Advances in Group Theory and Applications

7 (2019), pp. 143–157 ISSN: 2499-1287
DOI: 10.32037/agta-2019-006

Weak Engel Conditions on Linear Groups

B.A.F. Wehrfritz

(*Received Sept. 6, 2018; Accepted Oct. 30, 2018 — Communicated by F. de Giovanni*)

Abstract

We study several weak Engel conditions on linear groups, starting from the "almost Engel" condition of Khukhro and Shumyatsky. There the groups were Engel modulo certain finite subsets. Here we replace "finite subsets" by "Chernikov subgroups" in one case and "polycyclic-by-finite subgroups" in another. We also include a shorter proof of a theorem of Shumyatsky characterizing linear almost Engel groups.

Mathematics Subject Classification (2010): 20F45, 20H20

Keywords: Engel condition; linear group; matrix group

1 Introduction

A group G is *almost Engel* if for all $g \in G$ there exists a finite subset $E = E(g)$ of G such that for all $x \in G$ there is a positive integer m such that for all $n \geqslant m$ we have $[x,_n g] \in E$. If G is almost Engel, then for each $g \in G$ there is a unique minimal subset E as above, which is denoted by $\mathbf{E}(g)$ and called the *sink* of g (or the g-*sink*), and

$$\mathbf{E}(g) = \{x \in G \,:\, \text{for some } n \geqslant 1,\ x = [x,_n g]\}.$$

Subgroups and images of almost Engel groups are almost Engel. For all this see Khukhro and Shumyatsky's paper [5], especially its Section 2.

Shumyatsky in his paper [8] proves the following theorem.

Theorem 1.1 *Let* G *be an almost Engel linear group. Then*

i) G *is finite-by-hypercentral.*

ii) *If for each* $g \in G$ *there exists* $m = m(g)$ *such that for each* $x \in G$ *and* $n \geqslant m$ *we have* $[x,_n g] \in \boldsymbol{E}(g)$, *then* G *is finite-by-nilpotent.*

If G is any group
$$\{\zeta_i(G) : 0 \leqslant i \leqslant s\},$$
s some ordinal, denotes the upper central series of G with

$$\zeta(G) = \bigcup_{i \leqslant s} \zeta_i(G)$$

its hypercentre.

If $\mathfrak{X}$ is a class of groups, say that a group G is $\mathfrak{X}$-Engel if for each $g \in G$ there is an $\mathfrak{X}$-subgroup E of G such that for all $x \in G$ there is a positive integer m such that for all $n \geqslant m$ we have $[x,_n g] \in E$. Clearly if $\mathfrak{X}$ is subgroup and image closed, then so is the class $\mathfrak{X}$-Engel. We give below a proof of Shumyatsky's theorem above and then show how small modifications to that proof yields a proof of the following theorem (it is not convenient to produce one proof covering both cases, mainly since finite-Engel as defined above is not quite the same as almost Engel).

Theorem 1.2 *Let* G *be a Chernikov-Engel linear group. Then* $G/\zeta(G)$ *is Chernikov and* G *is Chernikov-by-nilpotent.*

Although if X is a group such that $X/\zeta(X)$ Chernikov, then X is Chernikov-by-hypercentral (see [13], Theorem B), the converse is false in general, even if X is Chernikov-by-abelian, see Example 3 of [13]. Thus we include both conclusions in the statement of Theorem 1.2. However, in our context the following more than suffices.

Proposition 1.3 *Let* G *be a linear group of degree* n. *If* G *is Chernikov-by-hypercentral, then* $G/\zeta(G)$ *is Chernikov. If* $G/\zeta(G)$ *is Chernikov, then* G *is Chernikov-by-(nilpotent of class at most max*$\{1, n-1\}$*).*

The next obvious step is to replace the class of Chernikov groups by the class **PF** of polycyclic-by-finite groups and consider linear **PF**-Engel groups. This immediately presents problems. Example 2 of [13] is a polycyclic-by-hypercentral group X with $X/\zeta(X)$ not polycyclic-by-finite. Moreover, this X has faithful linear representations of degree 2 in every characteristic. Example 1 of [13] is a group Y with $Y/\zeta(Y)$ polycyclic such that Y is not **PF**-by-hypercentral. As a further complication if G is a **PF**-Engel group and $g \in G$, there is no obvious canonical choice to play the role of the f-sink. However the methods of this paper do at least yield the following theorem.

