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On the Universal Group $\mathrm{PSL}(2, \mathbb{C})$

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Abstract

The group $\Gamma = \mathrm{PSL}(2, \mathbb{C})$ arises in a wide variety of contexts; hyperbolic geometry, automorphic function theory, number theory and group theory. Much of combinatorial group theory arose out of the study of discrete subgroups of $\Gamma = \mathrm{PSL}(2, \mathbb{C})$, in particular Fuchsian Groups and Kleinian groups. From the Poincaré polygon theorem surface groups can be faithfully represented in $\mathrm{PSL}(2, \mathbb{C})$. Extending this, most cyclically pinched one-relator groups can also be embedded in Γ. Recent results of Fine and Rosenberger ([61],[62]) show that all finitely generated fully residually free groups, the so called limit groups, can also be faithfully represented in this group. In this paper we survey the tremendous impact this single group has had on combinatorial group theory in particular and infinite group theory in general.

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1 Introduction

Much as the complex numbers $\mathbb{C}$ play a fundamental role in commutative algebra, the single group $\Gamma = \mathrm{PSL}(2, \mathbb{C})$ plays a unifying role in infinite group theory, especially in combinatorial group theory. This group arises in a wide variety of contexts; hyperbolic geometry, automorphic function theory, number theory and group theory to name just a few. This group is also the starting off point for many important areas within infinite group theory such as geometric group theory and hyperbolic groups. Much of combinatorial group theory arose out of the study of discrete subgroups of $\Gamma = PSL(2,\mathbb{C})$, in particular Fuchsian groups and Kleinian groups. From the Poincaré polygon theorem it follow that surface groups, that is the fundamental groups of compact surfaces, can be faithfully represented in $PSL(2,\mathbb{C})$. Extending this, most cyclically pinched one-relator groups, can also be embedded in Γ. Recent results show that all finitely generated fully residually free groups, the so called limit groups, can also be faithfully represented in this group. In this paper we explore and survey the tremendous impact this single group has had on combinatorial group theory in particular and infinite group theory in general.

In the next section we discuss the study of discrete groups. This arose originally out of the study of the fundamental groups of compact surfaces and the study of automorphic functions. Combinatorial group theory was initiated as a method to understand the infinite countable groups introduced by Poincaré much as finite groups arose originally out of the permutation methods in group theory. The article by Ackermann, Fine and Rosenberger [2] examined how the theory of surface groups has served as a motivating example for much of combinatorial group theory.

In Section 3 we look at Möebius transformations and the group $\Gamma = PSL(2,\mathbb{C})$ and how it appears in its various guises. We look at the classification of its elements which is tied to the classification of geometric

isometries. We also introduce trace formulas which tie the traces of elements

to group theoretic properties. In that section we then show many important classes of groups such as parafree groups and knot groups can be faithfully represented within $PSL(2,\mathbb{C})$.

The most important subgroups of $PSL(2,\mathbb{C})$ are the discrete ones. Fuchsian and Kleinian group arose in the study of automorphic functions and have continued to be important. They served as the motivation for much of what is now called geometric group theory. The original definitions of these classes of groups are entirely within our main group Γ. We describe these classes and discuss both classical and more recent results in Section 4. Number theoretically defined groups also are important in Γ and we describe a wealth of information on these.

Building on the theory of surface groups, the class of one-relator groups has been extensively studied. An important question is to determine when a one-relator group can be faithfully represented within $\mathrm{PSL}(2,\mathbb{C})$. We show that most cyclically pinched and conjugacy pinched one-relator groups can be faithfully represented. We then indicate how, what are termed essential representations, can be used to study general one-relator products.

The class of finitely generated fully residually free groups, also called limit groups, are the fundamental objects used in the solution of the Tarski problems by Kharlampovich and Myasnikov and by Sela. In the final section we show that there are constructive faithful representations of limit groups within Γ. Finally we present some open problems.

2 Countably discrete groups and combinatorial group theory

Group theory in general can be divided into three areas; finite groups, countable discrete groups and continuous groups. These three areas reflect how various groups were historically introduced. Further each area uses different methods to study groups. Of course there is overlap between the three areas.

Finite group theory which arose originally from Galois' work on the solvability of polynomial equations by radicals. Here as the name suggests the groups are finite, and the motivating examples are permutation groups on finite sets. One of the crowning achievements in finite group theory was the classification of all finite simple groups, the proof of which took thousands of published pages.

Continuous group theory arose out of differential equations and out of geometry and the attempt to translate the methods of Galois theory to the study of solutions of differential equations. Continuous groups and Lie groups became crucial when it was realized after Klein's Erlanger Program that any study of metric geometry required knowledge of the group of isometries of the geometry. These entailed studies of the so-called classical groups such as the Euclidean group and the orthogonal and unitary groups. The study of continuous group involved then the methods of Lie group theory and manifolds.

Finally combinatorial group theory grew out of the need to study the infinite discrete groups necessary to understand the combinatorial objects in low dimensional topology specifically originally surface groups and the other fundamental groups introduced in low-dimensional topology by Poincaré.

A *surface group* is the fundamental group of a compact orientable or non-orientable surface. If the genus of the surface is g then we say that the corresponding surface group also has genus g.

An orientable surface group S_g of genus $g \geqslant 2$ has a one-relator presentation of the form

$$S_g = \langle a_1, b_1, \dots, a_g, b_g; [a_1, b_1] \dots [a_g, b_g] = 1 \rangle$$

while a non-orientable surface group T_g of genus $g \geqslant 2$ also has a one-relator presentation - now of the form

$$T_g = \langle a_1, a_2, \dots, a_g; a_1^2 a_2^2 \dots a_g^2 = 1 \rangle.$$

Much of combinatorial group theory arose originally out of the theory of one-relator groups and the concepts and ideas surrounding the Freiheitssatz or Independence Theorem of Magnus (see section 5). Going backwards the ideas of the Freiheitssatz were motivated by the topological properties of surface groups. Surface groups have motivated a great many of the areas of exploration in combinatorial group theory and infinite discrete group theory. This surface group motivation comes from the rich interplay surface groups provide among group theory, topology, hyperbolic geometry and computer science. From topology, a surface group inherits many of its properties from topological properties of the surface for which it is the fundamental group. This raises the questions of which of these properties are actually algebraic, that is dependent on the group theoretic structure and/or the presentational form and independent of the topology.

From the perspective of this paper what is crucial is that surface groups admit faithful Fuchsian representations in $PSL(2, \mathbb{C})$, that is they can be represented faithfully as discrete subgroups of $PSL(2, \mathbb{C})$. Further this faithful representation can be into $PSL(2, \mathbb{R})$. The existence of this representation has several consequences. First a surface group is linear and hence inherits all properties of linear groups. This raises questions, concerning when a group with a one-relator presentation is actually linear. Secondly a Fuchsian group describes through the upper half-plane model of hyperbolic geometry a discrete group of isometries of the hyperbolic plane. It follows that surface groups have many properties related to this geometric interpretation and as before it raises the purely group theoretic question of which of these properties can be deduced purely from the presentation. Further the method used to determine if an element of a surface group can be trivial has led to small cancellation theory. This in recent years has been closely tied to computer science via the concept of an automatic group.

In another direction surface groups are closely tied to the solution of the Tarski problems by Kharlampovich and Myasnikov ([86],[87], [91],[92],[93],[90]) and independently by Sela ([141],[142],[144],[143], [145],[146]). Surface groups provide the primary examples of non-free elementary free groups, that is non-free groups that have the same elementary or first order theory as the class of nonabelian countable free groups. We show that any elementary free group and more generally any limit group has a faithful constructive representation within $PSL(2, \mathbb{C})$. The book [40] provides a standard reference for material necessary to understand the Tarski problems and their solution.

3 The group $PSL(2, \mathbb{C})$

The group $\Gamma = PSL(2, \mathbb{C})$ consist of all linear fractional transformations

$$z' = f(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are any complex numbers satisfying $ad - bc \neq 0$.

The map $f(z)$ is a bijective holomorphic function from the Riemann sphere to the Riemann sphere. An element of $PSL(2, \mathbb{C})$ is called a *Möbius transformation*.

The set of all Möbius transformations forms a group under composition. This group can be given the structure of a complex manifold in such a way that composition and inversion are holomorphic maps. The Möbius group is then a complex Lie group. The Möbius group is the automorphism group of the Riemann sphere.

The group Γ of Möbius transformations arises in a wide range of contexts especially relative to their orbit spaces and the ties to Riemann surfaces, hyperbolic orbifolds and hyperbolic geometry.

3.1 Möebius transformations and their classification

If the upper half plane is taken as a model for two-dimensional hyperbolic geometry with the metric

$$ds = \frac{|dz|}{y}$$

where $z = x + iy$, then isometries of hyperbolic spaces can be identified with Möbius transformations. Considering these as acting on the point at infinity, a Möbius transformation is defined on the extended complex plane (i.e. the complex plane augmented by the point at infinity).

This extended complex plane can, via stereographic projection, be thought of as a sphere, the Riemann sphere, or as the complex projective line. Every Möbius transformation is a bijective conformal map of the Riemann sphere to itself. Indeed, every such map is by necessity a Möbius transformation.

The set of all Möbius transformations forms a group under composition called the Möbius group identified with Γ. It is the automorphism group of the Riemann sphere (when considered as a Riemann surface).

The Möbius group is isomorphic to the group of orientation-preserving isometries of hyperbolic 3-space and therefore plays an important role when studying hyperbolic 3-manifolds.

In physics, the identity component of the Lorentz group acts on the celestial sphere in the same way that the Möbius group acts on the Riemann sphere. In fact, these two groups are isomorphic. An observer who accelerates to relativistic velocities will see the pattern of constellations as seen near the Earth continuously transform according to infinitesimal Möbius transformations. This observation is often taken as the starting point of twistor theory.

Certain subgroups of the Möbius group form the automorphism groups of the other simply-connected Riemann surfaces (the complex plane and the hyperbolic plane). As such, Möbius transformations play an important role in the theory of Riemann surfaces. The fundamental group of every Riemann surface is a discrete subgroup of the Möbius group (see Fuchsian group and Kleinian group in the next section). A particularly important discrete subgroup of the Möbius group is the Modular group; it is central to the theory of many fractals, modular forms, elliptic curves, Pellian equations and quadratic forms.

Möbius transformations can be more generally defined in spaces of dimension $n > 2$ as the bijective conformal orientation-preserving maps from the n-sphere to the n-sphere. Such a transformation is the most general form of conformal mapping of a domain. According to Liouville's theorem a Möbius transformation can be expressed as a composition of translations, similarities, orthogonal transformations and inversions.

A linear fractional transformation

$$z' = \frac{az+b}{cz+d}$$

where a, b, c, d are any complex numbers satisfying $ad - bc \neq 0$ can be considered as a pair of matrices $\pm A$ where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $A \in SL(2,\mathbb{C})$. The elements of Γ can be classified by the trace of A. Relative to hyperbolic geometry this classification is analogous to the classification of Euclidean isometries as translations, rotations, reflections and glide reflections. We have:

(1) A is *hyperbolic* if $tr(A) \in \mathbb{R}$ and $|tr(A)| > 2$;

(2) A is *parabolic* if $tr(A) \in \mathbb{R}$ and $|tr(A)| = 2$;

(3) A is *elliptic* if $tr(A) \in \mathbb{R}$ and $|tr(A)| < 2$;

(4) A is *loxodromic* if $tr(A) \notin \mathbb{R}$.

