecture for Automorphism Groups of Finite p-Groups

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Abstract

We give a history of the recently settled conjecture that if G is a finite, non-cyclic p-group with $|G| > p^2$, then $|G|$ divides $|\mathrm{Aut}\, G|$.

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1 Introduction

There are many conjectures—famous and not so famous—that have been proved or disproved over the years, even centuries. Some are associated with the mathematicians who formulated them (e.g. Fermat, Riemann, Goldbach, Poincare), some get descriptions (e.g. the classification, the word problem), and some get abbreviations (e.g. the PORC conjecture). Some are "globally" famous (especially when proved) and are written about in the "New York Times" (e.g. Fermat [44], the classification [28]), while others are "locally" famous (e.g. the Segal conjecture in homotopy theory [1]), and that's where ours sits. Here we discuss a 60 year old conjecture in group theory that held some prominent mathematicians in its grip and has finally been put to rest.

Let G be a finite, non-cyclic p-group of order greater than p^2 and let Aut G be its group of automorphisms. The conjecture that the order of G divides the order of Aut G has been in the literature for at least 60 years and was settled in 2015 by J. González-Sánchez and A. Jaikin-Zapirain [34] (while many readers will know how the story ends, we will let the tension build and reveal the conclusion in Subsection 4.4). The conjecture was never attributed to anyone in particular, nor did a name for it appear until 2006 when Bettina Eick referred to it as the "divisibility" conjecture [15]. In this paper we simply refer to it as "The Conjecture" and abbreviate it as **TC**. We give a history of the scholars who chipped away at **TC** and the techniques they used to prove it in certain cases. Only outlines of proofs are included, with some less detailed than others because of the complexity of the methods used.

This paper is organized as follows: Section 2 contains notation and conventions used throughout the paper, Section 3 gives historical context, Section 4 is the main part of the paper where we go through a chronology of **TC**, and the paper concludes in Section 5.

2 Notation and conventions

Commonly used notation and definitions are given below; others will be given in context throughout the paper.

- G is a finite group, and e is its identity.
- $H \leqslant G$ means H is a subgroup of G and $H \lhd G$ means H is a normal subgroup of G; $[G : H]$ is the index of H in G.
- If X is a subset of the group G, then $\langle X \rangle$ denotes the subgroup of G generated by the elements of X.
- p always denotes a prime number.
- $|G|$ is the cardinality of G, and $|G|_p$ is the highest power of p that divides $|G|$.
- $o(x)$ is the order of the element $x \in G$.
- $\exp(P)$ is the exponent of the p-group P.

- $Z(G)$ is the center of G; we will sometimes shorten the notation to Z.
- $[G, G]$ is the commutator subgroup of G: if $x, y \in G$ then $[x, y]$ means $xyx^{-1}y^{-1}$.
- $\Phi(G)$ is the Frattini subgroup of G.
- $\Omega_i(G) = \{x \in G \mid o(x) \mid p^i\}$ and $\mho_i(G) = \{x^{p^i} \mid x \in G\}$.
- The *lower central series* of G is the normal series
$$G = G_0 \rhd G_1 \rhd G_2 \rhd \ldots \rhd G_n \rhd \ldots$$
where $G_i = [G, G_{i-1}]$.
- The *upper central series* of G is the normal series
$$1 = Z_0 \lhd Z_1 \lhd Z_2 \lhd \ldots \lhd Z_n \lhd \ldots$$
where $Z_1 = Z(G)$ and for $i > 1$, Z_i is the unique subgroup of G such that $Z_i/Z_{i-1} = Z(G/Z_{i-1})$. Furthermore, Z_i can be described as
$$Z_i = \{x \in G \mid [x, G] \subseteq Z_{i-1}\}.$$
- A group is *nilpotent* if its upper central series terminates at G. The lower central series of a nilpotent group will terminate at $\{1\}$. The smallest n for which $G_n = \{1\}$ (equivalently $Z_n = G$) is the *nilpotence class* of G. Thus groups of nilpotence class 1 are abelian.
- $\operatorname{Aut} G$ is the group of automorphisms of G; $\operatorname{Inn} G$ is the subgroup of inner automorphisms of G; $\operatorname{Out} G$ is the factor group of outer automorphisms of G.
- $\operatorname{Aut}_c G$ is the group of *central automorphisms* of G, meaning that $\alpha(x)x^{-1} \in Z(G)$ for all $x \in G$.
- If $N \leqslant G$, then $\operatorname{Aut}_N G$ is the subgroup of automorphisms that normalize N, meaning that $\alpha(N) = N$.
- If $M \leqslant G$, then $\operatorname{Aut}^M G$ is the subgroup of automorphisms that centralize M, meaning that $\alpha(m) = m$ for all $m \in M$ (note that if $N \lhd G$ then $\operatorname{Aut}^{G/N} G$ is the subgroup of automorphisms

that induce the identity on G/N, hence satisfy $\alpha(g)g^{-1} \in N$ for all $g \in G$).

- If $N \lhd G$ then

$$\mathrm{Aut}^{N,G/N} G = \mathrm{Aut}^{N} G \cap \mathrm{Aut}^{G/N} G$$

and

$$\mathrm{Aut}_N^{G/N} G = \mathrm{Aut}_N G \cap \mathrm{Aut}^{G/N} G.$$

- If $H \leqslant G$ and $\phi \in \mathrm{Aut}\, H$, then an *extension* of ϕ to G is an automorphism $\hat{\phi} \in \mathrm{Aut}\, G$ such that

$$\hat{\phi}(h) = \phi(h)$$

for all $h \in H$.

We aim to quote theorems in their original form as much as possible, but have changed some notation to match ours for consistency.

3 Historical background: the years 1854–1954

The origin of **TC** is fuzzy but the first paper that addresses it in the given form was by E. Schenkman in 1955 [56], so we will concentrate on the 60 year period 1955–2015. We begin our story, however, in the mid-1800's.

3.1 Early group theory

In his detailed history of group theory, H. Wussing argued that the concept of an abstract group took hold in the 1880's [61]. E. Galois may have been the first to use the word "group", but it was A. Cayley who first conceptualized a group abstractly in 1854 [8]. However, according to Wussing, appreciation of a formal, axiomatic approach to group theory did not gain wide acceptance for nearly another three decades. At the turn of the 20th century, the "postulates" (as they were called at the time) defining a group were not yet honed to perfection, but both finite and infinite groups as we know them now were recognizable (see E. Moore's 1902 paper [49], for example).

Familiar figures such as W. Burnside, R. Dedekind, G. Frobenius, O. Hölder, L. Kronecker, W. von Dyck, and H. Weber worked in the period 1880–1920 and contributed much of the theory that students currently learn in a first abstract algebra course. Other well known figures in group theory such as N.H. Abel, E. Galois, C. Jordan, F. Klein, J.-L. Lagrange, P. Ruffini, and M.L. Sylow did their work prior to the formalization of group theory (as early as 1770 for Lagrange), mostly while studying permutations and algebraic solutions to equations. Their work was eventually generalized to the new, group-theoretic setting. For example, in 1887 Frobenius [29] directly proved for abstract groups Sylow's 1872 theorem that a permutation group of order divisible by p^k has a subgroup of order p^k [58]. Thus, group theory was in its infancy at this time but the structure of finitely generated abelian groups and Sylow's theorems about p-groups were well known.

3.2 Early automorphisms

In 1907 Arthur Ranum wrote "The great importance of the isomorphisms of a given group is largely due to the fact that they enable us to construct new groups of which the given group is an invariant sub-group. However, very little is known about isomorphisms in general, and even when the given group belongs to the simplest and most fundamental class, viz. of abelian (commutative) groups, the group of isomorphisms has not been thoroughly studied except in a few extremely special cases" [53]. Those extremely special cases included cyclic groups (see W. Burnside [7], Sections 168–170), and elementary abelian p-groups (E. Moore [48]). Ranum himself determined the automorphism group (what he termed an "i-group") of a general abelian group of order p^n.

As early as 1909 there was interest in the relationship between $|G|$ and $|\mathrm{Aut}\, G|$. H. Hilton [40] proved that if G is abelian of order p^r then

$$(p-1)p^{r-1} \;\big|\; |\mathrm{Aut}\, G| \;\big|\; (p^r-1)(p^{r-1}-1)\ldots(p-1)^{\frac{1}{2}r(r-1)}.$$

In 1936 Garrett Birkhoff and Philip Hall [4] asked "What can one infer about the order [of Aut G], simply from a knowledge of $|G|$?" They were not concerned with what eventually became **TC**; indeed, they obtained an upper bound on $|\mathrm{Aut}\, G|$ rather than a lower bound and mentioned that the search for lower bounds had been "unattempted".