Theorem 1.4 *Let G be a **PF**-Engel linear group of characteristic $p \geqslant 0$. Then G is soluble-by-finite. If $p > 0$ then G is nilpotent-by-finite and **PF**-by-hypercentral. If $p = 0$ then G is nilpotent-by-(**PF**-by-hypercentral).*

Proposition 1.5 *Let G be a linear group with $G/\zeta(G) \in$ **PF**. Then G is **PF**-by-hypercentral.*

This is a partial **PF** analogue to Proposition 1.3. As pointed out above, the converse of Proposition 1.5 is false in all characteristic. Of course every **PF**-by-hypercentral group is **PF**-Engel.

2 Proof of Theorem 1.1, part i)

Thus let G be an almost Engel linear group, say with $G \leqslant GL(n, F)$, n a positive integer and F a field. We break the proof into a series of steps.

a) G *is soluble-by-finite*.

The free group $\langle x, g\rangle$ of rank 2 on the exhibited generators is not almost Engel (the commutators $[x,_r g]$ for $r \geqslant 1$ are all distinct). It then follows from Tits's theorem (see [10], 10.17) that G is soluble-by-periodic. If S denotes the soluble radical of G, then S is (Zariski) closed and G/S is linear over F (see [10], 5.11 and 6.4). Thus to prove the claim we may assume G is periodic.

If $\mathrm{char} F = 0$ then G is abelian-by-finite (e.g., [10], 9.4). Suppose

$$\mathrm{char}\, F = p > 0$$

and assume G is not soluble-by-finite. With S as above, by [7], 5.1.5 there exists a subgroup $T \geqslant S$ of G with $K = T/S$ infinite, simple and (again by [10], 6.4) linear over F. Then K is of Lie type over some infinite locally finite field k_0 of characteristic p (e.g. [9]). Therefore for some infinite subfield k of k_0 the group K has a section isomorphic to $SL(2, k)$, see [1], 6.3.1.

$PSL(2,k) \cap Tr(2,k)$ is the split extension of the additive group k^+ of k by the multiplicative group k^* of K, where for $a \in k^*$ and $b \in k^+$ the action is given by $b^a = ba^2$. If $a \neq \pm 1$, then

$$|a^2 - 1| = m \geqslant 1 \quad \text{and} \quad b = b(a^2-1)^m = [b,_m a].$$

Therefore $b \in \mathbf{E}(a)$ for all such b, so $\mathbf{E}(a)$ is infinite and hence $SL(2, k)$, K and G are not almost Engel. This completes the proof of a).

b) *If* $G \leqslant Tr(n, F)$, *then* G *is finite-by-nilpotent*.

First consider $a \in F^*$ and $B \leqslant F^+$ with $Ba = B$ and assume the split extension $C = \langle a \rangle B$ is almost Engel. If $d \in \mathbf{E}(a) \backslash \{0\}$, then

$$d = [d,_e a] = d(a-1)^e$$

for some $e \geqslant 1$ and so $(a-1)^e = 1$. If $b \in B$, then

$$b = b(a-1)^{er}$$

for all $r \geqslant 1$, so $b \in \mathbf{E}(a)$, $B \subseteq \mathbf{E}(a)$ and B is finite. If $\mathbf{E}(a) = \{0\}$ and $b \in B\backslash\{0\}$, then $b(a-1)^r = 0$ for some r and $a = 1$. Hence B is finite or $a = 1$ (when trivially a centralizes B).

Let U denote the unipotent radical of G, so $G' \leqslant U$. Then G has a normal series

$$\langle 1 \rangle = U_0 \leqslant U_1 < \ldots \leqslant U_t = U$$

of length $t = n(n-1)/2$ such that each U_i/U_{i-1} is centralized by U and acted upon by G via an embedding of it into F^+ and a homomorphism of G/U into F^*. By our initial remark each U_i/U_{i-1} is finite or G-central.

A simple induction shows that G is finite-by-nilpotent. For example, suppose G/U_1 is finite-by-(nilpotent of class c). If U_1 is finite, then G is finite-by-nilpotent. Suppose U_1 is G-central. By Hall's theorem ([6], 4.25) the index of $\zeta_{2c}(G/U_1)$ is in G/U_1 is finite. Therefore

$$(G : \zeta_{2c+1}(G))$$

is finite and consequently G is finite-by-nilpotent by Baer's theorem ([6], Corollary 2 to 4.21).

c) G *is nilpotent-by-finite. If* G *is connected,* G *is nilpotent*.