A has finite order $p \geqslant 2$ if and only if

$$tr(A) = 2\cos(\frac{q\pi}{p})$$

with $1 \leqslant q \leqslant p$ and $\gcd(q,p) = 1$. There are certain trace identities that will play a fundamental role in the structure of subgroups of Γ that we describe now. Two elements $A, B \in \Gamma$ have a common fixed point if and only if $tr([A,B]) = 2$ where

$$[A,B] = ABA^{-1}B^{-1}$$

is the commutator of A, B. This is related to *elementary groups*. A subgroup of Γ is called *elementary* if any two elements of infinite order have a common fixed point. Let $x = tr(A), y = tr(B)$ and $z = tr(AB)$, then:

(1) $tr(AB^{-1}) = xy - z$;

(2) $tr([A, B]) = x^2 + y^2 + z^2 - xyz - 2$.

For the final trace identities, we define inductively the *Chebyshev polynomials* $S_n(t)$ by

$$S_0(t) = 0, S_1(t) = 1 \quad \text{and } S_n(t) = tS_{n-1}(t) - S_{n-2}(t)$$

for $n \geqslant 2$. further $S_n(t) = -S_{-n}(t)$ for $n < 0$.

(3) $A^n = S_n(x)A - S_{n-1}(x)I$ for $n \in \mathbb{N} \cup \{0\}$.

(4) $tr[A^n B^m] - 2 = S_n^2(x) S_m^2(y)(tr[A, B] - 2)$, for $n, m \in \mathbb{N} \cup \{0\}$.

4 Fuchsian and Kleinian groups

The classical theory of Γ has centered for the most part on discrete subgroups, Fuchsian and Kleinian groups and their orbit spaces, Riemann surfaces and hyperbolic 3-orbifolds respectively. A subgroup $G \subset \Gamma$ is *discrete* if G contains no sequence of non-trivial elements

$$T_n = \pm \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, T_n(z) = \frac{a_n z + b_n}{c_n z + d_n}$$

which converges to the identity

$$I = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

element-wise. We say that G acts *discontinuously* on an open set $S \subset \mathbb{C}$ if for each $z \in S$ the set $\{T(z) | T \in G\}$ have no accumulation point in S. A *discontinuous* subgroup of Γ is a subgroup of Γ which acts discontinuously on some open set in $\mathbb{C}$.

A discontinuous subgroup of Γ must be discrete. However a discrete subgroup need not be discontinuous. Consider for example the group $\mathrm{PSL}(2,\mathbb{Z}[i])$, the set of linear fractional transformations in Γ with entries from the Gaussian integers $\mathbb{Z}[i]$. This is clearly discrete but nowhere discontinuous in $\mathbb{C}$ (see [59]). However this cannot happen for real subgroups of Γ and we have the result

Theorem 4.1 *Let* G *be a subgroup of* $\mathrm{PSL}(2,\mathbb{R})$. *Then* G *is discrete if and only if* G *is discontinuous in the upper half plane.*

A subgroup $G \subset \Gamma$ is *elementary* if any two elements of infinite order $g,h \in G$ have a common fixed point (considered as linear fractional transformations). This is equivalent to $\mathrm{tr}([g,h]) = 2$ for any $g,h \in G$ of infinite order. The elementary discrete subgroups of Γ can be fully classified (see [65]).

It is important to know when a given subgroup of Γ is discrete or not. The following are general conditions for discreteness in terms of two generator and cyclic subgroups.

Theorem 4.2 (see [136] and [82]) (1) *Let* H *be a non-elementary subgroup of* $\mathrm{PSL}(2,\mathbb{C})$. *Then* H *is discrete if and only if each two-generator subgroup of* H *is discrete.*

(2) *Let* H *be a non-elementary subgroup of PSL*$(2,\mathbb{R})$. *Then* H *is discrete if and only if each cyclic subgroup of* H *is discrete.*

4.1 Fuchsian groups

A *Fuchsian group* is a non-elementary discrete (and hence discontinuous) subgroup of $\mathrm{PSL}(2,\mathbb{R})$, or a conjugate of such a subgroup in Γ. Since the real axis can be mapped onto any given circle by an element $T \in \Gamma$ we can equivalently define a Fuchsian group as a non-elementary discontinuous subgroup of Γ which fixes a circle C and maps the interior of C on itself Fuchsian groups in older literature were called *fixed circle groups*.

In this section we consider only subgroups of $\mathrm{PSL}(2,\mathbb{R})$. These act on the upper half-plane $\mathbb{H}^2$.

It is clear that a non-elementary subgroup of a Fuchsian group is also Fuchsian. We have the following theorem.

Theorem 4.3 *A non-elementary subgroup of a Fuchsian group* G *is itself Fuchsian. Further if* $G \subset \Gamma$ *and if* $[G:H] < \infty$ *and* H *is discontinuous then* G *is also discontinuous. It follows that a Fuchsian group cannot be a finite index subgroup of a non-discontinuous group.*

The theory of Fuchsian groups is extensive, both the algebraic theory and geometric/topological theory. In this paper we will survey newer results on subgroups and other classifications in Fuchsian groups but for the general theory we refer the reader to the books by Katok [85] and Fine and Rosenberger [59].

If G is a subgroup of $PSL(2, \mathbb{R})$ we say that $z_1, z_2 \in \mathbb{H}^2$ are *congruent under* G if there exists a $g \in G$ with $g(z_1) = z_2$. If G is discrete then a *fundamental domain* for G is a connected open subset $D \subset \mathbb{H}^2$ such that any two points in D are not congruent under G and every point in $\mathbb{H}^2$ is congruent to some point in the closure of D.

A classical and crucial result is that any finitely generated Fuchsian group has a fundamental domain D, called a *Ford Domain* or *Dirichlet region*, in $\mathbb{H}^2$ which is a non-Euclidean polygon, called a *Poincaré polygon* which defines a hyperbolic tiling or tesselation of the hyperbolic plane $\mathbb{H}^2$. Further the side-pairing transformations of D generate G and a special presentation for G, called a Poincaré presentation, can be read off from D. Poincaré proved that the procedure can be reversed starting with special non-Euclidean polygons to construct Fuchsian groups. The quotient space $\mathbb{H}^2/G$ is a Riemann surface. The Fuchsian group G is *co-compact* if this quotient space is compact.

The following result ties the theory of Fuchsian groups to combinatorial group theory. The complete proof and constructions can be found in the books [85],[21],[99]. A free side of D is a side that lies on the real axis.

Theorem 4.4 (1) *Let* G *be a nontrivial finitely generated Fuchsian group with a Ford domain* D *with no free sides. Then there exist* $t, t \geqslant 0$*, parabolic generators,* $p_1, \ldots, p_t$*,*$2g, g \geqslant 0$*, hyperbolic generators* $a_1, b_1, \ldots, a_g, b_g$*, and* $s, s \geqslant 0$*, elliptic generators of respective finite orders* $m_1, \ldots, m_s$ *if* $s > 0$ *with* $n = t + g + s > 0$ *such that* G *has the following presentation*

$$G = \langle p_1, \ldots, p_t, e_1, \ldots, e_s, a_1, b_1, \ldots, a_g, b_g\,; \\ e_1^{m_1} = \ldots = e_s^{m_s} = R = 1\rangle \tag{1}$$

where

$$R = p_1 \ldots p_t \cdot e_1 \ldots e_s \cdot [a_1, b_1] \ldots [a_g, b_g].$$

This presentation is called a Poincaré presentation for G*; the number* g *is called the genus of* G*; the sequence* $(g; m_1, \ldots, m_s; t)$ *is called the signature of* G*.*

The hyperbolic area of the Dirichlet region D *for* G *is given by*

$$\mu(D) = 2\pi \left(2g - 2 + t + \sum_{i=1}^{s} (1 - \frac{1}{m_i}) \right)$$

further $\mu(D)$ *is the hyperbolic area for any fundamental domain for* G *and hence we can define* $\mu(G) = \mu(D)$*; the number*

$$\chi(G) = -\frac{1}{2\pi}\mu(D)$$

is called the Euler characteristic of G*.*

(2) *Conversely given a signature* $(g; m_1, \dots, m_s; t)$ *with* $m_i \geqslant 2$ *then there exists a Fuchsian group* G *with that signature only if* $\mu(G) > 0$*.*

If $t \neq 0$ then group theoretically G is a free product of cyclics. If $t = 0$ then there are no parabolic elements in G and the corresponding group is cocompact. If $s = t = 0$ then the Fuchsian group is isomorphic to the fundamental group of a compact surface of genus g. Hence any orientable surface group has a faithful representation in $PSL(2, \mathbb{R})$ and hence in $PSL(2, \mathbb{C})$ as a Fuchsian subgroup.

An abstract group with a presentation of the form (1) is called an F-group. If $\mu(F) > 0$ then an F-group is a Fuchsian group. There has been a general study of F-groups (see [104]) and they have been shown to satisfy most properties of Fuchsian groups in general. In particular Hoare, Karrass and Solitar ([73],[74]) gave an algebraic proof based on the Reidemeister-Schreier process than any finite index subgroup of an F-group is again an F-group of the same type and any infinite index subgroup is a free product of cyclics.

A Fuchsian group with signature $(0; p, q, r; 0)$ and hence with presentation

$$\langle x, y; x^p = y^q = (xy)^r = 1 \rangle$$

is called a *ordinary triangle group*. It is the group generated by hyperbolic reflections in the sides of a hyperbolic triangle. In Section 4 we will look at the classification of Fuchsian triangle groups.

In [118] Magnus provided a survey of the uses of 2×2 complex matrices in combinatorial group theory. In that paper he describes many other classes of groups that have faithful and in many cases discontinuous representations in Γ. We close this section by mentioning some of the results from that paper. Riley [131],[132],[133] showed that certain knot groups have faithful representations in Γ, and further some of the images are discontinuous in hyperbolic 3-space and hence are discrete. Magnus also described that the class of parafree groups, that is groups with the same lower central series as free groups, introduced by G. Baumslag [12] with the presentation

$$G_{m,n} = \langle a, b, c; c = a^{-m}c^{-1}a^{m}ca^{-n}b^{-1}a^{n} \rangle \text{ with } mn \neq 0$$

have faithful representations in Γ, a fact proved by D. Forastiero [66]. Fine, Rosenberger and Stille had examined the isomorphism problem for this class of groups and in particular showed that the isomorphism problem is solvable for the subclass $G_{n,1}$ (see [63]).

Finally Magnus discussed the problem of determining rational faithful representations of Fuchsian groups, that is representations into $PSL(2, \mathbb{Q})$. In particular the group

$$G = \langle a, b; ([a, b])^2 = 1 \rangle$$

has a faithful discrete representation in $PSL(2, \mathbb{Q})$. The group G contains as subgroups all orientable surface groups of finite genus. It follows that all orientable surface groups have faithful rational Fuchsian representations.

4.2 Kleinian groups

A Kleinian group is a discrete subgroup of $\Gamma = PSL(2, \mathbb{C})$. Hence the class of Fuchsian groups is a subclass of the class of Kleinian groups. Since Γ has several different interpretations: as conformal transformations of the Riemann sphere, as orientation-preserving isometries of 3-dimensional hyperbolic space $\mathbb{H}^3$, and as orientation preserving conformal maps of the open unit ball B^3 in $\mathbb{R}^3$ to itself, a Kleinian group can be regarded as a discrete subgroup acting on one of these spaces.