Setting $|G| = g$, Birkhoff and Hall proved that $|\mathrm{Aut}\, G|$ is a divisor of $g^{r-1}\theta(g)$, where $\theta(g)$ denotes the order of the group of automorphisms of the elementary abelian group of order g, and r is the number of distinct prime factors of g. They used Sylow theory and group actions to prove the theorem, though the language of group actions had not yet been developed.

In 1952 I.N. Herstein and J. Adney [38] proved that if p^2 divides $|G|$ then p divides $|\mathrm{Aut}\, G|$. They first proved the theorem true for abelian groups: if G is cyclic or elementary abelian then $|\mathrm{Aut}\, G|$ is known and divisible by p; otherwise G has a cyclic factor of order at least p^2 and an order p automorphism of the factor can be extended to the whole group. Next they showed by contradiction that if a non-abelian group does not have an order p automorphism it must be a direct product of an abelian Sylow-p subgroup together with its normalizer. Then, as before, they extended an order p automorphism of the Sylow-p subgroup to the whole group.

In 1954 William Scott [57] mentioned that "several authors have given upper bounds on the order [of Aut G] in terms of [|G|]", but Birkhoff and Hall [4] "suggested the problem of determining a lower bound". Hilton did this for abelian groups [40] while Herstein and Adney gave the only general result by this time [38]. Scott improved upon the Herstein-Adney result just slightly by showing if p^3 divides $|G|$ then p^2 divides $|\mathrm{Aut}\, G|$. Taking S to be a Sylow-p subgroup of G of order p^n and letting

$$\mathcal{Z} = S \cap Z(G),$$

Scott showed (i) if $[S : \mathcal{Z}] \geqslant p^2$ then a simple calculation shows

$$|\mathrm{Inn}\, G| \geqslant p^2;$$

(ii) if $[S : \mathcal{Z}] = 1$ then $Z(G)$, which satisfies the theorem by Hilton's result, has a complement in G so any automorphism of $Z(G)$ can be extended to G; and (iii) if $[S : \mathcal{Z}] = p$ then any automorphism α' of S satisfying $\alpha'(s)s^{-1} \in Z(G)$ for all $s \in S$ can be extended to an automorphism of G; such automorphisms together with an inner automorphism generated by $s_0 \in S - Z(G)$ generate at least p^2 members of Aut G.

Although Scott's result appears to be just a small step in the direction of **TC**, his construction of what we now recognize as central automorphisms was prescient. We will see these again in Theorem 4.1 and especially Theorem 4.4. Additionally, Scott's conjecture that there is a function $f(n)$ so that p^n divides $|\mathrm{Aut}\, G|$ whenever $p^{f(n)}$ divides $|G|$ led to an extensive body of work that paralleled the development of **TC**.

4 Chronology of TC

4.1 Building a conjecture: the years 1955–1976

In the 1950's there was a clear interest in the relationship between $|G|$ and $|\mathrm{Aut}\, G|$. The first result that purports to confirm **TC** came in 1955 from E. Schenkman [56], although Schenkman certainly did not recognize his result as the start of a quest that would last for 60 years. He was motivated by a conversation with F. Haimo during which "the question arose as to whether a nilpotent group always possesses an outer automorphism". It turns out that there was a flaw in one of Schenkman's arguments (in Lemma 3), invalidating both his result on outer automorphisms and on **TC**. It was not until 1966 that C. Godino provided a correct proof of the existence of outer automorphisms in finite nilpotent groups of class 2 [33], and not until 1968 that R. Faudree (Schenkman's doctoral student at Purdue University) provided a correct proof of **TC** for finite nilpotent groups of class 2 [23].

Despite Schenkman's error, we recognize him here because his work clearly inspired that of Godino, Faudree, and several others who mimicked his proof technique for the subsequent 20 years.

Theorem 4.1 (Schenkman [56]) *If* G *is a finite non-Abelian group of prime power order whose commutator subgroup is in the center (i.e.* G *is nilpotent of class 2), then the order of* G *divides the order of the group of automorphisms of* G*.*

Before outlining Schenkman's specific ideas, we say something about proving **TC** in general. Obviously the crux is finding enough automorphisms to prove the result. The notions of inner and outer automorphisms were well known by the 1950's so the fact that

$$|G/Z(G)| = |\mathrm{Inn}\, G|$$

meant that one way to prove **TC** was to show

$$|Z(G)| \;\Big|\; [\mathrm{Aut}\, G : \mathrm{Inn}\, G] = |\mathrm{Out}\, G|, \tag{4.1}$$

or, equivalently, one can show $|Z(G)| \leqslant |\mathrm{Out}\, G|_p$ since G is assumed to be a p-group.

During the years 1955–1972, the prevailing method for proving Equation (4.1) was to control $Z(G)$ in some manner (e.g. Schenkman required that $Z(G) \leqslant [G, G]$ in Theorem 4.1; Ree required that

$$\exp(Z(G)) = p$$

in Theorem 4.3; Adney and Yen put other restrictions on the exponent of $Z(G)$ in Theorem 4.4) so that the size of Inn G was known, then consider the role of $Z(G)$ in determining non-inner automorphisms. For example, suppose one can construct $\gamma_x \in \mathrm{Aut}\, G$ for every $x \in Z(G)$. Setting $\Gamma = \langle \gamma_x \mid x \in Z(G) \rangle$, then

$$[\Gamma : \Gamma \cap \mathrm{Inn}\, G] = [\Gamma \cdot \mathrm{Inn}\, G : \mathrm{Inn}\, G] \;\Big|\; [\mathrm{Aut}\, G : \mathrm{Inn}\, G].$$

The smaller $\Gamma \cap \mathrm{Inn}\, G$ is, the closer one is to having $|Z(G)|$ divide

$$[\Gamma : \Gamma \cap \mathrm{Inn}\, G].$$

In particular, central automorphisms commute with all inner automorphisms so in some sense are "usually" not inner themselves. Thus, we will see $\mathrm{Aut}_c G$ playing a big role in proving **TC**. Central and inner automorphisms can be supplemented with additional automorphisms on a case-by-case basis, depending on the structure of G.

Now we go back to Theorem 4.1.

Proof — Schenkman's main technique was to build automorphisms and count them using the following ideas.

Construction 4.2 *Let G be a finite group.*

(i) *If M is a normal subgroup of G so that the coset aM is of order n and generates G/M, and if $z \in M \cap Z(G)$ satisfies $z^n = e$, then the mapping σ defined by the rule $\sigma(ma^r) = ma^r z^r$ is an automorphism of G.*

(ii) *More generally, if every $g \in G$ can be written uniquely as*

$$ca_1^{r_1} a_2^{r_2} \dots a_n^{r_n},$$

where $c \in [G, G]$, then there are conditions under which the homo-

morphism σ_x *defined by*

$$\sigma_x(ca_1^{r_1}a_2^{r_2}\dots a_n^{r_n}) = ca_1^{r_1}a_2^{r_2}\dots(a_i x)^{r_i}\dots a_n^{r_n}, \qquad (4.2)$$

is an automorphism of G, *where* $x \in Z(G)$.

In hindsight, we see that Construction 4.2 leads to central automorphisms, but Schenkman did not mention this. Nor did Schenkman list the conditions in Construction 4.2 (ii) correctly, but it turns out that one must have

$$G = \langle a_1, \dots, a_i x, \dots, a_n \rangle,$$

and $o(a_i x) = o(a_i)$ (see [23]). Nonetheless, Schenkman's procedure was to use Construction 4.2 several times to build automorphisms of G, the number of which he tried to track based on the orders of the generators of $[G, G]$ and on the orders of the corresponding direct factors of $G/[G, G]$. □

TC was nowhere to be seen in Schenkman's paper, but Scott's paper was listed in the references (although there is no mention of Scott's paper in the body of Schenkman's paper) so Schenkman was clearly aware of the general interest in relating $|G|$ to $|\mathrm{Aut}\, G|$. Because he was ultimately interested in outer automorphisms he skipped over class 1 p-groups (i.e. abelian p-groups) and went straight to class 2. We will soon see that A. Otto addressed the abelian case.

R. Ree (probably most famous for constructing the "Ree" groups: finite simple groups of Lie type over a finite field) published two papers on the topic of group automorphisms in the 1950's. In the first, he began by referencing the quest of Haimo and Schenkman to find outer automorphisms of nilpotent groups, and in 1956 he proved: *If* G *is* p*-group of exponent* p *and order greater than* 2, *then* G *has an outer automorphism* [54]. In his 1958 follow-up paper, Ree's sole purpose was to prove **TC** for p-groups of exponent p using his knowledge of Out G from the earlier paper.

Theorem 4.3 (Ree [55]) *If* G *is a finite* p*-group every element of which satisfies the equation* $x^p = e$, *and if* G *is of order greater than* p^2, *then the order of* G *divides the order of the group of automorphisms of* G.