Let G° denote the connected component of G containing 1.

Then G/G° is finite and G° is triangularizable (over some extension field of F) by a) and 5.8 and 5.11 of [10]. By b) there is a finite normal subgroup N of G° with G°/N nilpotent. Then $C = G^\circ \cap C_G(N)$ is nilpotent, closed ([10], 5.4) and of finite index in the connected group G°. Consequently $G^\circ = C$ and hence G° is nilpotent.

d) *G is finite-by-hypercentral. If G is finitely generated then G is finite-by-nilpotent.*

This completes the proof of part i) of Theorem 1.1, and it follows immediately from c), the lemma below and the well-known fact that finitely generated hypercentral groups are nilpotent (see for instance [12], 6.2). □

Lemma 2.1 *Let G be an almost Engel, nilpotent-by-finite group. Then G is finite-by-hypercentral.*

Proof — Now a group X is finite-by-hypercentral if, by [3], and only if, by [2], $X/\zeta(X)$ is finite (alternatively see Theorems A and E of [13]).

Let N be a normal subgroup of G of finite index that is nilpotent of class c and set $A = \zeta_1(N)$. We induct on c; if $c = 0$ the claim is vacuous. Let $c \geqslant 1$ and assume G/A is finite-by-hypercentral. Then $H/A = \zeta(G/A)$ has finite index in G/A by our initial remark.

Let $g \in G$. Then

$$\mathbf{E}(g) \cap A = \{a \in A \,:\, \text{for some } r = r(a) \geqslant 1,\ a = [a,_r\, g]\}.$$

If also $b \in \mathbf{E}(g) \cap A$ with $b = [b,_s\, g]$ and $s \geqslant 1$, then $ab = [ab,_{rs}\, g]$, since A is abelian. It follows that $\mathbf{E}(g) \cap A$ is a finite subgroup of A.

Let T be a (finite) transversal of N to G. If $g \in G$, then $g = ht$ for some $h \in N$ and $t \in T$. Then $[a, g] = [a, t]$ for all $a \in A = \zeta_1(N)$ and hence

$$E = \langle \mathbf{E}(g) \cap A \,:\, g \in G \rangle = \langle \mathbf{E}(t) \cap A \,:\, t \in T \rangle$$

is a finite normal subgroup of G contained in A. Also A/E consists of right Engel elements of G/E, so H/E also consists of right Engel elements of G/E. Therefore $H/E \leqslant \zeta(G/E)$, e.g. by Gruenberg's Theorems 1.2 and 1.4 of [4]. But then G/E is finite-by-hypercentral by our initial remark again. Consequently so is G. □

3 Proof of Theorem 1.1, part ii)

By part i) of Theorem 1.1 there is a finite normal subgroup L of G with G/L hypercentral. Then L is closed in G, so G/L is also linear (again [10], 6.4). By the minimal choice of $\mathbf{E}(g)$, for each $g \in G$,

$$\mathbf{E}(g) \subseteq L.$$

Thus

$$[x,_{m(g)}\ g] \in L$$

for all x and g in G. Thus G/L consists entirely of bounded left Engel elements. Consequently G/L is nilpotent by Gruenberg's theorems ([10], 8.15 and 8.2). □

Remark 3.1 In the above proof of ii) we use i). However, in the context of ii) the special case of i) needed is actually very simple. For, all finite subsets of G are closed and if $g \in G$ and $n \geqslant m(g)$, then

$$\phi : G \longrightarrow G$$

given by

$$x\phi = [x,_n\ g]$$

is continuous. Then $G^\circ\phi$ is connected, so $|G^\circ\phi| = 1$. But $1\phi = 1$. Thus $G^\circ\phi = \{1\}$, $[x,_n\ g] = 1$ for all $x \in G^\circ$ and G° consists entirely of right Engel elements of G. Then by 8.15 of [10] we have $G^\circ \leqslant \zeta(G)$ and G/G° is finite. Therefore G is finite-by-hypercentral.