The theory of general Kleinian groups was founded by Felix Klein (1883) and Henri Poincaré (1883), who named them after Felix Klein.

A standard reference for the theory of Kleinian groups both theoretically and geometrically is the book by B. Maskit [121].

Discreteness of a Kleinian group implies points in B^3 have finite stabilizers, and discrete orbits under the group G. The orbit G_p of a point p will typically accumulate on the boundary of the closed ball B^3

The boundary of the closed ball is called the *sphere at infinity*, The set of accumulation points of G_p is called the limit set of G, while the complement of the limit set is called the domain of discontinuity or the ordinary set or the regular set. A Kleinian group is called *elementary* if its limit set is finite, in which case the limit set has 0, 1, or 2 points. This is equivalent the definition given above.

There are several finiteness conditions for Kleinian groups which play an important role in their classification. A Kleinian group is said to be of *finite type* if its region of discontinuity has a finite number of orbits of components under the group action, and the quotient of each component by its stabilizer is a compact Riemann surface with finitely many points removed, and the covering is ramified at finitely many points.

A Kleinian group G has *finite covolume* if the orbit space $\mathbb{H}^3/G$ has finite volume. Any Kleinian group of finite covolume is finitely generated. A Kleinian group G is called *cocompact* if the orbit space $\mathbb{H}^3/G$ is compact. Cocompact Kleinian groups have finite covolume. Finally a Kleinian group is *geometrically finite* if it has an analogously defined fundamental polyhedron (in hyperbolic 3-space) with finitely many sides. If G is geometrically finite then a group presentation for G can be read off of the fundamental polyhedron. This is the Poincaré Polyhedron Theorem and is the 3-dimensional analog of the Poincaré Polygon Theorem for Fuchsian groups. If G is a geometrically finite Kleinian group and D is a fundamental domain for it then the sides of D are paired $\{s, s'\}$. The side pairing transformations $g_{s,s'}$ generate G. Further if $s = s'$ then $g_{s,s'}^2 = 1$. These are called *reflection relations*. In addition and given an edge $e \in D$ there is a cycle

$$h = h(e) = g_1 c \dots g_k$$

and in an integer t such that $h^t = 1$. These are called the *cycle relations*.

Theorem 4.5 *Let G be a geometrically finite Kleinian group with fundamental polyhedron D. Then the side pairing transformations generate G and the reflection relations and cycle relations provide a complete set of relations for G. Conversely if D is a polyhedron of finite covolume in $\mathbb{H}^3$ satisfying certain standard geometric conditions (see* [121]*) then the group G generated by the side pairing transformations is a Kleinian group (hence discrete) with fundamental polyhedron D.*

There are several different proofs of this important theorem. One which explains the necessary conditions on D can be found in Maskit's book [121].

5 Kleinian groups and hyperbolic three manifolds

The research on Kleinian groups has centered mainly on the relation to three-manifold topology. The fundamental group of an oriented hyperbolic three manifold has a faithful representation as a Kleinian group. Conversely any Kleinian group G which has no non-trivial torsion elements can be identified with the fundamental group of a hyperbolic three-manifold. When a Kleinian group G is isomorphic to the fundamental group of a hyperbolic three-manifold then the orbit space $\mathbb{H}^3/G$ is a Kleinian model of the manifold.

The classification of Kleinian groups, through their interpretation as fundamental groups of three-manifolds, play a crucial role in the study of *Thurston's Geometrization Theorem*. The geometrization conjecture stated that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure. The conjecture was proposed by William Thurston (1982), and imply several other conjectures, such as the Poincaré conjecture.

Thurston's hyperbolization theorem proved the conjecture for Haken manifolds (see [156]). Thurston's geometrization conjecture is the analogue of the uniformization theorem for two-dimensional surfaces, which states that every closed simply connected Riemann surface can be given one of three geometries (Euclidean, spherical, or hyperbolic).

Grigori Perelman proved the full geometrization conjecture in 2003 using Ricci flow with surgery. There are now several different papers with details of the proof (see [6],[125]). The famous Poincaré conjecture, which states that any closed simply connected three manifold is homeomorphic to the three-sphere, one of the designated millenium problems, is a corollary of Perelman's result.

6 Kleinian groups and number theory

Kleinian groups arise in number theory in several different ways. They are important in the the theory of modular functions. The two most extensively studied approaches to the connections between Kleinian groups and number theory are via $\mathrm{PSL}(2, R)$ where R is a discrete subring of the complex numbers $\mathbb{C}$ and via the relation to quaternion algebras. The second approach leads to the study of *arithmetic Fuchsian* and *arithmetic Kleinian groups* and will be discussed in the next subsection.

6.1 The modular group M

The most widely studied number theoretically defined Kleinian group is the *Modular group* $M = PSL(2, \mathbb{Z})$. The study of M began in the 19th century out of the study of modular forms. M and its subgroups were studied by Fricke and Klein [67] and a presentation for M was known to Dyck in the 1880's. In many aspects the study of M can be seen as the beginnings of combinatorial group theory.

Since $\mathbb{Z}$ is a discrete subring of $\mathbb{R}$ the group M is a Fuchsian group. As such M has close ties to the study of Riemann Surfaces. For example let S be a Riemann surface of genus 1. This is topologically a torus and depends on a single complex parameter τ. If S is identified with a parallelogram within $\mathbb{C}$ the parameter τ is the ratio of the sides. If S' is another such Riemann surface with parameter τ' then S and S' are conformally equivalent if and only if τ is M-congruent to τ' that is $T(\tau) = \tau'$ for some $T \in M$.

Since M is a Fuchsian group it has a a Ford domain in the upper half plane given by

$$D = \{(x, y); -1 < x < 1, y \geqslant x^2 + y^2 = 1\}$$

(see [59]). From the Ford domain and the Poincaré polygon theorem we have the following presentation for M.

Theorem 6.1 $M = \langle x, y; x^2 = y^3 = 1 \rangle$ *where*

$$x : z \mapsto -\frac{1}{z}, y : z \mapsto -\frac{1}{z+1}.$$

Group theoretically this is a free product of a two-cycle and a three-cycle and hence $M = \mathbb{Z}_2 \star \mathbb{Z}_3$.

We mention that there proofs of the structure of M that do not use the Poincaré polyhedron theorem (see [59])

Much of the algebraic structure of M can be deduced from the free product decomposition. We mention several results. Proofs can be found in [37].

Theorem 6.2 *Let* M *be the Modular group. Then:*

(1) *any element of finite order is conjugate to either* x *or* y*;*

(2) *any torsion-free subgroup of* M *must be a free group;*

(3) *any abelian subgroup of* M *is cyclic and isomorphic to either* $\mathbb{Z}_2, \mathbb{Z}_3$ *or* $\mathbb{Z}$*.*

Much has been done on subgroup classification within M. A large portion of this research has centered on the *congruence subgroup problem* abbreviated CSP.

Let R be ring and A an ideal in R. Then in $SL(n, R)$ the *principal congruence subgroup* mod A, denoted $SL(n, R)(A)$, is

$$\{T \in SL(n, R)(A); T \equiv I \text{ mod } A \text{ element-wise}\}.$$

A *congruence subgroup* of $SL(n, R)$ is a subgroup which contains a principal congruence subgroup. $SL(n, R)$ satisfies the *congruence subgroup property* if every subgroup of finite index is a congruence subgroup. Mennicke [121] proved that for $n \geqslant 3$ the groups $SL(n, \mathbb{Z})$ satisfy the CSP while Bass, Milnor and Serre [8] proved the CSP holds for $SL(n, R)$ for $n \geqslant 3$ and for a wide class of rings. Serre extended this to $SL(2, R)$ [148] where R an algebraic number ring which contains a unit of infinite order. Among the algebraic number rings the sole possible exceptions are M and the groups Γ_d where

$$\Gamma_d = PSL(2, O_d)$$

where O_d is the ring the integers in the quadratic imaginary number field $Q(\sqrt{-d})$ with $d > 0$.

For M the principal congruence subgroup of level n denoted by $M^{(n)}$ consists of those integral matrices congruent to the identity mod n element-wise. Each $M^{(n)}$ is a normal subgroup of finite index (the kernel of the homomorphism from $PSL(2, \mathbb{Z})$ to $PSL(2, \mathbb{Z}_n)$ and each $M^{(n)}$ is a free group. For M, the CSP asks whether every normal subgroup of finite index is a congruence subgroup.

Theorem 6.3 M *does not satisfy the CSP, that is there are subgroups of finite index which do not contain a principal congruence subgroup.*

There are direct proofs of this result that can be found in the books by Newman [127] and Fine [37]. A separate proof using the theory of profinite groups and which also works for the groups Γ_d was done by Lubotzky [102]. A nice constructive group theoretical proof can be found in [136].

6.2 The Bianchi groups

Besides the Modular group the class of number theoretically defined Kleinian groups that have been of most interest are the groups

$$\Gamma_d = PSL(2, O_d)$$

where d is a positive square-free integer and O_d is the ring of integers in the quadratic imaginary number field $\mathbb{Q}(\sqrt{-d})$. These are known collectively as the *Bianchi groups* named after L. Bianchi who worked on their geometry in the nineteenth century [22]. They share many properties with the Modular group. If $d = 1, 2, 3, 7, 11$ the rings O_d have a Euclidean algorithm and the corresponding groups Γ_d are the *Euclidean Bianchi groups*. The algebraic properties of the Euclidean Bianchi groups were worked out by Fine (see [37] or [38]). Algebraically they are quite similar to M, especially $\Gamma_1 = PSL(2, \mathbb{Z}[i])$ where $\mathbb{Z}[i]$ are the Gaussian integers. Γ_1 is known as the *Picard group* (see [37]). As far as the algebraic structure they seem to break down into four classes

$$\{\Gamma_1\}, \{\Gamma_3\}, \{\Gamma_2, \Gamma_7, \Gamma_{11}\}$$

and $\Gamma_d, d \neq 1, 2, 3, 7, 11$. A method of R. Swan [152] allows one to determine finite presentations for each Γ_d. A computer program was developed by R. Riley to implement Swan's method for the Γ_d. They are not discontinuous in $\mathbb{H}^3$ so the Poincaré polyhedron theorem does not apply.

Interest in the class Γ_d arose from work of Serre [148]. A group G satisfies property FA if whenever G acts on a tree X the set of fixed points is nonempty. Serre showed that the groups Γ_d if $d \neq 3$ do not satisfy FA and this raised the question as to whether these groups admit nontrivial splittings as free products with amalgamation. Fine proved this to be true in the Euclidean cases and then Frohman and Fine [68],[69] using Swan's method proved it in general.

Theorem 6.4 ([68],[69]) *For each* $d \neq 3$ *the Bianchi group* Γ_d *admits a splitting as a nontrivial free product with amalgamation.*

Further the exact structure of the factors can be determined. In general they are graph products of finite groups.