Proof — When G is non-abelian there is a maximal normal subgroup $N \lhd G$ such that G/N is cyclic of order p, generated by aN, and $Z(G) \leqslant Z(N)$. It is routine to see that $\phi : Z(N) \to Z(N)$ defined by $\phi(x) = [x, a]$ is a homomorphism with $Z(G) \leqslant \operatorname{Ker}\phi$. Then,

$$|Z(G)| \;\Big|\; |\operatorname{Ker}\phi| = [Z(N) : \operatorname{Im}\phi].$$

Since all non-trivial elements have order p, Ree's version of Construction 4.2 (i) showed that for each $x \in Z(N)$, $\sigma_x(na^r) = na^r x^r$ defines an element of $\operatorname{Aut} G$. Ree proved that the associated homomorphism $\sigma : Z(N) \to \operatorname{Aut} G$ is injective and $\sigma(Z(N)) \cap \operatorname{Inn} G = \sigma(\operatorname{Im}\phi)$. Thus

$$\begin{aligned}[Z(N) : \operatorname{Im}\phi] &= [\sigma(Z(N)) : \sigma(\operatorname{Im}\phi)] \\ &= [\sigma(Z(N)) : \sigma(Z(N)) \cap \operatorname{Inn} G] \\ &= [\sigma(Z(N)) \cdot \operatorname{Inn} G : \operatorname{Inn} G].\end{aligned}$$

Since

$$|Z(G)| \;\Big|\; [Z(N) : \operatorname{Im}\phi]$$

and

$$[\sigma(Z(N)) \cdot \operatorname{Inn} G : \operatorname{Inn} G] \;\Big|\; [\operatorname{Aut} G : \operatorname{Inn} G],$$

we see that $|Z(G)|$ divides $[\operatorname{Aut} G : \operatorname{Inn} G]$, so by Equation 4.1 **TC** is proved in this case.

When G is abelian of order p^d, then

$$|\operatorname{Aut} G| = p^{\binom{d}{2}} \prod_{i=0}^{d-1} (p^{d-i} - 1).$$

Since $d \geqslant 3$, we see that $|G| \;\big|\; |\operatorname{Aut} G|$ in the (elementary) abelian case too. □

While Ree's 1958 paper was focused on what became **TC**, his original motivation was a question about outer automorphisms. By contrast, in 1965 Adney and Yen stated that their (only) purpose was "to investigate the relationship between the order of G and the order of [Aut G] under certain circumstances".

Adney and Yen were unable to prove **TC** for all p-groups of class 2, but did prove it for some purely non-abelian p-groups of class 2. Such PN *groups*, as they are known, have no abelian direct factors.

Theorem 4.4 (Adney and Yen [2]) *The order* $|G|$ *divides* $|\operatorname{Aut} G|$ *if* G *is a purely non-abelian* p*-group of class* 2*,* p *odd, satisfying one of the following* *conditions:*

(i) Z *is cyclic;*

(ii) $\exp(Z) = \exp([G, G])$*;*

(iii) $\exp(Z) \geqslant \exp(G/[G, G])$*;*

(iv) $\mathrm{Aut}_c G$ *is abelian.*

Although their work was later eclipsed by Faudree, whose Theorem 4.9 shows that **TC** holds for all non-abelian p-groups of class 2, Adney and Yen brought to light the importance of using central automorphisms to prove **TC**.

Proof — Every central automorphism α determines a homomorphism $f_\alpha : G \to Z(G)$ defined by $f_\alpha(x) = \alpha(x)x^{-1}$ so there is an injective homomorphism $\mathrm{Aut}_c G \to \mathrm{Hom}(G, Z(G))$. Adney and Yen proved the oft-cited theorem that the injection is a bijection if and only if G is PN. Furthermore, when G is a PN p-group, $\mathrm{Aut}_c G$ is a p-group.

Under the first three conditions in the theorem, Adney and Yen decomposed the center as $Z(G) = A \times B$ where $\exp A > \exp B = p^m$, then used the structure of $Z(G)$ in relation to $[G, G]$ to ultimately compute the bound

$$|\mathrm{Hom}(G, Z(G))| \geqslant \frac{|G|}{p^{b-m}},$$

where $p^b = \exp[G, G]$.

In case $|G| > |\mathrm{Hom}(G, Z(G))|$, they constructed enough non-central p-power automorphisms to prove **TC** using the following decomposition of G and subsequent construction of automorphisms of G.

Construction 4.5 *Suppose*

(i) $[G, G] = \langle u \rangle \times U$*, where* $o(u) = \exp[G, G] > p^m \geqslant \exp U$*,*

(ii) $[g, h] = u$ *and* $h^{p^{b+m}} = e$*.*

Let $H = \langle g, h \rangle$ *and* $L = \{x \in G \mid [g, x], [h, x] \in U\}$*. Then* $G = HL$ *and* σ_k *defined by*

$$\sigma_k(g^s h^t x) = (gh^{p^k})^s h^t x$$

is in $\mathrm{Aut}^Z G$ *when* $k \geqslant m$*. Furthermore,* $[\Sigma : \Sigma \cap \mathrm{Aut}_c G] = p^{b-m}$ *where* Σ *is generated by the* σ_k*.*

By choosing g and h carefully, Adney and Yen showed

$$|\Sigma \cdot \mathrm{Aut}_c G| = [\Sigma : \Sigma \cap \mathrm{Aut}_c G]|\mathrm{Aut}_c G| \geqslant p^{b-m} \frac{|G|}{p^{b-m}} = |G|.$$

It was quite a bit more difficult to prove the theorem under the fourth condition, but the basic idea was the same: determine a lower bound for $|\mathrm{Aut}_c G|$ and make up the difference with Construction 4.5. □

In 1966 Otto [51] noted "In recent years there has been an increased interest in the relationship between the order of a finite group G and the order of the automorphism group [Aut G] of G". He was the first to publish a proof that if the p-group G is abelian then $|G|$ divides $|\mathrm{Aut}\, G|$ if and only if G is non-cyclic of order greater than p^2. His most important contribution to **TC** was a reduction in the problem: the theorem below shows that to prove **TC** it suffices to prove it for PN-groups.

Theorem 4.6 (Otto [51]) *If the* p*-group* G *is the direct product* $P \times B$ *of the two subgroups* P *and* B *where* P *is abelian of order* p^r *and* B *is a* PN*-group, then* $p^r \cdot |\mathrm{Aut}\, B|_p$ *divides* $|\mathrm{Aut}\, G|$.

Proof — Let $T = \mathrm{Aut}(P) \times \mathrm{Aut}(B)$ then $|T|_p = |\mathrm{Aut}(P)|_p|\mathrm{Aut}(B)|_p$ and every pair in T determines a unique automorphism of G. If P is neither cyclic nor of order p^2, then $|P| \bigm| |\mathrm{Aut}(P)|_p$, so

$$p^r \cdot |\mathrm{Aut}\, B|_p \bigm| |T|_p \bigm| |\mathrm{Aut}\, G|.$$

If P is either cyclic or of order p^2, then Otto focused on counting central automorphisms. Specifically, since $|\mathrm{Aut}\, P|_p = p^{r-1}$ he determined

$$|T \cdot \mathrm{Aut}_c G|_p = \frac{|T|_p |\mathrm{Aut}_c G|_p}{|T \cap \mathrm{Aut}_c G|_p} = \frac{p^{r-1}|\mathrm{Aut}\, B|_p |\mathrm{Aut}_c G|_p}{p^{r-1}|\mathrm{Aut}_c B|_p}.$$

As long as

$$|\mathrm{Aut}_c G|_p > p^{r-1}|\mathrm{Aut}_c B|_p \tag{4.3}$$

then

$$p^r \cdot |\mathrm{Aut}\, B|_p \bigm| |T \cdot \mathrm{Aut}_c G|_p \bigm| |\mathrm{Aut}\, G|.$$

A careful use of Fitting's formula [25] for the number of central automorphisms of B and of G proved Equation (4.3). □

The work of Adney and Yen not only alerted Otto to the idea of decomposing G as a direct product of an abelian factor with a PN factor, but also gave him a launching point by having shown that $\mathrm{Aut}_c G$ is a p-group. We will see that this fact lived on, and played a role in nearly every other proof of the **TC**.

Otto further proved **TC** holds for p-groups of maximal class and order at least p^4, and for certain PN groups. In doing so, he obtained bounds on the order of $\mathrm{Aut}_c G$ for PN groups, showing "the influence of the center and commutator factor group".

Also published in 1966 were Godino's correction of Schenkman's attempt to prove the existence of outer automorphisms for all finite nilpotent groups of class 2 [33], and Gaschütz' famous theorem that translates from the original German to the following theorem.

Theorem 4.7 (Gaschütz [30]) *Every finite non-abelian* p*-group has an outer automorphism of* p*-power order.*

Gäschutz was neither motivated by nor even mentioned **TC**, but we acknowledge his contribution here for two reasons: first, his result is a highly cited and important result in group theory; second, it established the following case of **TC**.