Remark 3.2 We cannot improve finite-by-hypercentral to finite-by-nilpotent in the lemma, even for linear groups, which is all we need. The infinite, locally dihedral 2-group is hypercentral, abelian-by-finite but not finite-by-nilpotent. It is also linear of degree 2 in any characteristic except 2. The wreath product of a Prüfer 3-group by a cyclic group of order 3 has similar properties and is linear of degree 3 and characteristic 2. If $G \leqslant GL(2, F)$, where $\operatorname{char} F = 2$ and G is finite-by-hypercentral, then $(G : \zeta(G))$ is finite and $\zeta(G) = \zeta_1(G)$ by 8.6 of [10]. Then G' is finite and G is finite-by-abelian.

Remark 3.3 Let G be a subgroup of $GL(n, F)$ and for the moment, and only for simplicity of notation, assume F is algebraically closed. Let H denote the Hirsch-Plotkin radical $\eta(G)$ of G. The Fitting subgroup of G is nilpotent (see [10], 8.2) closed in G (see [10], 5.9

and 5.11) and of finite index in H (see [10], 8.2). Therefore H is closed in G.

Let H_u denote the set of unipotent elements of H and H_d the set of diagonalizable elements of H. Then H_u and H_d are subgroups of H (see [10], 7.11). They are also closed in H and hence in G; for

$$H_u = \{x \in H : (x-1)^n = 0\}$$

and $(H_d)^\circ$ is diagonalizable (see [10], 5.8 and 7.1), so its closure in H is too. Hence $(H_d)^\circ$ and therefore H_d are closed in H.

If $x \in GL(n, F)$, let

$$x = x_u x_d = x_d x_u$$

be the Jordan decomposition of x in $GL(n, F)$ (see [10], 7.2). Set

$$K = \langle x_u, x_d : x \in H \rangle.$$

Then K is locally nilpotent normalized by G and

$$K = K_u \times K_d = K_u H = K_d H$$

(see [10], 7.14 and 7.11). Now $K \cap G = H$, so

$$K_u \cap G = H_u \quad \text{and} \quad K_d \cap G = H_d.$$

Hence

$$G/H_u = G/(K_u \cap G) \simeq K_u G/K_u = KG/K_u$$

and

$$H/H_u \simeq K/K_u \simeq K_d.$$

Further K is the Hirsch-Plotkin radical of KG by 7.16 of [10], so as above K_u and K_d are closed in KG. Thus G/H_u is isomorphic to a linear group over F such that the image of H/H_u is a d-group. Similarly, G/H_d is isomorphic to a linear group over F such that the image of H/H_d is unipotent (use [10], 6.4, 6.5 and 6.6).

4 Proof of Proposition 1.3

If G is Chernikov-by-hypercentral there exist normal subgroups $A \leqslant T$ of G with A divisible, abelian and Chernikov, T/A finite and G/T hypercentral. Then $\zeta(G/A)$ has finite index in G/A. Also A is diagonalizable, so $(G : C_G(A))$ is finite (see [10], 7.1 and 1.12). Hence there is a normal subgroup L of G of finite index with $A \leqslant \zeta_1(L)$, L/A hypercentral in G/A and L hypercentral. Thus

$$L \leqslant \eta(G) \cap C_G(A) = \eta(C_G(A)) = K$$

say, where $\eta(X)$ denotes the Hirsch-Plotkin radical of X. Then K is a hypercentral group and, by 8.2 ii) and 5.11 ii) of [10], is closed of finite index in G.

Let K_u (resp. K_d) denote the set of unipotent (resp. diagonalizable) elements of K. Then G/K_u (resp. G/K_d) is isomorphic to a linear group over the algebraic closure of F, such that the image of K/K_u (resp. K/K_d) consists only of diagonalizable (resp. unipotent) elements (apply Remark 3.3 to $C_G(A)$ and extend these embeddings to G as in [10], 2.3).

Clearly $A \leqslant K_d$, so LK_d/K_d is G-hypercentral and of finite index in G/K_d. Also G is soluble-by-finite. Then K/K_u contains an abelian normal subgroup B/K_u of finite index in G/K_u (see [10], 3.6). Now A is divisible, so $A \leqslant B$ and

$$B/K_u = (AK_u/K_u) \times (D_1/K_u)$$

for some $D_1 \leqslant G$. But G/K is finite. Hence if

$$D = \bigcap_{g \in G} (D_1)^g,$$

then D is a normal subgroup of G containing K_u and B/D and hence G/D are Chernikov. Now L/A is G-hypercentral, so

$$LK_u/AK_u \leqslant K/AK_u$$

is G-hypercentral. Clearly

$$AK_u \cap D = K_u,$$

so

$$(LK_u \cap D)/K_u$$

is G-hypercentral and

$$G/(LK_u \cap D)$$

is Chernikov. Consequently

$$H = LK_d \cap LK_u \cap D$$

is G-hypercentral and G/H is Chernikov. Thus $G/\zeta(G)$ is Chernikov.