Theorem 6.5 ([68]) *For all* $d \neq 3$ *we have*

$$\Gamma_d = PE_2(O_d) \star_{H_d} G_d$$

where $PE_2(O_d)$ *is the projective elementary group (see* [37]*) and* G_d *is a subgroup depending on* d*. For each* d*,* H_d *is a free product with amalgamation of two copies of* M*.*

Prior to the proof of Fine and Frohman there was a great deal of evidence for the existence of a splitting using results of K. Vogtman [156] and Culler and Shalen [30]. It was shown that beyond the Euclidean Bianchi groups Γ_d admits a nontrivial splitting if

$$d = 6, 15, 19, 23, 31, 39, 47, 71.$$

Along these lines K. Kingston [95] worked out the algebraic structure of Γ_d when O_d has class number one. There was also this general theorem of H. Bass [7].

Theorem 6.6 *Let* G *be a finitely generated subgroup of* $GL(2, \mathbb{C})$*. then one of the following must occur:*

(1) *There is an epimorphismn* $f : G \to \mathbb{Z}$ *such that* $f(U) = 0$ *for all unipotent elements* $U \in G$*;*

(2) G *is an amalgamated product*

$$G = G_1 \star_H G_2$$

with $G_1 \neq H \neq G_2$ *such that very finitely generated unipotent subgroup of* G *is contained in a conjugate of* G_1 *of* G_2*;*

(3) G *is conjugate to a group of upper triangular matrices all of whose diagonal elements are roots of unity;*

(4) G *is conjugate to a subgroup of* $GL(2, A)$ *where* A *is a ring of algebraic integers.*

Hence almost all Kleinian groups admit nontrivial splittings.

We have mentioned that Serre proved [149] that among the algebraic number rings the sole possible exceptions to the CSP are M and the Γ_d. Serre [147] and then independently Lubotzky using the theory of profinite groups proved the next theorem.

Theorem 6.7 *For each* d*, the Bianchi group* Γ_d *does not satisfy the CSP, that is there are subgroups of finite index which do not contain a principal congruence subgroup.*

Direct proofs of this were given in the Euclidean cases by Fine (see [37]).

We mention that the Bianchi groups differ a great deal from their counterparts over real quadratic fields. In the real case the corresponding rings of integers are not discretely normed and hence not Kleinian groups.

We mention that the Bianchi groups arise in many other contexts especially in connection with three manifolds and representations of knot groups (see [132],[133]) as well as in connection with the classification arithmetic Fuchsian and Kleinian groups.

6.3 Arithmetic Kleinian groups

Fuchsian and Kleinian groups also arise number theoretically via a connection to quaternion algebras. These lead to what are called *arithmetic Kleinian groups*.

Two discrete subgroups G_1, G_2 of Γ are *commensurable* if their intersection is of finite index in both G_1 and G_2. They are commensurable in the the wide sense if G_1 is commensurable with a conjugate of G_2. Macbeath [105] proved that the field generated by the traces of the elements of a discrete subgroup G of Γ depends only on the commensurability class of G.

A *quaternion algebra* over a field k is a 4-dimensional non-commutative algebra A over k. The bais elements $1, i, j, ij$ have defining relations

$$i^2 = d_1, j^2 = d_2, ij = -ji$$

with $d_1, d_2 \in k$. If k is an algebraic number field within $\mathbb{C}$ then the quaternion algebra over k with defining constants d_1, d_2 is denoted by

$$A = \left(\frac{d_1, d_2}{k}\right).$$

Suppose k is algebraic number field in $\mathbb{C}$ which is ramified at all but one Archimedean place. Then a group G is an *arithmetic Kleinian group* if there exists a representation $\sigma : A \to GL(2, \mathbb{C})$ and an order $O \subset A$ such that G is commensurable with $P\sigma(O^1)$ where O^1 is the group of elements of norm 1 in O and P is the natural map

$$P : SL(2, \mathbb{C}) \to PSL(2, \mathbb{C}) = \Gamma.$$

A Kleinian group is derived from the quaternion algebra if it is of finite index in some $P\sigma(O^1)$. If k is a totally real field the G is an arithmetic Fuchsian group. Takeuchi [153] completely characterized the arithmetic Fuchsian groups.

Theorem 6.8 *Let F be a Fuchsian group of finite covolume. Then F is derived from a quaternion algebra if and only if*

(1) *The field $k = \mathbb{Q}(\mathrm{tr}(\gamma^2); \gamma \in F)$ is an algebraic number field and $\mathrm{tr}(F) = \{\mathrm{tr}(\gamma) | \gamma \in F\}$ consists of algebraic integers*

(2) *For every nonidentity $\mathbb{Q}$-homomorphishm $\phi : k \to \mathbb{C}$ the set $\phi(\mathrm{tr}(F))$ is bounded in $\mathbb{C}$.*

Maclachlan and Reid along with many others including Gehring, Helling, Hilden, G. Martin, Mennicke, Lozano and Rosenberger have worked extensively on arithmetic Kleinian groups (see [71]) and the references there), and have extended Takeuchi's results to characterize arithmetic Kleinian groups.

Theorem 6.9 ([106]) *Let G be a Kleinian group. Then:*

(1) *if G is arithmetic then G^2 is derived from a quaternion algebra;*

(2) *if G has finite covolume then G is derived from a quaternion algebra if and only if*

 (a) *$k = \mathbb{Q}(\mathrm{tr}(\gamma^2); \gamma \in G)$ is a algebraic number field and $\mathrm{tr}(G) = \{\mathrm{tr}(\gamma; \gamma \in G\}$ consists of algebraic integers, and*

 (b) *for every $\mathbb{Q}$-isomorphism $\phi : k \to \mathbb{Q}$ such that $\phi \neq \mathrm{Id}$ or complex conjugation then $\phi(\mathrm{tr}(G))$ is bounded in $\mathbb{C}$.*

Here $G^2 = \langle g^2 | g \in G\rangle$.

There has been a great deal of work on the classification of arithmetic Kleinian groups especially in terms of Fuchsian subgroups. Much of this is tied to the Bianchi groups discussed in the last section. We mention the following results.

Theorem 6.10 *Every non-elementary Fuchsian subgroup of Γ_d is arithmetic and arises from a quaternion algebra*

$$A = \left(\frac{-d, D}{\mathbb{Q}}\right)$$

for some D and has finite covolume as a Fuchsian group. Further each Γ_d contains infinitely many wide commensurability classes of non-elementary Fuchsian subgroups.

More generally:

Theorem 6.11 *Every non-elementary Fuchsian subgroup of an arithmetic Kleinian group is a subgroup of an arithemetic Fuchsian group.*

Finally:

Theorem 6.12 *The classes of Bianchi groups* Γ_d *are the commensurability classes of non-co-compact arithmetic Kleinian groups.*

We discuss additional results on the classification of Kleinian groups in the next section.

7 Classification of Kleinian groups

There has been considerable work on the classification of Kleinian groups. This has taken two different approaches. The first is by various different types of generators and the second is in combination with arithmeticity. Machlachlan and Rosenberger [111] showed that for fixed $n \geqslant 1$ there are up to conjugacy in $\mathrm{PGL}(2, \mathbb{R})$ only finitely many arithmetic Fuchsian groups which are generated by $\leqslant n$ elements. They also determined the commensurability classes of arithmetic Fuchsian surface groups of genus 2 (see [113]).

By a result of Machlachlan and Martin [110] there are up to conjugacy in Γ only finitely many two-generator arithmetic Kleinian groups generated by two elliptic elements. However in distinction to the case of Fuchsian groups there are infinitely many conjugacy classes three-generator arithmetic Kleinian groups.

The two-generator arithmetic Fuchsian groups were completely classified by Takeuchi [153],[154],[155] and Maclachlan and Rosenberger [111],[112]. There are only 85 arithmetic triangle groups. There has also been a classification of arithmetic Fuchsian groups with various different types of signatures (see [71]). There is an ongoing research project Gehring, Machlachlan and Martin (see [71] and the references there) to classify the two-generator arithmetic Kleinian groups. They showed that up to conjugacy in Γ there are only 41 non-cocompact arithmetic Kleinian group which can be generated by two elements of finite order.

Maclachlan, Martin and Mckenzie [109] described and classified all cocompact arithmetic Kleinian groups generated by two elliptic elements of finite order which have quadratic invariant trace field. There are 26 of them, 23 of which are Dehn filings of knot or link complements. The remaining are generalized triangle groups. Maclachlan and Martin [110] also identify all non-elementary Kleinian groups with two elliptic generators whose commutator is also elliptic.

Conder, Gehring, Maclachlan and Martin (see [30] and [72] have a research program to identify all two-generator arithmetic Kleinian groups. They showed there are up to conjugacy only finitely many two-generator arithmetic Kleinian groups generated by a pair of elliptic or parabolic elements. Up to conjugacy there are exactly 4 arithmetic Kleinian groups generated by two parabolics elements. These are all knot and link complements. There are exactly 14 conjugacy classes of two-generator arithmetic Kleinian groups with one generator elliptic and the other parabolic. In their commensurability classes are exactly 5 Bianchi groups Γ_d with $d = 1, 2, 3, 7, 15$.

Further work on Kleinian groups with two generators can be found in the work of Klimenko [96], Klimenko and Sakuna [98] and in the thesis of Q. Zhang (see [97]). Klimenko and Kopteva ([97] and the references there) have a program to classify two-generator Kleinian groups whose generators have real parameters. Fine and Rosenberger describe all generating pairs of two-generator Fuchsian groups (see the book [59] and the references there).

7.1 Cyclically pinched and conjugacy pinched one-relator groups

Surface groups have been pivotal in the development of combinatorial group theory (see [2]). From the point of view of this paper what has been crucial is that they have faithful representations in our primary group Γ and hence are linear. This has been generalized in several ways, in particular in terms of cyclically pinched and conjugacy pinched one-relator groups.

If $g \geqslant 2$ then each orientable surface group S_g has a free product with amalgamation decomposition of the form

$$S_g = F_1 \underset{U=V}{\star} F_2$$

where F_1 is the free group on $a_1, b_1, \ldots, a_{g-1}, b_{g-1}$, F_2 is the free group on a_g, b_g and

$$U = [a_1, b_1] \ldots [a_{g-1}, b_{g-1}], V = ([a_g, b_g])^{-1}.$$

In general a *cyclically pinched one-relator group* is a group with a finite presentation of the form

$$G = F_1 \underset{U=V}{\star} F_2$$

where F_1, F_2 are free groups and U, V represent nontrivial elements in the respective free groups. Hence any orientable surface group of genus $g \geqslant 2$ falls in the larger class of cyclically pinched one-relator groups.

A *conjugacy pinched one-relator group* is the HNN analog of a cyclically pinched one-relator group. This is a group with a finite presentation of the form

$$G = \langle t, F; t^{-1}Ut = V \rangle$$

where F is a free group and U, V are nontrivial elements in F. An orientable surface group with $g \geqslant 2$ can also be expressed as a conjugacy pinched one-relator group (see the book [40]).

Before continuing we mention when groups of these types are actually free groups. For cyclically pinched groups this follows from the primitivity of elements generating the amalgamated subgroups.

Theorem 7.1 *Let* G *be a cyclically pinched one-relator group so that*

$$G = \langle a_1, \dots, a_p, b_1, \dots, b_q; U = V \rangle$$

where $U \in \langle a_1, \dots a_p \rangle, V \in \langle b_1, \dots b_q \rangle$. *Then* G *is a free group if and only if* U *is primitive in* $\langle a_1, \dots a_p \rangle$ *or* V *is primitive in* $\langle b_1, \dots b_q \rangle$.