Theorem 4.8 *If* G *is a non-cyclic* p*-group of order* p^2 *and* $|Z(G)| = p$, *then* $|G| \mid |\mathrm{Aut}\, G|$.

Proof — By Gaschütz' theorem, $p \mid |\mathrm{Out}\, G|$, hence $p|\mathrm{Inn}\, G| \mid |\mathrm{Aut}\, G|$, where $p|\mathrm{Inn}\, G| = |Z(G)||G/Z(G)| = |G|$. □

The 1960's ended with Faudree's correction of Schenkman's Theorem 4.1.

Theorem 4.9 (Faudree [23]) *If* G *is a finite nonabelian nilpotent class two* p*-group, then the order of* G *divides the order of* Aut G.

Proof — Faudree began by determining the conditions under which Construction 4.2 (ii) led to central automorphisms of G. He knew the number of such automorphisms and let them generate a subgroup T of Aut G. The order of T did not suffice, as Schenkman originally claimed, so Faudree refined Construction 4.5 just enough so that he could squeeze out a few more non-central automorphisms. Where Adney and Yen concentrated on the index

$$[\Sigma : \Sigma \cap \mathrm{Aut}_c G],$$

Faudree instead constructed five individual automorphisms of G,

$$f_1, \ldots, f_5,$$

with specific orders, then proved

$$|G| \bigm| |VT| \bigm| |\mathrm{Aut}\, G|,$$

where $V = \langle f_1, \ldots, f_5 \rangle$. □

Faudree only published a few papers related to group theory, and just the one related to **TC**, but we would be remiss if we did not mention that he was a powerhouse in combinatorics, which certainly helped him count automorphisms. According to MathSciNet, Faudree published 254 papers, 244 of which are classified as "combinatorics" and 50 of which were coauthored with Paul Erdös.

The 1970's saw the most action of any decade regarding **TC**. Of the eight papers directly addressing **TC**, four of them involved R. Davitt, who was Otto's doctoral student at Lehigh University. Even though Otto did not explicitly state or propose **TC**, he certainly understood that it was true in many cases and he likely encouraged Davitt to dive in. Between the two of them, they wrote six papers addressing **TC** between 1966 and 1980. Of the four published in the 1970's, two were written jointly and two were single-authored by Davitt.

Davitt opened up the 70's recognizing the development of **TC** in [9]. The first lines of his paper stated "It is well known that if G is a finite noncyclic Abelian p-group of order greater than p^2, then the order $|G|$ of G divides the order of the automorphism group [Aut G] of G. This result has recently been extended to other classes of finite p-groups". He went on to say that the purpose of his paper was to prove the following theorem about metacyclic p-groups, where G is *metacyclic* if it is an extension of a cyclic group by a cyclic group.

Theorem 4.10 (Davitt [9]) *If* $p \neq 2$ *and* G *is a non-cyclic metacyclic* p*-group of order greater than* p^2*, then* $|G|$ *divides* $|\mathrm{Aut}\, G|$*.*

Proof — Davitt's paper began with the construction of automorphisms of a general p-group G in a manner reminiscent of Construction 4.2 (i). First he showed that if G has a normal subgroup K so that G/K is cyclic, generated by aK, and if $e \neq a^{p^m} \in \Omega_n(Z(G))$ the mapping θ defined by

$$\theta(a^j k) = a^{j(p^m+1)}k$$

is in $\mathrm{Aut}^K G$ and has order $\frac{o(a)}{p^m}$. Second, if G is assumed to be regular (i.e. for all $a, b \in G$,

$$(ab)^p = a^p b^p c^p$$

for some $c \in [\langle a, b\rangle, \langle a, b\rangle]$) then the hypotheses can be loosened a bit: if $x \in \Omega_n(Z(K))$, then the mapping ϕ, defined by

$$\phi(a^j k) = (ax)^j k$$

is in $\mathrm{Aut}^K G$ and has order $o(x)$.

Since G is metacyclic in this theorem (and regular when $p > 2$), one can choose K to be cyclic then construct θ and ϕ, but Davitt used both cyclic and non-cyclic subgroups, depending on the specific metacyclic structure of G, to build automorphisms of particular orders so that subgroups of $\mathrm{Aut}\, G$ of the form $S = \langle\theta, \phi, \mathrm{Inn}\, G\rangle$ had order equaling $|G|$. □

Building off of Theorem 4.10, Davitt and Otto jointly proved the following generalization in 1971.

Theorem 4.11 (Davitt and Otto [12]) *If* G *is a finite* p*-group* ($p > 2$) *such that* $G/Z(G)$ *is metacyclic and nontrivial, then* $|G|$ *divides* $|\mathrm{Aut}\, G|$*.*

Proof — When $G/Z(G)$ is metacyclic it has a cyclic normal subgroup $\langle bZ\rangle$, for some $b \in G$, and corresponding cyclic factor group. The abelian subgroup $M = \langle b, Z\rangle$ is normal in G and has cyclic quotient. Since G can be assumed to have nilpotency class greater than 2 (by Theorem 4.9) and was proved to be regular, Davitt and Otto were able to compute the size of

$$|G/M||M/Z| = |G/Z| = |\mathrm{Inn}\, G|.$$

Since G can be assumed to be PN (by Theorem 4.4) we know

$$|\mathrm{Aut}_c G| = |\mathrm{Hom}(G, Z)| = |\mathrm{Hom}(G/[G, G]), Z)|.$$

The authors computed a lower bound for $|\mathrm{Hom}(G/[G, G]), Z)|$ in a manner similar to the original one used by Adney and Yen. They also computed a lower bound for $|R|$ where $R = \mathrm{Inn}\, G \cdot \mathrm{Aut}_c G$ because they were able to compute

$$|\mathrm{Inn}\, G \cap \mathrm{Aut}_c G| = |Z_2/Z|.$$

Finally, using one of Davitt's automorphism constructions from [9]—which, again, was a version of Construction 4.2 (i)—the authors built different subgroups of Aut G depending on the metacyclic structure of G/Z. These subgroups are of the form $\langle \theta, R \rangle$ and have order divisible by $|G|$. □

The first time **TC** appeared in print was in 1972 when Davitt wrote "It is natural to conjecture that if G is a finite non-cyclic p-group of order greater than p^2, then [$|G|$] divides [$|\mathrm{Aut}\, G|$]" [10]. The explicit statement showed that mathematicians were growing in confidence about the veracity of **TC**. In the 1972 paper, Davitt used his experience with regular p-groups from his previous two papers to study p-abelian p-groups, where a group G is n-*abelian*, for some integer n, if $(xy)^n = x^n y^n$ for all $x, y \in G$.

Theorem 4.12 (Davitt [10]) *If* G *is a non-cyclic* p*-Abelian* p*-group of order greater than* p^2*, then* $|G| \bigm| |\mathrm{Aut}\, G|$.

Proof — As in [9], Davitt began this paper by constructing automorphisms of G, whose orders he could compute. Finding inspiration from Adney and Yen—who used a factorization of $[G, G]$ to decompose G, then build automorphisms of G in Construction 4.5—Davitt used a factorization of $\mho_1(G)$ to decompose G, then build automorphisms of G. In particular, if $\mho_1(G) = \langle a^p \rangle \times M$ for $a \in G$ of order p^{n+1} and $L = \{x \in G \mid x^p \in M\}$, then $G = \langle a \rangle L$. The mapping σ, defined by $\sigma(a^k l) = a^{k(p+1)} l$ is in $\mathrm{Aut}^L G$ and has order p^n; the mapping ϕ_x, defined by $\phi_x(a^k l) = (ax)^k l$, is also in $\mathrm{Aut}^L G$ and has order $o(x)$ where $x \in \Omega_n(Z(L))$.

Letting $R = \langle \sigma \rangle$ and $S = \{\phi_x \mid x \in \Omega_n(Z(L))\}$, Davitt proved

$$R \cdot S \leqslant \mathrm{Aut}^{\Omega_1(Z(G))} G \leqslant \mathrm{Aut}\, G.$$

By a result of C. Hobby, when G is p-abelian $\mho_1(G) \leqslant Z(G)$ [41]. In a series of technical lemmas and eventually a proof by induction on the number of factors in the abelian group $\mho_1(G)$, Davitt finally showed

$$|G| \bigm| |V| \bigm| |\mathrm{Aut}^{\Omega_1(Z(G))} G| \bigm| |\mathrm{Aut}\, G|,$$

where V is essentially $R \cdot S \cdot \operatorname{Inn} G$. □

Davitt credited Ree for some of the techniques used to prove Theorem 4.12, and certainly we can see that Ree's Theorem 4.3 is a corollary of Davitt's more general result since groups of exponent p are trivially p-abelian.