Conversely, now assume $G/\zeta(G)$ is Chernikov. Set

$$n' = \max\{1, n-1\},$$

so

$$n - 1 \leqslant n' \leqslant n.$$

By (2) of [11] the subgroup

$$Z = [\zeta(G),_{n'} G]$$

is a diagonalizable $n!$-group and as such is abelian and Chernikov. Also

$$\zeta(G/Z) = \zeta(G)/Z = \zeta_{n'}(G/Z),$$

so by 4.23 and 4.21 (Corollary 2) of [6], the $n'+1$ term

$$X/Z = \gamma^{n'+1}(G/Z)$$

of the lower central series of G/Z is Chernikov. Therefore X is Chernikov and G/X is clearly nilpotent of class at most n'.

This concludes the proof of Proposition 1.3. □

Recall that $\{\gamma^j(X) : j \geqslant 1\}$ denotes the lower central series of the group X.

Corollary 4.1 *Let G be a linear group of degree n with $T = \gamma^{i+1}(G)$ a Chernikov group, i some non-negative integer. Set $n' = max\{1, n-1\}$. Then $G/\zeta_{2i}(G)$, $\gamma^{n'+1}(G)$ and $G/\zeta_{2n'}(G)$ are all Chernikov.*

Proof — Basically we repeat the proof of Proposition 1.3, but here in view of Hall's Theorem (see [6], 4.25) we have

$$(G/A : \zeta_{2i}(G/A))$$

finite. Thus we can choose our L as before but now with

$$L/A \leqslant \zeta_{2i}(G/A).$$

Hence

$$LK_d/K_d \quad \text{and} \quad (LK_u \cap D)/K_u$$

are both G-hypercentral with G-central height at most $2i$. Therefore H here is normal in G with G/H Chernikov and $H \leqslant \zeta_{2i}(G)$. Thus $G/\zeta_{2i}(G)$ is Chernikov. Finally Proposition 1.3 yields that $\gamma^{n'+1}(G)$ is Chernikov and consequently $G/\zeta_{2n'}(G)$ is Chernikov by the first claim of the corollary. □

There do exist groups G for which $\gamma^{i+1}(G)$ is Chernikov while $G/\zeta_{2i}(G)$ is not, even if $i = 1$ (see [13], Example 4). The analogue of the corollary for the class **PF** is always true, viz. if G is any group with $\gamma^{i+1}(G) \in$ **PF** (resp. **P**), then $G/\zeta_{2i}(G) \in$ **PF** (resp. **P**); see [6], vol.1, p.119.

5 Proof of Proposition 1.5

Suppose G is a subgroup of $GL(n, F)$ with $G/\zeta(G) \in$ **PF**. Then

$$S/\zeta(S) \in \mathbf{PF}$$

for every section S of G. By [11] the subgroup $[\zeta(G),_{n'} G]$ is a diagonalizable $n!$-group. The latter are all Chernikov groups, so G has a divisible abelian Chernikov normal subgroup $A \leqslant \zeta(G)$ with G/A of finite central height. By 4.21 (Corollary 2) of [6] there is a normal subgroup N of G containing A with G/N nilpotent and $N/A \in$ **PF**.

Suppose G has a diagonalizable normal subgroup D of finite index. Clearly $A \leqslant D \cap N$, so

$$D \cap N = A \times B$$

for some finitely generated abelian subgroup B. Since G/D is finite, so

$$B^G = \langle B^g : g \in G \rangle \leqslant D$$

is also a finitely generated abelian group. Clearly

$$D \cap N = AB^G$$

and G/N is nilpotent. Hence

$$D/B^G \leqslant \zeta(G/B^G).$$

Consequently G/B^G is finite-by-hypercentral by [3]. Therefore G is **PF**-by-hypercentral.

In general G is soluble-by-finite. Then G contains a triangularizable normal subgroup of G of finite index containing the unipotent radical U of G, (see [10], 3.6 and its proof). The previous paragraph applied to G/U yields that G/U is **PF**-by-hypercentral.