The situation is a bit more complicated for conjugacy pinched one-relator groups.

Theorem 7.2 *Let* G *be a conjugacy pinched one-relator group so that*

$$G = \langle a_1, \dots a_p, t; tUt^{-1} = V \rangle$$

where $U, V \in \langle a_1, \dots a_p \rangle$ *with* $p \geqslant 1$ *and* $U \neq 1, V \neq 1$. *Then* G *is a free group if and only if one of the following holds:*

(1) $\langle a_1, \dots a_p \rangle$ *has a basis* $\{U, x_1, \dots, x_{p-1}\}$ *such that* V *is conjugate in* $\langle a_1, \dots, a_p \rangle$ *to some* $V_1 \in \langle x_1, \dots x_{p-1} \rangle$;

(2) $\langle a_1, \dots a_p \rangle$ *has a basis* $\{V, x_1, \dots, x_{p-1}\}$ *such that* U *is conjugate in* $\langle a_1, \dots, a_p \rangle$ *to some* $U_1 \in \langle x_1, \dots x_{p-1} \rangle$.

Cyclically pinched and conjugacy pinched one-relator groups share many general properties with surface groups. This is especially true with linearity results, that is results also shared by linear groups. Wehfritz [157] showed that a cyclically pinched one-relator group where neither U nor V is a proper power has a faithful representation over a commutative field and is hence linear. Using a result of Shalen [150] and generalized by Fine and Rosenberger [56] if neither U nor V is a proper power then a cyclically pinched one-relator group has a faithful representation in $PSL(2, \mathbb{C})$ (see [63] and [64]). Further under the same conditions Fine, Kreuzer and Rosenberger [49] showed that there is faithful representation in $PSL(2, \mathbb{R})$. We will say more about this in Section 4. In particular cyclically pinched one-relator groups are residually finite and coherent, that is finitely generated subgroups are finitely presented, a result originally due to Karrass and Solitar [83]. We summarize many of these results in the following theorem.

Theorem 7.3 *Let G be a cyclically pinched one-relator group. Then:*

(1) G *is residually finite* ([13]*);*

(2) G *has a solvable conjugacy problem (see* [101]*) and is conjugacy separable* ([36]*);*

(3) G *is subgroup separable* ([25] *and generalized in* [1]*);*

(4) *if neither U nor V is a proper power then G has a faithful representation over some commutative field* ([158]*);*

(5) *if neither* U *nor* V *is a proper power then* G *has a faithful representation in* $PSL(2, \mathbb{C})$ *([56]) and* $PSL(2, \mathbb{R})$ ([49]*);*

(6) *if either* U *or* V *is not a proper power then* G *is hyperbolic* ([22], [83],[88]*);*

(7) *if either* U *or* V *is not a proper power then* G *has a faithful representation in* $PSL(2, \mathbb{R})$ ([49]*);*

(8) *if neither* U *nor* V *is in the commutator subgroup of its respective factor then* G *is free-by-cyclic* ([15]*);*

(9) *in* $p + q > 2$ *then* G *is SQ-universal* ([139]*). In particular* G *contains a nonabelian free group.*

Recall that a group G is *SQ-universal* if every countable group can be embedded as a subgroup of a quotient of G. SQ-universality is one measure of largeness for an infinite group (see [59]).

We remark that (9) is not correct if $p + q = 2$ and

$$G = \langle a, b; a^2 = b^2 \rangle;$$

it is correct if $p + q = 2$ and

$$G = \langle a, b; a^r = b^s \rangle$$

with $r, s > 1$ and $r + s > 4$.

Rosenberger using Nielsen cancellation, has given a positive solution to the isomorphism problem for cyclically pinched one-relator groups.

Theorem 7.4 ([138]) *The isomorphism problem for any cyclically pinched one-relator group is solvable; given a cyclically pinched one-relator group* G *there is an algorithm to decide in finitely many steps whether an arbitrary one-relator group is isomorphic or not to* G.

Although not directly connected to cyclically pinched one-relator groups we mention that Dahmani and Guiradel [33] have proved that all one-relator groups with torsion have a solvable isomorphism problem. This is an outgrowth of their solution to the isomorphism problem for hyperbolic groups with torsion. Sela (see [139]) had earlier proved the solvability of the isomorphism problem for torsion-free hyperbolic groups.

Conjugacy pinched one-relator groups are the HNN analogs of cyclically pinched one-relator groups and are also motivated by the structure of orientable surface groups S_g. In particular suppose

$$S_g = \langle a_1, b_1, \ldots, a_g, b_g; [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle \text{ with } g \geqslant 2.$$

Let $b_g = t$ then S_g is an HNN group of the form

$$S_g = \langle a_1, b_1, \ldots, a_g, t; tUt^{-1} = V \rangle$$

where $U = a_g$ and $V = [a_1, b_1] \ldots [a_{g-1}, b_{g-1}] a_g$. We now discuss a generalization of this.

Structurally such a group is an HNN extension of the free group F on $a_1, \ldots, a_n$ with cyclic associated subgroups generated by U and V with both U and V nontrivial and is hence the HNN analog of a cyclically pinched one-relator group. Groups of this type arise in many different contexts and share many of the general properties of the cyclically pinched case. However many of the proofs become tremendously more complicated in the conjugacy pinched case as compared to the cyclically pinched case. Furthermore in most cases additional conditions on the associated elements U and V are necessary. To illustrate this we state a result (see [59]) which gives a partial solution to the isomorphism problem for conjugacy pinched one-relator groups.

Theorem 7.5 *Let*

$$G = \langle a_1, \ldots, a_p, t; tUt^{-1} = V \rangle$$

be a conjugacy pinched one-relator group and suppose that neither U *nor* V *is a proper power in the free group on* $a_1, \ldots, a_p$*. Suppose further that there is no Nielsen transformation from* $\{a_1, \ldots, a_p\}$ *to a system* $\{b_1, \ldots, b_p\}$ *with* $U \in \{b_1, \ldots, b_{p-1}\}$ *and that there is no Nielsen transformation from* $\{a_1, \ldots, a_p\}$ *to a system* $\{c_1, \ldots, c_p\}$ *with* $V \in \{c_1, \ldots, c_{p-1}\}$*. Then:*

(1) G *has rank* $p + 1$ *and for any minimal generating system for* G *there is a one-relator presentation;*

(2) *the isomorphism problem is solvable.*

More information about both cyclically pinched one-relator groups and conjugacy pinched one-relator groups is in ([59] or [64]).

From a result of Mal'cev [120] the residual finiteness of the S_g implies that the S_g are hopfian, that is they cannot be isomorphic to a proper factor group.

7.2 The surface group conjecture

A complete group theoretic characterization of surface groups was provided by the solution of the *surface group conjecture* by Ciobanu, Fine and Rosenberger [29] building on work of H. Wilton [158]. In the Kourovka notebook [122] Melnikov proposed the following problem.

Question 7.6 *Suppose that* G *is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Must* G *be a surface group?*

As asked by Melnikov the answer is no. Recall that the Baumslag-Solitar groups $BS(m,n)$ are the groups

$$BS(m,n) = \langle a, b; a^{-1}b^m a = b^n \rangle$$

with $m \neq 0$ nad $n \neq 0$. If $|m| = |n|$ or either $|m| = 1$ or $|n| = 1$ these groups are residually finite. They are hopfian if $|m| = 1$ or $|n| = 1$ or both m and n have the same prime factors. In all other cases they are non-hopfian. If either $|m| = 1$ or $|n| = 1$ then every subgroup of finite index is again a Baumslag-Solitar group and therefore a one-relator group. It follows that besides the surface groups the groups $BS(1,m)$ also satisfy the hypotheses Melnikov's question. We then have the following conjecture.

Conjecture 7.7 (Surface Group Conjecture A) *Suppose that* G *is a residually finite non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group. Then* G *is either a surface group or a Baumslag-Solitar group* $B(1,m)$ *for some integer* m*.*

We note that the groups $B(1,1)$ and $B(1,-1)$ are surface groups. In surface groups not isomorphic to $B(1,1)$ or $B(1,-1)$, subgroups of infinite index must be free groups and there are noncyclic free groups. This is not true in the groups $BS(1,m)$. To avoid the Baumslag-Solitar groups, if they are not isomorphic to $B(1,1)$ or $B(1,-1)$, Surface Group Conjecture A, was modified to (see [29]):

Conjecture 7.8 (Surface Group Conjecture B) *Suppose that* G *is a non-free, non-cyclic one-relator group such that every subgroup of finite index is again a one-relator group and every subgroup of infinite index is a free group and* G *contains nonabelian free groups as subgroups of infinite index. Then* G *is a surface group.*

Using the structure theorem for fully residually free groups in terms of its JSJ decomposition (see [40] and the references there), Fine, Kharlampovich, Myasnikov, Remeslennikov and Rosenberger [47] made some progress on these conjectures. Finally Ciobanu, Fine and Rosenberger [29] building on work of H. Wilton [158] settled the surface group conjecture if G is assumed to be either a cyclically pinched one-relator group or a conjugacy pinched one-relator group.

We say that a group G satisfies *property IF* if every subgroup of infinite index is free. Recall that the standard one-relator presentation for a surface group allows for a decomposition as a cyclically pinched one-relator group and as a conjugacy pinched one-relator group (see [64]). In particular the following results are proved in [47].

Theorem 7.9 *Suppose that* G *is a finitely generated fully residually free group with property IF. Then G is either a free group or a cyclically pinched one relator group or a conjugacy pinched one-relator group.*

Corollary 7.10 *Suppose that* G *is a finitely generated fully residually free group with property IF. Then G is either free or every subgroup of finite index is freely indecomposable and hence a one-relator group.*

Further if the surface group conjecture is true then a group satisfying the conditions of the conjecture must be hyperbolic or free abelian of rank 2. The following is also proved.

Theorem 7.11 ([39]) *Let* G *be a finitely generated fully residually free group with property IF. Then either* G *is hyperbolic or* G *is free abelian of rank* 2*.*

In light of these results the following modified version of the surface group conjecture was given.

Conjecture 7.12 (Surface Group Conjecture C) *Suppose that* G *is a finitely generated non-free freely indecomposable fully residually free group with property IF. Then G is a surface group.*

Finally we note that although the focus in [39] was on property IF there has been some evidence for the Surface Group Conjecture based on the subgroups of finite index. Note that an orientable surface group of genus $g \geqslant 2$ with the presentation

$$G = \langle a_1, b_1, \ldots, a_g, b_g; [a_1, b_1] \ldots . [a_g, b_g] \rangle$$

also has a presentation

$$G = \langle x_1, \ldots, x_n; x_1 \ldots x_n x_1^{-1} \ldots x_n^{-1} = 1 \rangle$$

with n even. Curran [31] has proved the following.

Theorem 7.13 *Let* G *be a one-relator group with the presentation*

$$G = \langle x_1, \ldots, x_n; x_1^{\nu_1} \ldots x_n^{\nu_n} x_1^{-\nu_1} \ldots x_n^{-\nu_n} = 1 \rangle.$$

Then, if n is odd, there exist normal subgroups of finite index which do not have one-relator presentations. In particular if

$$G = \langle x_1, \ldots, x_n; x_1 \ldots x_n x_1^{-1} \ldots x_n^{-1} = 1 \rangle$$

then every subgroup of finite index is again a one-relator group if and only if n is even and hence G is a surface group.