Also published in 1972 was the final joint paper of Davitt and Otto, and Otto's last publication entirely. The authors once again focused attention on **TC** rather than the more nebulous "relationship" between $|G|$ and $|\operatorname{Aut} G|$. They wrote that "many [recent papers] have shown that for a certain class of finite p-groups, the order of the group divides the order of the automorphism group". Combining techniques from their previous three joint and single-authored papers, they proved the following theorem about modular p-groups, where a group G is *modular* (also known as *Iwasawa*) if its lattice of subgroups is modular.

Theorem 4.13 (Davitt and Otto [13]) *If G is a finite non-Abelian modular p-group ($p > 2$), then $|G|$ divides $|\operatorname{Aut} G|$.*

Proof — Once again, the authors used a series of detailed computations to obtain a bound on the order of a p-subgroup of $\operatorname{Aut} G$, namely $R = \operatorname{Aut}_c G \cdot \operatorname{Inn} G$. In this paper, they were aided by K. Iwasawa's result [43] that a modular p-group has an abelian normal subgroup A with corresponding cyclic factor group G/A (Iwasawa's original proof had some gaps that were filled in by F. Napolitani and Z. Janko—see [3] for a discussion). Interestingly, showing $|R| \geqslant |G|$ depended only on the power structure of G, A, and $Z(G)$; it was not necessary to supplement R with other automorphisms. □

We will come back to Davitt one last time in Subsection 4.2, but now turn our attention to a new approach to proving **TC** that came in the mid-70's based on a classification scheme by Philip Hall.

Hall spent a lifetime advancing the theory of algebra, particularly in the area of p-groups. In 1934 he wrote "The pages which follow represent the first stages of an attempt to construct a systematic general theory of groups of prime-power order. It is widely recognised, I believe, that the astonishing multiplicity and variety of these groups is one of the main difficulties which beset the advance of finite-group-theory" [36]. In 1940 he added "it seems unlikely that it will be possible to compass the overwhelming variety of prime-power groups within the bounds of a single finite system of formulae" [37]. Nevertheless, he developed a new system of classifying p-groups via the notion of isoclinicism.

Two groups G_1 and G_2 are *isoclinic* if there are isomorphisms

$$\alpha : G_1/Z(G_1) \to G_2/Z(G_2) \text{ and } \beta : [G_1, G_1] \to [G_2, G_2]$$

so that whenever $\alpha(x_1 Z(G_1)) = x_2 Z(G_2)$ and $\alpha(y_1 Z(G_1)) = y_2 Z(G_2)$, for $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$, then

$$\beta([x_1, y_1]) = [x_2, y_2].$$

Hall proved that G is isoclinic with a subgroup N if and only if $G = NZ(G)$.

In 1975, K. Hummel proved the next theorem, and the following year J. Buckley—who was Hummel's doctoral advisor at Western Michigan University—put the result in the context of isoclinicism.

Theorem 4.14 (Hummel [42]) *If a* p*-group* G *is the central product of nontrivial subgroups* H *and* A*, where* A *is abelian and* $|H| \bigm| |\mathrm{Aut}\, H|$*, then* $|G| \bigm| |\mathrm{Aut}\, G|$.

Proof — As we have come to expect, Hummel began by considering a maximal normal subgroup H in G so that G/H is cyclic, generated by aH, where $a \in Z(G)$. He proved a variation on Construction 4.2 (i), showing $\alpha \in \mathrm{Aut}\, H$ extends to $\hat{\alpha} \in \mathrm{Aut}\, G$ if and only if

$$\alpha(a^p) = a^{sp} h_0^p,$$

where $0 < s < p$ and $h_0 \in Z(H)$; in which case $\hat{\alpha}$ is defined by

$$\hat{\alpha}(a^k h) = a^{sk} h_0^k \alpha(h).$$

Still assuming G/H is cyclic, the heart of Hummel's argument showed that if $|H|$ divides $|\mathrm{Aut}\, H|$, then $|G|$ divides $|\mathrm{Aut}\, G|$. Rather than counting central and inner automorphisms as previous authors had done, Hummel used a clever group action argument. He recognized that the set E of extendable automorphisms of H is the stabilizer of the cyclic subgroup generated by a power of

$$\bar{a} \in Z(H)/\mho_1(Z(H))$$

under an action of $\mathrm{Aut}\, H$ on the set X of all cyclic subgroups of $Z(H)/\mho_1(Z(H))$. An upper bound on the index of the stabilizer is given by $|X|$. Since $Z(H)/\mho_1(Z(H))$ is elementary abelian of rank r, we know that

$$|X| = \frac{p^r - 1}{p - 1} < p^r.$$

Hence $[\operatorname{Aut} H : E] < p^r$ and we see that $|\operatorname{Aut} H|_p$ divides $p^{r-1}|E|_p$. Since it is assumed that $|H| \mid |\operatorname{Aut} H|$, we know

$$|G| = p|H| \mid p^r|E|_p$$

and it suffices to show that $p^r|E|_p$ divides $|\operatorname{Aut}_H G|$.

Consider the restriction map

$$\rho : \operatorname{Aut}_H G \to \operatorname{Aut} H.$$

Clearly $\operatorname{Im} \rho = E$ and Hummel showed $|\operatorname{Ker} \rho| = p^r$. Thus

$$p^r|E|_p \mid |\operatorname{Aut}_H G|$$

and the result is proved when G/H is cyclic.

The general theorem is now easily proved by induction on n, where $[A : H \cap A] = p^n$. □

A direct product is a central product so Otto's Theorem 4.6 is a special case of the theorem above.

Hummel noted that G is the central product of nontrivial subgroups H and A, where A is abelian, if and only if

$$Z(G) \nleqslant \Phi(G),$$

hence to prove **TC** one need only consider p-groups satisfying $Z(G) \leqslant \Phi(G)$.

Under Hummel's hypothesis, G and H are isoclinic. In 1976, Buckley (who credited P. Weichsel for the inspiration) proved the dual case, establishing **TC** when G is isoclinic to G/H and $|G/H|$ divides $|\operatorname{Aut} G/H|$. Buckley introduced new cohomological techniques to the process of establishing **TC**. We include his contribution below, but note that the paper in which it appears has had consequences well beyond the scope of **TC**. In particular, a generalization of Hummel's work identifying the image of a homomorphism in the Wells Exact Sequence [60] as the stabilizer of a certain action had far reaching consequences (see [14] for survey of this).

Theorem 4.15 (Buckley [5]) *Let* G *be a finite* p*-group and* $1 < N < G$.

(i) *If* G *and* N *are isoclinic and* $|N| \mid |\mathrm{Aut}(N)|$ *then* $|G| \mid |\mathrm{Aut}_N G|$.

(ii) *If* $N \lhd G$, G *and* G/N *are isoclinic, and* $|G/N| \mid |\mathrm{Aut}(G/N)|$ *then* $|G| \mid |\mathrm{Aut}_N G|$.

Proof — Let G be an extension of N by Q so that there is an exact sequence

$$\mathbf{e} : 1 \to N \xrightarrow{\mu} G \xrightarrow{\epsilon} Q \to 1$$

with μ injective, ϵ surjective, and $\mathrm{Im}\,\mu = \mathrm{Ker}\,\epsilon$. Such an extension is associated with a coupling $\chi : Q \to \mathrm{Out}\,N$. An automorphism of the extension $\mathbf{e}$ is an element of $\mathrm{Aut}_N G$. Every $\gamma \in \mathrm{Aut}_N G$ induces automorphisms $\overline{\gamma} \in \mathrm{Aut}\,Q$ and $\gamma|_N \in \mathrm{Aut}\,N$. Thus there is a homomorphism

$$\rho : \mathrm{Aut}_N G \to \mathrm{Aut}\,Q \times \mathrm{Aut}\,N.$$

The image of ρ is called the group of *inducible* pairs. To understand $\mathrm{Im}\,\rho$, Wells introduced the notion that $(\sigma, \tau) \in \mathrm{Aut}\,Q \times \mathrm{Aut}\,N$ is a *compatible* pair if the diagram below commutes

$$\begin{array}{ccc} Q & \xrightarrow{\sigma} & Q \\ \chi \downarrow & & \downarrow \chi \\ \mathrm{Out}\,N & \xrightarrow{c_{\overline{\tau}}} & \mathrm{Out}\,N \end{array}$$

where $c_{\overline{\tau}}$ is the inner automorphism of $\mathrm{Out}\,N$ induced by $\overline{\tau} = \tau\mathrm{Inn}\,N$. Denote the subgroup of $\mathrm{Aut}\,Q \times \mathrm{Aut}\,N$ consisting of all compatible pairs by $\mathrm{Comp}(\chi)$.

Wells proved that there is an exact sequence

$$0 \to \mathrm{Der}(Q, Z(N)) \to \mathrm{Aut}_N G \xrightarrow{\rho} \mathrm{Comp}(\chi) \xrightarrow{\omega} H^2(Q, Z(N)) \qquad (4.4)$$

where ω is a set map and $\mathrm{Der}(Q, Z(N))$ is the group of derivations

$$\delta : Q \to Z(N)$$

satisfying $\delta(q_1 q_2) = \delta(q_1)^{q_2} + \delta(q_2)$ for all $q_1, q_2 \in Q$.