We apply Remark 3.3. With $H = \eta(G)$, clearly $U = H_u$, and there is an isomorphism of G/H_d onto a linear group L (over the algebraic closure of F and say of degree m) such that the image of H/H_d is unipotent. Clearly $\zeta(G) \leqslant H$. Hence [11] applied to L yields that

$$[\zeta(G),_m G] \leqslant H_d.$$

Also $G/\zeta(G) \in$ **PF**, so L has finite central height. Hence there is a normal subgroup $M \geqslant H_d$ of G with G/M nilpotent and M/H_d a **PF**-group (by [6], 4.21 yet again). Finally

$$U \cap H_d = H_u \cap H_d = \langle 1 \rangle.$$

Therefore G is **PF**-by-hypercentral. □

6 Proof of Theorem 1.2

We copy the structure of our proof above of Shumyatsky's theorem. Thus let G be a Chernikov-Engel subgroup of $GL(n, F)$. Since Chernikov groups satisfy the minimal condition on subgroups, for each $g \in G$ we have a unique minimal subgroup E as in the definition of Chernikov-Engel, which we again denote by $\mathbf{E}(g)$.

a) G *is soluble-by-finite.*

Of course, Chernikov groups are soluble-by-finite. Repeat the proof of the previous a). The free group $\langle x, g \rangle$ and the split extension k^*k^+ for k an infinite locally finite field (the action being via squares) are not Chernikov-Engel, in the latter case since each $\mathbf{E}(g) \leqslant k^+$, which is elementary abelian; whence the $\mathbf{E}(g)$ are all finite.

b) *If* $G \leqslant Tr(n, F)$, *then* G *is finite-by-nilpotent.*

For if $g \in G$, then $\mathbf{E}(g) \leqslant G'$, which here is torsion-free or of finite exponent. Hence each $\mathbf{E}(g)$ is finite, G is almost Engel and the previous b) applies directly.

c) G *is nilpotent-by-finite (and* G° *is nilpotent).*
Just copy the proof of the earlier c), but using the new a) and b).

d) $G/\zeta(G)$ *is Chernikov and* G *is Chernikov-by-nilpotent.*
This will complete the proof of Theorem 1.2. To prove it we modify the proof of the lemma above. By c) there is a nilpotent normal subgroup N of G of finite index. By 5.11 ii) of [10] we may choose N closed in G. Then $A = \zeta_1(N)$ is also closed in G and therefore G/A is isomorphic to a linear group (see [10], 6.4). Hence by induction on the class of N we may assume that G/A is Chernikov modulo

$$H/A = \zeta(G/A).$$

If $g \in G$ clearly $\mathbf{E}(g) \cap A$ is Chernikov, so if T is a transversal of N to G, then

$$E = \langle \mathbf{E}(g) \cap A \,:\, g \in G \rangle = \langle \mathbf{E}(t) \cap A \,:\, t \in T \rangle$$

is a Chernikov normal subgroup of G. Therefore H/E consists of right Engel elements of G/E and

$$H/E \leqslant \zeta(G/E)$$

by Theorems 1.2 and 1.4 of [4] again (note that we do not claim and do not need that G/E is necessarily isomorphic to some linear group). Therefore G/E is Chernikov-by-hypercentral by Theorem B of [13] and hence G is Chernikov-by-hypercentral. Finally $G/\zeta(G)$ is Chernikov and G is Chernikov-by-nilpotent by Proposition 1.3. □

7 Proof of Theorem 1.4

Again we copy our proof of Theorem 1.1. Thus let G be a **PF**-Engel subgroup of $GL(n, F)$. With the notation as in previous paragraphs labeled a) the groups $\langle x, g \rangle$ and k^*k^+ (with action via squares) are clearly not **PF**-Engel. Thus as previously we obtain a) as below.

a) G *is soluble-by-finite;* G *is unipotent-by-abelian-by-finite.*

b) *If* $G \leqslant Tr(n, F)$*, and char* $F > 0$*, then* G *is finite-by-nilpotent.*
For G' has finite exponent and hence G here is almost Engel. Thus b) holds. Then exactly as in previous cases we obtain the following variant of c)

c) *If char* $F > 0$*, then* G *is nilpotent-by-finite.*

d) *If char* $F > 0$*, then* G *is* ***PF****-by-hypercentral.*

As we have done twice before, we can split a nilpotent normal subgroup N of G of finite index given into unipotent and d-group cases. If N is unipotent, then N has finite exponent, every **PF** subgroup of G is finite and G is almost Engel and hence finite-by-hypercentral by Shumyatsky's theorem. If N is a d-group, then N is diagonalizable, so G is abelian-by-finite and hence G is **PF**-by-hypercentral by the following lemma. That G is nilpotent-by-(**PF**-by-hypercentral) if char $F = 0$ also follows from the lemma below and a) above. □