Using the following result of Wilton combined with results of Guildenhuys, Kharlampovich and Myasnikov and the Karrass-Solitar subgroup theorems for free products with amalgamation, Ciobanu, Fine and Rosenberger [29] settled Surface group Conjecture C and the general conjecture for cyclically pinched and conjugacy pinched one-relator groups.

Theorem 7.14 ([158]) *Let G be a hyperbolic one-ended cyclically pinched one-relator group or a hyperbolic one-ended conjugacy pinched one-relator group. Then either G is a surface group, or G has a finitely generated non-free subgroup of infinite index.*

Let G be a finitely generated group. Let $S \subset G$ be a finite generating set of G and let $\Gamma(G, S)$ be the Cayley graph of G with respect to S. Then the number of ends of G is defined as

$$e(G) = e(\Gamma(G, S)).$$

It was proved by Stallings that a finitely generated group G has more than one end if and only if the group G admits a nontrivial decomposition as an amalgamated free product or an HNN extension over a finite subgroup.

In [29] it was proved that surface group conjecture C is true.

Theorem 7.15 *Suppose that G is a finitely generated non-free freely indecomposable fully residually free group with property IF. Then G is a surface group. That is Surface Group Conjecture C is true.*

Thus fully residually free and property IF completely characterize surface groups.

Theorem 7.16 *G is a surface group if and only if G is finitely generated, non-free, indecomposable, fully residually free and satisfies property IF.*

The main result in [29] is that the Surface Group Conjecture is true if G is a cyclically pinched or conjugacy pinched one-relator.

Theorem 7.17

(1) *Let* G *be a cyclically pinched one-relator group satisfying property IF. Then* G *is a free group or a surface group.*

(2) *Let* G *be a conjugacy pinched one-relator group satisfying property IF. Then* G *is a free group, a surface group or a solvable Baumslag-Solitar group.*

We finally remark that Ponzoni had some success with Surface Group Conjecture A for two-generator one-relator groups (see [128]).

8 One-relator products and groups of F-type

One of the cornerstones of combinatorial group theory is the theory of one-relator groups which is direct outgrowth of the theory of surface groups. In this section we present several generalization of both one-relator groups and Fuchsian groups via group presentations. These generalizations have been widely researched and also use representations into Γ.

The most fundamental result in the theory of one-relator groups is the Freiheitssatz or independence theorem of Magnus.

Theorem 8.1 (The Freiheitssatz) *let* $G = \langle x_1, \dots, x_n; R\rangle$ *be a one-relator group with* R *a cyclically reduced word in the free group on* $\{x_1, .., x_n\}$ *that involves all the generators. Then the subgroup of* G *generated by any proper subset of the generators is a free group with these generators as a free basis.*

Magnus [113] proved this in 1929 although he claims it was known earlier to Max Dehn who suggested the theorem to Magnus. A comprehensive discussion of the Freiheitssatz and its impact on general combinatorial group theory can be found in the paper [55] or in the book [59].

The theory of one-relator groups has been generalized to the theory of *one-relator products*. Let $\{G_i\}i \in I$ be an indexed collection of groups. Then a *one-relator product* is the quotient $G = A/N(R)$ where A is the free product $A = \star_i A_i$ and $N(R)$ is the normal closure of a single non-trial word R in the free product. The groups A_i are called the *factors* and R is the *relator*. As in the one-realtor case we say that R involves A_I if R has a non-trivial syllable from A_I.

If $R = S^m$ with S a non-trivial cyclically reduced word in the free product and $m \geqslant 2$ then R is a *proper power*. In this formulation a one-relator group is a one-relator product of free groups.

8.1 The Freiheitssatz and essential representations

We say that the one-relator product satisfies a Freiheitssatz (abbreviated FHS) if each factor injects into G via the identity map. There are two approaches to the Friehitssatz for one-relator products, The first is to impose coniditons on the factors and the second is to impose conditions on the relator. A group G is locally indicable if every finitely generated subgroup has an infinite cyclic quotient. Howie [75] and independently Brodskii [24] proved that a one-relator product of locally indicable factors satisfies the Freiheitssatz.

In the case where $r = S^m$ small cancellation theory (see [59]) shows that the FHS holds if $m \geqslant 7$. Gonzales-Acuna and Short proved the FHS for $m = 6$ while Howie ([75],[76]) proved it for

$$m = 4, 5 \quad \text{and} \quad m = 3$$

(see [34]) with additional conditions. In general, the case $m = 2$ is open.

Fine, Howie and Rosenberger [43] and Baumslag, Morgan and Shalen [16] handled special cases of $m = 2$ using special representations into Γ tying this to the rest of this paper.

The closest one-relator products to discrete subgroups of Γ are the *one-relator products of cyclics*. These are groups of the form

$$G = \{x_1, \ldots, x_n; x_i^{e_i} = 1, i = 1, \ldots, n, R^m = 1\}, \quad n \geqslant 2,$$

where $e_i = 0$ or $e_i \geqslant 2$ for $i = 1, \ldots, n$ and R is a cylically reduced word in the free product of cyclics on

$$\{x_1, \ldots, x_n\}$$

involving all x_i and $m \geqslant 1$.

An *essential representataion* of a one relator product of cyclics is a representation $\rho : G \to \Gamma$ such that for each $i = 1, \ldots n$ $\rho(x_i)$ has order e_i and $\rho(R)$ has order m. Fine, Howie and Rosenberger [43] used essential representations to study one-relator products of cyclics.

Theorem 8.2 *If $m \geqslant 2$ then any one-relator products of cyclics has an essential representation that is faithful on the free product of cyclics* *on* $\{x_1, .., x_{n-1}\}$*. In particular the Freiheitssatz holds for any one-relator product of cyclics.*

Although one-relator products of cyclics may not be linear, a research program was initiated by Fine, Howie and Rosenberger and including many others to study linearity properties of groups of this type (see [59]). These properties included the Tits Alternative, that is that very subgroup is either virtually solvable or contains a non-abelian free subgroup, the virtual torsion-free property of Selberg and the existence of an Euler characteristic. These results can be found in the book (see [59]).

8.2 The generalized triangle and generalized tetrahedron groups

An ordinary triangle group $T(p, q, r)$ is a group with q presentation

$$T(p, q, r) = \langle a, b; a^p = b^q = (ab)^r = 1 \rangle$$

This is the orientation preserving subgroup of the group generated by reflections in the sides of spherical, hyperbolic or Euclidean triangle. the finite tringle groups are completely classified (see [59]). In particular $T(p, q, r)$ is finite if and only if $p, q, r \geqslant 2$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

the important relevant results about ordinary triangle group are summarized in the next theorem.

Theorem 8.3 *Let*

$$G = \langle a, b; a^p = b^q = (ab)^r = 1 \rangle$$

with $p, q, r \geqslant 2$ *and let* $s(G) = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$*. Then*

(1) G *admits a faithful representation into* Γ*. The image group is Fuchsian if and only if* $s(G) < 1$*;*

(2) G *is finite if and only if* $s(G) > 1$*;*

(3) *If* $s(G) < 1$ *then* G *contains an orientable surface group of genus* $g \geqslant 2$ *as a subgroup of finite index (in particular* G *has a subgroup of finite index which maps onto a nonabelian free group);*

(4) *The elements* $a, b, (ab)$ *have exact orders* p, q, r *respectively;*

(5) G *is virtually torsion-free;*

(6) G *satisfies the Tits alternative.*

A natural generalization of the ordinary triangle groups are two-generator one-relator products of cyclics. These are called *generalized triangle groups* and have presentations of the form

$$\langle a, b; a^p = b^q = (R(a, b)^r = 1\rangle$$

where $R(a, b)$ is a nontrivial cyclically reduced word in the free product on a and b and $p, q, r \geqslant 2$. These arise in many different places, especially in presentations of knot and link groups (see [71]). What has been of research interest concerning the generalized triangle groups is the extension of the linearity properties of the ordinary triangle groups. In particular when does a generalized triangle group have a faithful representation into Γ. Fine, Howie and Rosenberger [43] proved that a generalized triangle group always has an essential representation into Γ. Using this essential representation has been the main technqiue in studying the generalized triangle groups and this work has mainly centered on the classification of the finite generalized triangle groups and the Tits Alternative.

Howie, Metafsis and Thomas [81] and Levai, Rosenberger and Souvignier [100] completely classified the finite generalizied triangle groups.

Theorem 8.4 *Besides the finite ordinary triangle groups there are, up to isomorphsim only 12 finite generalized triangle groups. The largest has the presentation* $\langle a, b; a^2 = b^3 = (ababab ab^2ab^2abab^2ab^2)^2 = 1\rangle$ *and has order* $2^{20}3^45$.

Howie, Metafsis and Thomas [80] also determined the algebraic structure of the finite generalized triangle groups.

Relative to the Tits alternative we have the following theorem.

Theorem 8.5 *The Tits alternative holds for any generalized triangle group with* $m > 2$.

It has been conjectured that it also hold if $m = 2$. Considerable results on the case $m = 2$ was done by J.Howie [79] and Howie and Konovalov [80]. In a series of papers J. Howie ([76],[77],[79]) gave positive answers for the cases $m = 2$ and $p, q \geqslant 3$. There also partial results for $m = 2$ and $p = 2$,

Although there are no general results there are many examples of generalized triangle groups that have faithful discrete representations into Γ. Some of these examples have images with finite covolume.

The essential representation technique has also been used effectively to study a related class of group *the generalized tetrahedron groups*. These group generalize the ordinary tetrahedron groups. An ordinary tetrahedron group is the orientation preserving subgroup of the group generated by reflections in the sides of spherical, hyperbolic or Euclidean tetrahedron. Such a group has a presentation of the form

$$T = \langle a, b, c; a^p = b^q = c^r = (ab^{-1})^m = (ca^{-1})^n = (bc^{-1})^t = 1\rangle$$

with $p, q, r, m, n, t \geqslant 2$. A generalized tetrahedron group is a group with a presentation

$$T = \langle a, b, c; a^p = b^q = c^r = (R_1(a, b))^m = (R_2(a, c))^n = (R_3(b, c))^t = 1\rangle$$

with $p, q, r, m, n, t \geqslant 2$ and $R_a(a, b), R_2(a, c), R_3(b, c)$ are non-trivial cyclically reduced words in the free products on the generators they involve.

Group theoretically such a group is a triangular product of the generalized triangle groups

$$\langle a, b; a^p = b^q = (R_1(a, b))^m = 1\rangle$$
$$\langle a, c; a^p = c^r = (R_2(a, c))^n = 1\rangle$$
$$\langle b, c; b^q = c^r = (R_3(b, c))^T = 1\rangle$$

with edge amalgamations over the cyclic subgroups $\langle a\rangle, \langle b\rangle, \langle c\rangle$.

Fine, Levin, Roehl and Rosenberger showed that any generalized tetrahedron group admits an essential representation into Γ. These groups have been studied by using a combination of essential representations and Gersten-Stallings angles in triangular products. A complete list of all finite generalized tetrahedron groups is given in the papers [44] and [46]. If $(m, n, t) \neq (2, 2, 2)$ then the Tits alternative holds [45]. In that paper there are many special cases considered for $(m, n, t) = (2, 2, 2)$.