Buckley improved upon Wells' exact sequence by showing that $\mathrm{Comp}(\chi)$ acts on the set of equivalence classes of extensions of N by Q that have coupling χ, denoted $\mathcal{E}_\chi(Q, N)$, and that $\operatorname{Im}\rho$ is the stabilizer of the class of $[\mathbf{e}]$ under this action.

When G and N are isoclinic, $Q = G/N$ acts trivially on $Z(N)$ so

$$|\operatorname{Ker}\rho| = |\mathrm{Der}(Q, Z(N))| = |\mathrm{Hom}(Q, Z(N))| = p^m \tag{4.5}$$

for some $m \geqslant 1$. Buckley showed that $[\mathbf{e}]$ belongs to an invariant subset of order p^m, thus $\operatorname{Im}\rho$ can be viewed as the stabilizer of $[\mathbf{e}]$ under the action of $\operatorname{Aut} Q \times \operatorname{Aut} N$ on a set of size p^m. As long as $\mathbf{e}$ is non-trivial, we see that

$$[\operatorname{Aut} Q \times \operatorname{Aut} N : \operatorname{Im}\rho] < p^m$$

so $|\mathrm{Aut}_N G|_p = |\operatorname{Ker}\rho| \cdot |\operatorname{Im}\rho|_p \geqslant p|\operatorname{Aut} Q|_p \cdot |\operatorname{Aut} N|_p$. Furthermore, G/N is abelian so $|G/N| \mid p|\operatorname{Aut} G/N|_p$. Thus if $|N| \mid |\operatorname{Aut} N|$, then $|G| \mid |\mathrm{Aut}_N G|$ and **TC** is proved when G and N are isoclinic.

Similarly, when G and G/N are isoclinic, N is abelian so

$$|N| \mid p|\operatorname{Aut} N|_p.$$

If $|G/N| \mid |\operatorname{Aut} G/N|$, then $|G| \mid |\mathrm{Aut}_N G|$ and **TC** is proved. □

Where Hummel reduced the task of proving **TC** to considering p-groups G satisfying $Z(G) \leqslant \Phi(G)$, Buckley's theorem further reduced the problem to one of considering p-groups G satisfying $\Omega_1(Z(G)) \leqslant [G, G]$.

We end this section with a collection of highlights in chronological order.

Year	**Author**	**Result**
1955	Schenkman	The first result related to **TC**
1966	Otto	To prove **TC** need only consider PN groups
1968	Faudree	To prove **TC** need only consider groups of nilpotence class > 2
1972	Davitt	First mention of **TC** in print
1976	Buckley	To prove **TC** need only consider groups with $\Omega_1(Z(G)) \leqslant [G, G]$

Table 4.1: Highlights 1955-1976

4.2 The lean years: 1977–2005

The build-up to **TC** was fairly intense with about a dozen authors contributing to it between 1955 and 1976, but through the 1980's just one new mathematician published work related to **TC** and only two papers appeared in the 1990's. The theorems proved in this time period chipped away at **TC**, proving that it held in more and more cases, but none reduced the problem in any significant way.

A first example of this more case-by-case analysis of **TC** came from Davitt in 1980, and was his final contribution to the problem.

Theorem 4.16 (Davitt [11]) *If* G *is a finite non-cyclic* p*-group of order greater than* p^2 *such that* $|G : Z| \leqslant p^4$ *then* $|G|$ *divides* $|\mathrm{Aut}\, G|$.

Proof — Davitt first assumed that a group G not only satisfies the hypotheses of the theorem, but its nilpotency class is greater than 2 by Theorem 4.9, $Z(G) \leqslant \Phi(G)$ by Theorem 4.14, G is PN by Theorem 4.6, G is not p-abelian by Theorem 4.12, and $|Z(G)| > p$ by Theorem 4.8. Under all these conditions, he proved that if G is regular then $G/Z(G)$ is metacyclic, hence G satisfies **TC** by Theorem 4.11. This established the theorem for all primes $p \geqslant 5$, thus more than half of the paper dealt with irregular 2- and 3-groups.

Letting $R = \mathrm{Inn}\, G \cdot \mathrm{Aut}_c G$, Davitt proved $|R| \geqslant |G|$ by carefully analyzing the structure of G, even relying on the Hall-Senior tables [35] in the $p = 2$ case. He neither built additional automorphisms, as in some of the previous theorems, nor used cohomological methods. □

As an immediate corollary, **TC** holds for all finite non-cyclic p-groups G with $p^3 \leqslant |G| \leqslant p^5$.

T. Exarchakos extended Theorem 4.16 to all finite non-cyclic p-groups G with $p^3 \leqslant |G| \leqslant p^6$ in [21], and N. Gavioli extended it to $p^3 \leqslant |G| \leqslant p^7$ in [31]. J. Flynn et al. concentrated on 2-groups, and extended the results to $2^3 \leqslant |G| \leqslant 2^9$ in [26].

Exarchakos was quite active in the 1980's (actually beginning in 1979). He essentially split his time between improving the bounds for the "Scott" function $f(n)$ mentioned at the end of Section 2 ([17], [18],[6]) and proving **TC** for several more classes of groups. We highlight some of the latter results next, where Exarchakos calls G an "LA" (large automorphism) group if it satisfies **TC**.

Theorem 4.17 (Exarchakos)

(i) (Cor 1.1, [19]) *If the Frattini subgroup* $\Phi(G)$ *of* G *is cyclic, then* G *is an LA-group.*

(ii) (Thm 2, [19]) *Let* G *have a normal subgroup* M *which has maximal class. Then* G *is an LA-group, if either,*

(a) G/M *is elementary abelian, or*

(b) M *has index* p^2 *in* G.

(iii) (Thm 3, [19]) *Let* M *be a maximal subgroup of G. If* M *contains a normal subgroup* H *of order* p *such that* M/H *is of maximal class, then* G *is an LA-group.*

(iv) (Thm 1, [20]) *Let* G *be a finite group of order* p^n *and class* c. *If* G_i/G_{i+1} *is cyclic of order* p^r *for all* $i = 1, 2, \ldots, c-1$ *and*

$$\exp(G/[G,G]) = p^r,$$

then $|G|$ *divides* $|\mathrm{Aut}\,G|$.

(v) ([21]) *If* G *is a finite non-abelian group of order* p^n, p *a prime number and* $n \leqslant 6$, *then* $|\mathrm{Aut}\,G|_p \geqslant |G|$.

Proof — The bulk of Exarchakos' work in the three papers [19], [20], and [21] involved an analysis of $|\mathrm{Aut}_c G|$. Since G can be assumed to be PN,

$$|\mathrm{Aut}_c G| = |\mathrm{Hom}(G/[G,G], Z(G))|.$$

The various hypotheses gave Exarchakos control over the invariants in the decompositions for the abelian groups $G/[G,G]$ and $Z(G)$ so that he could compute lower bounds for $|\mathrm{Aut}_c G|$ and $|\mathrm{Inn}\,G|$. Then he showed $|G|$ divides $|\mathrm{Aut}_c G \cdot \mathrm{Inn}\,G|$. □

TC appeared in the Kourovka Notebook [47] for the first time in 1992, but other than the contributions of Gavioli in 1993 and Flynn et al. in 1994, there was a lull in attention to **TC** until the mid-2000's.

4.3 Almost all p-groups satisfy TC: the years 2006–2014

Thus far we have seen attempts to prove **TC** based on the order of G, on the nilpotence class of G, and on the isoclinicism family of G. Each of these classifications took **TC** one step closer to being proved, but none settled the problem. A new tactic occurred in 2006 when B. Eick used coclass theory to prove that almost all 2-groups satisfy **TC**.

If P is a p-group, then the *coclass* of P is $n - c$ where $|P| = p^n$ and P has nilpotence class c. Coclass theory was introduced by

C. Leedham-Green and M. Newman in 1980 as a way of classifying p-groups [46]. They proposed five conjectures on the structure of p-groups of a fixed coclass that were subsequently proved—and are now called the *coclass theorems*—in a multi-year research project. The work on coclass theory is ongoing, but in 2002 Leedham-Green and S. McKay wrote the most comprehensive treatment of p-groups [45] since Hall in [37].

That almost all 2-groups of coclass r satisfy **TC** is an immediate corollary to the following theorem.

Theorem 4.18 (Eick [15]) *For every* $s \in \mathbb{N}$ *there exists* $o(r,s) \in \mathbb{N}$ *such that* $2^s|G| \mid |\mathrm{Aut}\, G|$ *for all 2-groups* G *of coclass* r *and order at least* $o(r,s)$.

Proof — To prove this result, Eick used a stunning combination of graph theory, profinite group theory (specifically, pro-p-groups), and group cohomology.