Lemma 7.1 *Let* A *be an abelian normal subgroup of finite index in the* ***PF****-Engel group* G*. Then* G *is* ***PF****-by-hypercentral.*

Proof — Let T be a (finite) transversal of A to G. For each $t \in T$ choose the subgroup $E(t) \in$ **PF** as in the definition of **PF**-Engel and set

$$A(t) = A \cap E(t).$$

Thus for each $a \in A$ there exists $m \geqslant 1$ such that if $n \geqslant m$ then

$$[a,_n t] \in A(t).$$

Note that if $g \in G$ and $a \in A$, there is $m = m(g, a) \geqslant 1$ such that if $n \geqslant m$ then

$$[a,_n t^g] \in A(t)^g.$$

Suppose $g = bu$, where $b \in A$ and $u \in T$. Then $A(t)^g = A(t)^u$ and if $t^g \in At'$ where $t' \in T$, then

$$[a,_n t^g] = [a,_n t'].$$

Set $\mathbf{A}(t)$ equal to the intersection of the $A(s)^g$ for all $s \in T$ and $g \in G$ with $s^g \in At$. Then for all $a \in A$ there exists $m \geqslant 1$ such that if $n \geqslant m$ then

$$[a,_n t] \in \mathbf{A}(t).$$

Now $s^g \in At$ if and only if $s^u \in At$. Hence $\mathbf{A}(t)$ is the intersection only of the $A(s)^u$ for s and u in T with $s^u \in At$. Clearly $\mathbf{A}(t) \in$ **PF**. Also if $x \in G$, then

$$\mathbf{A}(t)^x = \mathbf{A}(t')$$

whenever $At^x = At'$.

Set

$$B = \langle \mathbf{A}(t) : t \in T \rangle \leqslant A.$$

Clearly $B \in \mathbf{PF}$. Also B is normal in G. Hence A/B consists entirely of right Engel elements of G/B. Consequently $A/B \leqslant \zeta(G/B)$ and hence G/B is finite-by-hypercentral. Therefore G is **PF**-by-hypercentral. □

REFERENCES

[1] R.W. Carter: "Simple Groups of Lie Type", *Wiley*, London (1972).

[2] C. Casolo – U. Dardano – S. Rinauro: "Variants of theorems of Baer and Hall on finite-by-hypercentral groups", *J. Algebra* 452 (2016), 279–287.

[3] M. de Falco – F. de Giovanni – C. Musella – Y.P. Sysak: "On the upper central series of infinite groups", *Proc. Amer. Math. Soc.* 139 (2011), 385–389.

[4] K.W. Gruenberg: "The upper central series in soluble groups", *Illinois J. Math.* 5 (1961), 436–466.

[5] E.I. Khukhro – P. Shumyatsky: "Almost Engel compact groups", *J. Algebra* 500 (2018), 439–456.

[6] D.J.S. Robinson: "Finiteness Conditions and Generalized Soluble Groups", *Springer*, Berlin (1972).

[7] M. Shirvani – B.A.F. Wehrfritz: "Skew Linear Groups", *Cambridge University Press*, Cambridge (2008).

[8] P. Shumyatsky: "Almost Engel linear groups", *Monatsh. Math.* 186 (2018), 711–719.

[9] S. Thomas: "The classification of the simple periodic linear groups", *Arch. Math. (Basel)* 41 (1983), 103–116.

[10] B.A.F. Wehrfritz: "Infinite Linear Groups", *Springer*, Berlin (1973).

[11] B.A.F. Wehrfritz: "Hypercentral unipotent subgroups of linear groups", *Bull. London Math. Soc.* 11 (1973), 310–313.

[12] B.A.F. Wehrfritz: "Groups and Ring Theoretic Properties of Polycyclic Groups", *Springer*, Berlin (2009).

[13] B.A.F. Wehrfritz: "Variants of theorems of Schur, Baer and Hall", *Ric. Mat.* 67 (2018), 331–337.

B.A.F. Wehrfritz
Queen Mary University of London
London E1 4NS
London (England)
e-mail: b.a.f.wehrfritz@qmul.ac.uk