More results on these groups can be found in the book [59].

8.3 Groups of F-type

Another class of groups that are embeddable in Γ and are very close to Fuchsian groups are *groups of F-type*.

A *group of F-type* is a group with a presentation of the form

$$G = \langle a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n;$$

$$a_1^{e_1} = \cdots = a_m^{e_m} = b_1^{f_1} = \cdots = b_n^{f_n} = 1, u = v\rangle$$

where $1 \leqslant n, m$, $e_i = 0$ or $e_i \geqslant 2$ for $i = 1, 2, \ldots, m$, $f_j = 0$ or $f_j \geqslant 2$ for $j = 1, 2, \ldots, n$,

$$u = u(a_1, a_2, \ldots, a_m)$$

is cyclically reduced and of infinite order in the free product on $a_1, a_2, \ldots, a_m$ and $v = v(b_1, b_2, \ldots, b_n)$ is cyclically reduced and of infinite order in the free product on $b_1, b_2, \ldots, b_n$. In more general language they can be described as cyclically pinched one-relator products of cyclics. Hence cyclically pinched one-relator groups (see [59]) are groups of F-type.

For this paper we assume that group of F-type does not decompose as a free product of cyclic groups.

Groups of F-type were introduced by Fine and Rosenberger in [55] as natural algebraic generalizations of Fuchsian groups via presentations. Any finitely generated co-compact Fuchsian group of geometric rank > 2, is a group of F-type via its Poincaré presentation. What ties this to the rest of the paper is the result fo Fine and Rosenberger [55] that any group of F-type such that u and v are not proper powers in the respective free products has a faithful representation into Γ. From this many nice properties of groups of F-type were obtained in particular properties generalizing those of Fuchsian groups. Fine, Moldenhauer and Rosenberger [53] extend this to show that any group of F-type that is hyperbolic has a faithful representations into $\mathrm{PSL}(2, \mathbb{R})$. A group of F-type is hyperbolic unless u is a proper power or a product of two elements of order 2 and v is also a proper power or a product of two elements of order 2 (see [83]).

From the faithful representations of hyperbolic groups of F-type into $\mathrm{PSL}(2, \mathbb{R})$ we obtain the linearity properties of Fuchsian groups. Recall that a group is *commutative transitive* abbreviated *CT* if commutativity is transitive on nonidentity elements. Any subgroup of $PSL(2,\mathbb{R})$ is commutative transitive (see [40]). We summarize these.

Theorem 8.6 *Let G be a hyperbolic group of F-type Then:*

(1) G *is virtually torsion-free.*

(2) G *is residually finite and hopfian.*

(3) G *is commutative transitive.*

(4) *If $e_i \geqslant 2$ then a_i has order exactly e_i. The analogous result holds for the b_j.*

(5) *Any element of finite order is conjugate to a power of some a_i or some b_j.*

(6) *Any finite subgroup of G is cyclic and conjugate to a subgroup of a certain $\langle a_i \rangle$ or a certain $\langle b_j \rangle$.*

(7) *If G is a cyclically pinched one-relator group and if neither u nor v is in the commutator subgroup of its respective factor then G is free-by-cyclic (see [19]).*

(8) *In $m+n>2$ then G is SQ-universal. In particular G contains a nonabelian free group.*

For proofs of these see [61], [43] or [59]. We note that (2) was proved originally under the restriction that either u or v is is not a proper power. Using results of Allenby [3] this restriction can be removed.

Closely tied to commutative transitivity is the concept of being CSA. A group G is *CSA* or *conjugately separated abelian* if maximal abelian subgroups are malnormal. These concepts have played a prominent role in the studies of fully residually free groups, limit groups and discriminating groups (see [39] and [42]). They also play a role in the solution to the Tarski problems (see the next section) CSA always implies CT. In general the class of CSA groups is a proper subclass of the class of CT groups however they are equivalent in the presence of residual freeness. For hyperbolic groups of F-type we have.

Theorem 8.7 *Let G be a hyperbolic group of F-type. Assume that G is torsion-free or has only odd torsion, that is, e_i is odd if $e_i \geqslant 2$ and f_j is odd if $f_j \geqslant 2$. Then G is CSA.*

Recall that linear groups satisfy the Tits alternative, that is they either contain a free subgroup of rank 2 or are virtually solvable. From linearity and an examination of the possible solvable cases we have for groups of F-type.

Theorem 8.8 *Let* G *be a group of F-type. The either* G *contains a free subgroup of rank* 2 *or* G *is solvable with one of the following presentations*

(1) $\langle a, b; a^2b^2 = 1\rangle$,

(2) $\langle a, b, c; a^2 = b^2 = abc^2 = 1\rangle$,

(3) $\langle a, b, c, d; a^2 = b^2 = c^2 = d^2 = abcd = 1\rangle$.

Further if G *is not solvable then* G *is SQ-universal.*

Note that the SQ-universality follows in the non-solvable cases because G has a subgroup of finite index that maps onto a free group of rank 2.

Recall that a group G is *conjugacy separable* if given any nontrivial $g, h \in G$ that are not conjugate then there exists a finite quotient G^* of G where the images of g and h are still not conjugate, Conjugacy separability implies residual finiteness. Further G is *subgroup separable* or LERF if given any subgroup $H \subset G$ and $g \notin H$ then there exists a finite quotient G^* of G with $g^* \notin H^*$ with g^*, H^* the images of g, H in G^*.

Using results of Allenby [3], Allenby and Tang [5] and Niblo [128] and Aab and Rosenberger [1] groups of F-type were shown to be both conjugacy separable and subgroup separable (see [56], [43] or [59]).

Theorem 8.9 *Let* G *be a group of F-type. Then:*

(1) G *is conjugacy separable;*

(2) G *is subgroup separable.*

Dahmani and Guiradel [33] proved that all hyperbolic groups have a solvable isomorphism problem. Sela ([140]) had proved this earlier for torsion-free hyperbolic groups. In general hyperbolic groups have solvable word problem and conjugacy problem. These then apply to the hyperbolic groups of F-type. Earlier Rosenberger [138] using Nielsen cancellation showed that the isomorphism problem is solvable in the class of cyclically pinched one-relator groups in general without the additional condition of hyperbolicity.

Theorem 8.10 *For a hyperbolic group of F-type*

(1) *the word problem is solvable;*

(2) *the conjugacy problem is solvable;*

(3) *the isomorphism problem in the class of hyperbolic groups of F-type is solvable.*

9 Limit groups and the Tarski problems

Alfred Tarski in 1940 made three well-known conjectures concerning nonabelian free groups. We call these the *Tarski Problems* or *Tarski Conjectures* and they asked, among other things, whether all nonabelian free groups satisfy the same first-order or elementary theory. A proof of these conjectures was given by Kharlampovich and Myasnikov ([86],[87],[91],[92],[93],[90]) and independently by Sela ([141],[142],[144],[143],[145],[146]). Both proofs were monumental works involving the development of several new areas of mathematics. In the case of Kharlampovich and Myasnikov this was called it algebraic geometry over groups. Sela called it *diophantine geomtry*. In their proofs both sets of authors completely classified groups which have the same first order or elementary theory as the class of free groups. These are known as *elementary free groups*. Fine and Rosenberger ([147],[149]) tied this to our group Γ by proving that any elementary free group has a faithful representation in Γ. One of their two proofs is constructive using the JSJ decomposition of an elementary free group. We explain all these ideas in this section.

A first-order or elementary sentence in group theory has logical symbols

$$\forall, \exists, \vee, \wedge, \sim$$

but no quantification over sets. A first-order theorem in a free group is a theorem that says a first-order sentence is true in all nonabelian free groups.

We start with a first-order language appropriate for group theory. This language, which we denote by L_0, is the first-order language with equality containing a binary operation symbol $\cdot$ a unary operation symbol $^{-1}$ and a constant symbol 1. A *universal sentence* of L_0 is one of the form

$$\forall \overline{x}\{\phi(\overline{x})\}$$

where $\overline{x}$ is a tuple of distinct variables, $\phi(\overline{x})$ is a formula of L_0 containing no quantifiers and containing at most the variables of $\overline{x}$. Similarly an *existential sentence* is one of the form

$$\exists \overline{x}\{\phi(\overline{x})\}$$

where $\overline{x}$ and $\phi(\overline{x})$ are as above. A *universal-existential sentence* is one of the form

$$\forall \overline{x} \exists \overline{y}\{\phi(\overline{x}, \overline{y})\}.$$

Similarly defined is an *existential-universal sentence*. It is known that every sentence of L_0 is logically equivalent to one of the form

$$Q_{1 x_1} \dots Q_{n x_n} \phi(\overline{x})$$

where

$$\overline{x} = (x_1, \dots, x_n)$$

is a tuple of distinct variables, each Q_i for $i = 1, \dots, n$ is a quantifier, either $\forall$ or $\exists$, and $\phi(\overline{x})$ is a formula of L_0 containing no quantifiers and containing free at most the variables $x_1, \dots, x_n$. Further vacuous quantifications are permitted. Finally a *positive sentence* is one logically equivalent to a sentence constructed using (at most) the connectives $\vee, \wedge, \forall, \exists$.

If G is a group then the *universal theory* of G consists of the set of all universal sentences of L_0 true in G. We denote the universal theory of a group G by $Th_{\forall}(G)$. Since any universal sentence is equivalent to the negation of an existential sentence it follows that two groups have the same universal theory if and only if they have the same *existential theory*. The set of all sentences of L_0 true in G is called the *first-order theory* or the *elementary theory* of G. We denote this by $Th(G)$. We note that being *first-order* or *elementary* means that in the intended interpretation of any formula or sentence all of the variables (free or bound) are assumed to take on as values only individual group elements - never, for example, subsets of nor functions, on the group in which they are interpreted.

We say that two groups G and H are *elementarily equivalent* (symbolically $G \equiv H$) if they have the same first-order theory, that is

$$Th(G) = Th(H).$$

Group monomorphisms which preserve the truth of first-order formulas are called elementary embeddings. Specifically, if H and G are groups and

$$f : H \to G$$

is a monomorphism then f is an *elementary embedding* provided whenever

$$\phi(x_0, \dots, x_n)$$

is a formula of L_0 containing free at most the distinct variables

$$x_0, \dots, x_n$$

and $(h_0, \dots, h_n) \in H^{n+1}$ then $\phi(h_0, \dots, h_n)$ is true in H if and only if

$$\phi(f(h_0), \dots, f(h_n))$$

is true in G. If H is a subgroup of G and the inclusion map

$$i : H \to G$$

is an elementary embedding then we say that G is an *elementary extension* of H.

Two very important concepts in the elementary theory of groups, are *completeness* and *decidability*. Given a nonempty class of groups $\mathcal{X}$ closed under isomorphism we say that its first-order theory is *complete* if given a sentence ϕ of L_0 either ϕ is true in every group in $\mathcal{X}$ or ϕ is false in every group in $\mathcal{X}$. The first-order theory of $\mathcal{X}$ is *decidable* if there exists a recursive algorithm which, given a sentence ϕ of L_0 decides whether or not ϕ is true in every group in $\mathcal{X}$.