To begin with, a p-group of coclass $r = n - c$ is associated with a graph $\mathcal{G}(p,r)$ whose vertices are the isomorphism types of p-groups of coclass r and there is a directed edge from G to a descendant H if $H/N \simeq G$, where N is the last non-trivial term in the lower central series of H. The group G is the unique ancestor for H in the forest $\mathcal{G}(p,r)$. The subgraph generated by all descendants of a group G is a *maximal coclass tree* if it contains exactly one infinite path. There is a one-to-one correspondence between infinite pro-p-groups of coclass r and maximal coclass trees.

The coclass theorems show that there are only finitely many isomorphism types of infinite pro-p-groups of coclass r, so there are only finitely many maximal coclass trees in $\mathcal{G}(p,r)$. Furthermore, almost all p-groups of coclass r are contained in a maximal coclass tree. Hence, to prove her main theorem, Eick showed that a maximal coclass tree in $\mathcal{G}(p,r)$ can contain only finitely many counterexamples to **TC**.

Eick's joint work with Leedham-Green in [16] essentially said that 2-groups of fixed coclass exhibit periodic behavior that can be described by a "finite set of data". To prove Theorem 4.18, Eick showed that the orders of the automorphism groups of these 2-groups also exhibit periodic behavior: as the orders of the 2-groups "grow by a constant factor ... the orders of the corresponding automorphism groups also grow by a constant, but larger factor." □

Eick's work suggested that it would be difficult to find counterexamples to **TC**, so the search to prove it for more families of groups continued. In 2007, S. Fouladi, A.R. Jamali, and R. Orfi also used coclass theory to attack **TC**.

Theorem 4.19 (Fouladi, et al. [27]) *Let* G *be a finite* p*-group of coclass 2. Then* $|G|$ *divides* $|\mathrm{Aut}\, G|$.

Proof — The authors first showed that under the coclass 2 assumption and all the previous results, G satisfies the two conditions

$$p \leqslant |Z(G)| \leqslant p^2 \text{ and } p^{n-3} \leqslant |[G,G]| \leqslant p^{n-2},$$

where $|G| = p^n$. They then proceeded much as authors did in earlier decades: certain cases were relatively easy to dispense with because the order of $\mathrm{Hom}(G/[G,G], Z(G))$ was computable, and for the remaining cases special automorphisms were constructed.

The novel idea in this paper was that the authors showed both

$$[\mathrm{Aut}_{Z_2}^{G/Z_2} G : \mathrm{Aut}_{Z_1}^{G/Z_1} G] \text{ and } [\mathrm{Aut}_{Z_{n-3}}^{G/Z_{n-3}} G : \mathrm{Aut}_{Z_{n-4}}^{G/Z_{n-4}} G]$$

are divisible by p^2. The first assertion came from constructing two automorphisms lying in $\mathrm{Aut}_{Z_2}^{G/Z_2} G - \mathrm{Aut}_{Z_1}^{G/Z_1} G$ that generate a subgroup of $\mathrm{Aut}\, G$ of order p^2. The second assertion came from observing that two inner automorphisms σ_s and σ_t, associated with two particular generators of G, show that

$$[\mathrm{Aut}_{Z_{n-3}}^{G/Z_{n-3}} G : \mathrm{Aut}_{Z_{n-4}}^{G/Z_{n-4}} G] \geqslant$$
$$[\mathrm{Aut}_{Z_{n-4}}^{G/Z_{n-4}} G\langle \sigma_s, \sigma_t \rangle : \mathrm{Aut}_{Z_{n-4}}^{G/Z_{n-4}} G] \geqslant p^2.$$

Using these results, the authors proved that $\mathrm{Aut}\, G$ has the following normal series

$$1 < \mathrm{Aut}_{Z_1}^{G/Z_1} G < \mathrm{Aut}_{Z_2}^{G/Z_2} G \leqslant \ldots \leqslant \mathrm{Aut}_{Z_{n-4}}^{G/Z_{n-4}} G$$
$$< \mathrm{Aut}_{Z_{n-3}}^{G/Z_{n-3}} G \leqslant \mathrm{Aut}\, G.$$

When $|G| \geqslant p^6$, **TC** is proved since $|\mathrm{Aut}_{Z_1}^{G/Z_1} G| = p^2$ under the particular circumstances. □

Using yet another approach to finding families of p-groups that satisfy **TC**, M. Yadav focused on Camina pairs in 2007, where (G, N), $N \lhd G$, is a *Camina pair* if for all $x \in G - N$, $N \subseteq [x, G]$.

Theorem 4.20 (Yadav [62]) *Let G be a finite p-group such that $(G, Z(G))$ is a Camina pair. Then $|G|$ divides $|\mathrm{Aut}\, G|$.*

Proof — The proof of the theorem rested on Yadav's Theorem 3.1 which states that when $(G, Z(G))$ is a Camina pair, one of the following holds: (i) there exists a maximal subgroup M of G such that

$$Z(M) = Z(G),$$

or (ii) the elementary abelian groups $Z_2(G)/Z(G)$ and $G/\Phi(G)$ have the same order.

Putting Yadav's work into the context of the Wells exact sequence (Equation (4.4)) we see that under the first condition above

$$\mathrm{Aut}^{M,G/M} G = \mathrm{Ker}\, \rho = \mathrm{Hom}(G/M, Z(G)).$$

Since G/M has order p and $Z(G)$ is elementary abelian,

$$|\mathrm{Hom}(G/M, Z(G))| = |Z(G)|.$$

Yadav proved $\mathrm{Ker}\, \rho \cap \mathrm{Inn}\, G = \{1\}$ so $|\mathrm{Ker}\, \rho||\mathrm{Inn}\, G| = |G|$ and **TC** is proved in this case.

Now assume the second condition. Since $(G, Z(G))$ is a Camina pair, $Z(G) \leqslant [G, G]$ and G is PN. Thus

$$|\mathrm{Aut}_c G| = |\mathrm{Hom}(G/[G, G], Z(G))| = |\mathrm{Hom}(G/\Phi(G), Z(G))|.$$

Setting $|G/\Phi(G)| = p^d$ and $|Z(G)| = p^r$, we see that $|\mathrm{Aut}_c G| = p^{rd}$. Since $\mathrm{Aut}_c G \cap \mathrm{Inn}\, G \simeq Z_2/Z(G)$, Yadav noted

$$\begin{aligned} |\mathrm{Aut}_c G \cdot \mathrm{Inn}\, G| &= \frac{|\mathrm{Aut}_c G||\mathrm{Inn}\, G|}{|\mathrm{Aut}_c G \cap \mathrm{Inn}\, G|} \\ &= \frac{|\mathrm{Aut}_c G||G/Z(G)|}{|Z_2/Z(G)|} = \frac{|\mathrm{Aut}_c G||G|}{|Z_2|}. \end{aligned}$$

Thus, it suffices to show $|\mathrm{Aut}_c G| \geqslant |Z_2|$. Now $|Z_2/Z(G)| = |G/\Phi(G)|$ implies $|Z_2| = p^{r+d}$. Since $r, d \geqslant 1$ we know that $p^{rd} \geqslant p^{r+d}$, so the theorem is proved. □

Two other papers in this time period added to the types of p-groups for which **TC** holds: Exarchakos returned with coauthors G. Dimakos and G. Baralis in 2010 to prove the result for all non-abelian groups with cyclic center [22], and in 2012 A. Thillaisundaram proved the following theorem about p-central p-groups, where G is p-*central* if $G^p \leqslant Z(G)$.

Theorem 4.21 (Thillaisundaram [59]) *For p an odd prime, let G be a non-abelian p^2-abelian p-central p-group, with $|G| \geqslant p^3$. Suppose that $Z(G)$ is of the form*

$$Z(G) \simeq \frac{\mathbb{Z}}{p^{e_1}\mathbb{Z}} \times \ldots \times \frac{\mathbb{Z}}{p^{e_n}\mathbb{Z}}$$

where $3 \leqslant e_1 \leqslant \ldots \leqslant e_n$ and $n \geqslant 3$. Then $|G|$ divides $|\mathrm{Aut}\, G|$.

Proof — Thillaisundaram showed that $|\mathrm{Out}\, G|_p \geqslant |Z(G)|$ by computing a lower bound for the order of a Sylow-p subgroup of $\mathrm{Out}\, G$ using extensions of $\mathrm{Aut}\, Z(G)$ to $\mathrm{Aut}^{G/Z} G$ that are non-inner.