The positive solution to the Tarski Problems, given by Kharlampovich and Myasnikov and independently by Sela (see [40]) is given in the next three theorems:

Theorem 9.1 (Tarski 1) *Any two nonabelian free groups are elementarily equivalent. That is any two nonabelian free groups satisfy exactly the same first-order theory.*

Theorem 9.2 (Tarski 2) *If the nonabelian free group H is a free factor in the free group G then the inclusion map $H \to G$ is an elementary embedding.*

In addition to the completeness of the theory of the nonabelian free groups the question of its *decidability* also arises. The *decidability* of the theory of nonabelian free groups means the question of whether there exists a recursive algorithm which, given a sentence ϕ of L_0, decides whether or not ϕ is true in every nonabelian free group. Kharlampovich and Myasnikov, in addition to proving the two above Tarski conjectures also proved the following.

Theorem 9.3 (Tarski 3) *The elementary theory of the nonabelian free groups is decidable.*

Prior to the solution of the Tarski problems it was asked whether there exist non-free *elementary free groups*, that is whether there exists non-free groups that have exactly the same first-order theory as the class of nonabelian free groups. The answer was yes, and both the Kharlampovich-Myasnikov solution and the Sela solution provide a complete characterization of the finitely generated elementary free groups. In the Kharlampovich-Myasnikov formulation these are given as a special class of what are termed NTQ groups ([86],[87],[91], [92],[93],[90]). The primary examples of non-free elementary free groups are the orientable surface groups of genus $g \geqslant 2$ and the nonorientable surface groups of genus $g \geqslant 4$.

If S_g denotes the orientable surface group of genus g recall that S_g has a one-relator presentation with a quadratic relator.

$$S_g = \langle a_1, b_1, \ldots, a_g, b_g; [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle.$$

Groups with presentations similar to this play a major role in the structure theory of fully residually free groups and NTQ groups (see [40]) .

We note that the solution to the Tarski Problems implies that any first-order theorem holding in the class of nonabelian free groups must also hold in most surface groups. In many cases proving these results directly is very nontrivial.

Theorem 9.4 (see [40],[41],[42]) *An orientable surface group of genus $g \geqslant 2$ is elementary free, that is has the same elementary theory as the class of nonabelian free groups. Further the nonorientable surface groups N_g for $g \geqslant 4$ are also elementary free.*

We need several other concepts. Let $\mathbb{X}$ be a class of groups. Then a group G is *residually* $\mathbb{X}$ if given any nontrivial element $g \in G$ there is a homomorphism

$$\phi : G \to H$$

where H is a group in $\mathbb{X}$ such that $\phi(g) \neq 1$. A group G is *fully residually* $\mathbb{X}$ if given finitely many nontrivial elements $g_1, \ldots, g_n$ in G

there is a homomorphism

$$\phi : G \to H,$$

where H is a group in $\mathbb{X}$, such that $\phi(g_i) \neq 1$ for all $i = 1, \dots, n$. Fully residually free groups have played a crucial role in the study of equations and first-order formulas over free groups. In Sela's solution to the Tarski problems finitely generated fully residually free groups are called *limit groups*. The *universal theory* of a group G consists of all universal sentences true in G. All nonabelian free groups share the same universal theory and a group G is called *universally free* if it shares the same universal theory as the class of nonabelian free groups.

As we introduced in the last section, a group G is *commutative transitive* or *CT* if commutativity is transitive on the set of nontrivial elements of G. That is if

$$[x, y] = 1 \quad \text{and} \quad [y, z] = 1$$

for nontrivial elements $x, y, z \in G$ then $[x, z] = 1$. A subgroup H of a group G is *malnormal* if

$$x^{-1}Hx \cap H = \{1\}$$

if $x \notin H$. A group G is *CSA* if maximal abelian subgroups are malnormal. CSA implies commutative transitivity but there exist CT groups that are not CSA. For example it can be shown that a noncyclic one-relator group G with torsion is CT but not CSA if G has elements of order 2 (see [54]). Another example of a CT group that is not CSA is the infinite dihedral group

$$G = \langle a, b; a^2 = b^2 = 1 \rangle.$$

It is straightforward that free products of abelian groups are CT and hence G is CT. On the other hand the commutator subgroup G' is the cyclic subgroup of G generated by ab. A nonabelian CSA group cannot have a nontrivial abelian normal subgroup and hence G is not CSA.

Remeslennikov [130] and independently Gaglione and Spellman [70] proved the following remarkable theorem which became one of the cornerstones in the proof of the Tarski problems (see [40]).

Theorem 9.5 *Suppose* G *is nonabelian and residually free. Then the following are equivalent:*

(1) G *is fully residually free;*

(2) G *is commutative transitive;*

(3) G *is universally free.*

Therefore the class of nonabelian fully residually free groups coincides with the class of residually free universally free groups. The equivalence of (1) and (2) in the theorem above was proved originally by Benjamin Baumslag [9], where he introduced the concept of fully residually free. Any finitely generated elementary free group being universally free must satisfy this theorem and hence be fully residually free.

In [29] classes of groups $\mathbb{X}$ were studied for which being fully residually $\mathbb{X}$ is equivalent to being residually $\mathbb{X}$ and commutative transitive, thus extending Baumslag's result.

Fine, Gaglione, Rosenberger and Spellman in [52] and [41] proved a series of results about elementary free groups. Many of the properties of elementary free groups are not first order. The next theorem summarizes many of these results. The proofs can be found in [52], [41] and [42].

Theorem 9.6 *Let* G *be a finitely generated elementary free group. Then:*

(1) (Magnus's Theorem) *if* $N(R) = N(S)$ *if* $R, S \in G$ *it follows that* R *is conjugate to either* S *or* S^{-1}*;*

(2) G *has cyclic centralizers of non-trivial elements. It follows that if* $x, y \in G$ *and* x,y *commute then both* x *and* y *are powers of a single element* $w \in G$*;*

(3) *if* $x, y, u, v \in G$ *with* $[x, y] \neq 1$ *and* u, v *in the subgroup generated by* x, y *it follows that if* $[x, y]$ *is conjugate to a power of* $[u, v]$ *within* $\langle x, y\rangle$*, that is there exists a* k *with* $[x, y] = g([u, v]^k)g^{-1}$ *for some* $g \in \langle x, y\rangle$*, and* $[x, y^m] = [u, v^n]$ *it follows that* $m = n$*. Further if* $m = n \geqslant 2$ *then* y *is conjugate within* $\langle x, y\rangle$ *to* v *or* v^{-1}*;*

(4) G *is conjugacy separable;*

(5) G *is hyperbolic and stably hyperbolic;*

(6) G *is a Turner group, that is the test elements in* G *are precisely those elements that do not fall in a proper retract;*

(7) *if* G *is freely indecomposable then the automorphism group of* G *is tame.*

What ties this section to the main theme of the paper is the following result that says any finitely generated fully residually free group and hence any elementary free group can be embedded isomorphically in Γ; this was proved in [61],[62].

Theorem 9.7 *Let* G *be a finitely generated fully residually free group. Then* G *admits a faithful representation into* Γ*. Further a faithful representation can be effectively constructed.*

Since elementary free groups are fully residually free we get

Corollary 9.8 *Any finitely generated elementary free group admits a faithful representation into* Γ*.*

Fine and Rosenberger [53],[63] give two different proofs of this result (see also [40]). The first is constructive and uses the JSJ decomposition of any elementary free group. The second uses an embedding of any elementary free group in a nonstandard free group (see [40]). This result places the whole theory of elementary free groups within Γ.

In [159], answering questions posed by Bestvina, Wilton gave the following result.

Theorem 9.9 *A finitely generated fully residually free group can be embedded in any algebraic group over* $\mathbb{R}$ *which contains nonabelian free groups. In particular any finitely generated fully residually free group has a faithful representation in* $\mathrm{PSL}(2, \mathbb{R})$*.*

Wilton's proof uses the algebraic geometry over groups; the proof is not constructive, for a different proof see [40].

10 Some open problems

We present some open problems on Γ, discrete groups and onerrelator products.

Question 10.1 *Let* G *be a finitely generated non-elementary subgroup of* $\mathrm{PSL}(2,\mathbb{C})$ *or* $\mathrm{PSL}(2,\mathbb{R})$*. Is* G *finitely presented?*

We mention that it should be finitely presented by Hilbert's ideal theorem for polynomial rings.

Question 10.2 *Complete the result that generalized triangle and tetrahedron groups satisfy the Tits alternative in general. For the tetrahedron case there are many open cases.*

Question 10.3 *Finish the classification of Kleinian groups with real parameters, that is, complete the program of Klimenko and Kopteva* [97].

We mentioned that all arithmetic Fuchsian groups with two generators and all arithmetic Fuchsian surface groups of genus 2 are classified.

Question 10.4 *Classify, up to conjugacy, all arithmetic Fuchsian groups with three generators, especially those of genus* 0*.*

Question 10.5 *Does the FHS hold for all one-relator products with torsion-free factors.*

Question 10.6 *Under what conditions is a subgroup of* Γ *hyperbolic as a group?*

Question 10.7 *Can a generalized triangle group contain a cocompact Fuchsian subgroup.*

(a) *Given a subgroup of* Γ *which provides a faithful representation of a Fuchsian group as an abstract group determine conditions when the image group is discrete.*

(b) *Let* G *be a generalized triangle group or generalized tetrahedron group. Assume that* G *has a faithful representation within* Γ*. When is the image group discrete.*

Question 10.8 *Classify the algebraic properties of the non-Euclidean Bianchi groups by class number. In particular classify the amalgam structure and the structure of normal subgroups.*

Question 10.9 *Are the non-Euclidean Bianchi groups conjugacy separable.*

Question 10.10 *Given a one-relator product of cyclics with proper power relator* R^m*. Does the FHS hold in general if* $m = 2$*.*

Question 10.11 *When does a generalized triangle group have a faithful representation in* Γ.

Question 10.12 *Let* G *be a group of F-type. Determine conditions on* U, V *such that*

(a) *subgroups of finite index are again groups of F-type;*

(b) *torsion-free subgroups of finite index are one-relator groups;*

(c) *subgroups of infinite index are free products of cyclics.*

Question 10.13 *When is the automorphism group of a group of F-type tame?*

Question 10.14 *The conjugacy problem for general one-relator groups.*

Question 10.15 *Which cyclically pinched one-relator groups are fully residually free or elementary free?*

Question 10.16 *Which generalized tetrathedron groups can be generated by two elements?*

Question 10.17 *Which groups of F-type can be generated by fewer that* $n + m - 1$ *generators?*

Question 10.18 *Let* G *be a generalized triangle or generalized tetrahedron group.*

(a) *Give conditions under which* G *is decomposable as a free product with amalgamation.*

(b) *Give conditions under which* G *is SQ-universal.*

(c) *Is the word problem for* G *solvable in general?*

Question 10.19 *Let* $G = \langle a, b; a^p = b^q = R(a, b) = 1 \rangle, 1 < p, q$. *When is G trivial, non-trivial, finite or infinite?*

We remark that the respective three-generator group

$$G = \langle a, b, c; a^p = b^q = c^r = R(a, b, c) = 1 \rangle, \quad 1 < p, q, r,$$

is always non-trivial. This follows from the fact that the dimension of character space for representations into $PSL(2, \mathbb{C})$ is greater or equal 0 (see also [78]).

Question 10.20 *Which cyclically pinched one-relator groups contain a surface group?*

Question 10.21 *Solve the surface group conjecture in general.*

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