Using the characterization of the automorphism group of a finite abelian group due to Hillar and Rhea [39], Thillaisundaram counted the number of automorphisms of $Z(G)$ that reduce to $I_{n\times n}$ mod p. Under certain circumstances she showed how to extend such a non-trivial $\theta \in \mathrm{Aut}\, Z(G)$ to $\hat{\theta} \in \mathrm{Aut}^{G/Z}(G)$ by using Wells' bijection between $\mathrm{Aut}_N G$ and triples $(\theta, \phi, \chi) \in \mathrm{Aut}\, N \times \mathrm{Aut}\, G/N \times N^{G/N}$ satisfying compatibility conditions [60] (Thillaisundaram credited Passi et al. [52] for this correspondence, but it is originally due to Wells). Again, by carefully counting the extensions, the main theorem is proved. □

4.4 TC is settled: the year 2015

The bombshell hit in 2015 when J. González-Sánchez and A. Jaikin-Zapirain [34] showed that **TC** is false! In the 60 years since Schenkman's first stab at **TC** not a single mathematician wondered publicly whether it would turn out to be false. In retrospect, the title of the González-Sánchez and Jaikin-Zapirain paper — "Finite p-groups with small automorphism group"—makes it obvious that one should have been searching among groups with small automorphism group for a counterexample, compared with the large automorphism groups of **TC** as coined by Exarchakos.

Theorem 4.22 (González-Sánchez and Jaikin-Zapirain [34]) *For each prime* p *there exists a family of* p*-groups* $\{U_i\}$ *such that*

$$\lim_{i\to\infty} |U_i| = \infty \text{ and } \limsup_{i\to\infty} \frac{|\operatorname{Aut} U_i|}{|U_i|^{40/41}} < \infty.$$

In particular, for every prime p*, there exists a nonabelian finite* p*-group* G *such that* $|\operatorname{Aut} G| < |G|$.

Proof — As they pointed out in their paper, the construction comes in two steps: first, find an infinite, finitely generated, pro-p group U for which

$$\dim(\operatorname{Aut} U) < \dim(U);$$

second, as

$$U = \varprojlim U_i \quad \text{and} \quad \operatorname{Aut} U = \varprojlim \operatorname{Aut} U_i,$$

where the U_i are finite p-groups, show that $|\operatorname{Aut} U_i| < |U_i|$ when i is large.

To accomplish step one, they noted that if U is a uniform pro-p group then

$$\dim(\operatorname{Aut} U) = \dim_{\mathbb{Q}_p} \operatorname{Der}(\mathbf{L}(U))$$

where $\mathbf{L}(U)$ is the Lie $\mathbb{Q}_p$-algebra associated with U and $\operatorname{Der}(\mathbf{L}(U))$ its algebra of $\mathbb{Q}_p$-derivations. Examples of Lie algebras $\mathbf{L}$ with

$$\dim(\operatorname{Der}(\mathbf{L})) < \dim(L)$$

exist, and one such example was used to build U.

Continuous cohomology of pro-p-groups was used for step two. The authors showed that

$$H^1_{\text{cts}}(U, \mathbf{log}(U)) \simeq \operatorname{Der}(\mathbf{log}(U))$$

is finite, where $\mathbf{log}(U)$ is the Lie ring corresponding to U under the Lazard correspondence. This implies there is a uniform upper bound for $|H^1(U, L_i)|$, where

$$L_i = \mathbf{log}(U)/p^i\mathbf{log}(U).$$

The upper bound was used to show that

$$[\mathrm{Aut}\, U_i : \mathrm{Inn}\, U_i],$$

where $U_i = U/U^{p^i}$, has an upper bound too. Finally, their computation of an upper bound for $|\mathrm{Inn}\, U_i|$ finished off the proof. □

Theorem 4.22 closed the door on **TC**, but opened the door on new techniques in group theory. In his review of [34], S. Ghoraishi wrote "This article contains some new methods in group theory. Therefore I recommend that young researchers in the field read it carefully and learn its methods for further use" [32].

5 What's next?

The result in [34] put an end to **TC**, but not necessarily to the search for all finite non-cyclic p-groups of order greater than p^2 that do satisfy **TC**. Indeed, the 2015 bombshell was followed quickly by two additions to the cases for which **TC** holds.

Theorem 5.1 (Fernández-Alcober and Thillaisundaram [24])

(i) *Let* G *be a finite non-cyclic* p*-group with* $|G| \geqslant p^3$ *and with an abelian maximal subgroup* A. *Then* $|G|$ *divides* $|\mathrm{Aut}\, G|$.

(ii) *Let* G *be a finite* p*-group with elementary abelian center, such that*

$$C_G(Z(\Phi(G))) \neq \Phi(G) \text{ and } Z(M) \supset Z(G)$$

for all maximal subgroups M *of* G. *Then* $|G|$ *divides* $|\mathrm{Aut}\, G|$.

Proof — For the first result, the authors used the same tactic as Yadav in Theorem 4.20, which was to show $|\mathrm{Aut}_c G| \geqslant |Z_2|$. Depending on the size of $\exp(G/[G,G])$ relative to $\exp Z(G)$, they showed

$$|\mathrm{Hom}(G/[G,G], Z(G))| \geqslant |Z_2|,$$

which suffices when G is assumed to be PN.

For the second result, the authors noted that groups for which

$$Z(G) \leqslant \Phi(G)$$

fall into three categories: (i) G has a maximal subgroup M such that $Z(M) = Z(G)$, (ii) $Z(M) \gneq Z(G)$ for all maximal subgroups of G and either (ii.a) $C_G(Z(\Phi(G))) \neq \Phi(G)$ or (ii.b) $C_G(Z(\Phi(G))) = \Phi(G)$.

In case (i),

$$|\mathrm{Out}\, G|_p \geqslant |\mathrm{Aut}^{M,G/Z} G| = |\mathrm{Hom}(G/M, Z(M))| = |\Omega_1(Z(M))| = |Z(G)|$$

so **TC** is proved by Equation 4.1.

In case (ii.a), the authors used an extension of Gaschütz' Theorem 4.7 that says p divides $|C_{\mathrm{Out}\, G} Z(G)|$. In the case at hand, there is a non-inner automorphism $\beta \in \mathrm{Aut}^Z S$, where $S \leqslant G$, of p-power order that can be extended to a non-inner automorphism $\gamma \in \mathrm{Aut}^Z G$ by a result due to O. Müller [50]. The authors showed $\gamma \notin \mathrm{Aut}^M G$ and letting $\overline{\gamma}$ be its image in Out G, they further showed that

$$\overline{\gamma} \notin \mathrm{Out}^{M,G/M} G.$$

Then

$$|\mathrm{Out}\, G|_p \geqslant |\langle \overline{\gamma}, \mathrm{Out}^{M,G/M} G \rangle| \geqslant |Z(G)|$$

and **TC** is proved in this case. □

Disproving a long-standing conjecture is never as interesting as proving one. Who among us would be excited to find out that the Goldbach Conjecture is wrong? But any sort of let down is likely short-lived because mathematicians' quest for truth rises above sentimentality.

TC held a community of mathematicians—those who attempted to prove it, and those (as with this author) who merely watched intently from the sidelines—in a tight grip for a number of years. Surely interest in **TC** will wane, but the mathematics it spawned will live on.

We end with a list of the most easily stated cases for which **TC** holds. The cases are not mutually exclusive.

Result	**Author and citation**	**Year**
G satisfying $\lvert Z(G)\rvert = p$	Gaschütz [30]	1966
If $G = A \times B$, where A is abelian, B is PN and $\lvert B\rvert \mid \lvert \mathrm{Aut}\, B\rvert$	Otto [51]	1966
p-groups of maximal class	Otto [51]	1966
p-groups of class 2	Faudree [23]	1968
metacyclic p-groups, p odd	Davitt [9]	1970

G satisfying $G/Z(G)$ is metacyclic, p odd	Davitt and Otto [12]	1971
p-abelian p-groups	Davitt [10]	1972
modular p-groups, p odd	Davitt and Otto [13]	1972
If $G = AH$ is a central product with A abelian and $\lvert H\rvert \mid \lvert \operatorname{Aut} H\rvert$	Hummel [42]	1975
If $N \lhd G$ with $N \cap [G,G] = \{1\}$ and $\lvert G/N\rvert \mid \lvert \operatorname{Aut} G/N\rvert$	Buckley [5]	1976
G satisfying $[G : Z(G)] \leqslant p^4$	Davitt [11]	1980
G satisfying $p^3 \leqslant \lvert G\rvert \leqslant p^5$ $\leqslant p^6$ $\leqslant p^7$	Davitt [11] Exarchakos [21] Gavioli [31]	1980 1989 1993
G satisfying $\Phi(G)$ is cyclic	Exarchakos [19]	1981
p-groups of coclass 2	Fouladi, Jamali, and Orfi [27]	2007
G satisfying $xZ(G) \subset x^G$ for all $x \in G - Z$	Yadav [62]	2007
If G has an abelian maximal subgroup	Fernández-Alcober and Thillaisundaram [24]	2016
G satisfying $Z(G)$ is elementary abelian and $C_G(Z(\Phi(G))) \neq \Phi(G)$	Fernández-Alcober and Thillaisundaram [24]	2016

Table 5.1: Non-cyclic p-groups of order $> p^2$ for which **TC** holds

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