Advances in Group Theory and Applications
© 2018 AGTA - www.advgrouptheory.com/journal
6 (2018), pp. 23–53
ISSN: 2499-1287
DOI: 10.32037/agta-2018-003

Atomic Topological Groups *

Dikran Dikranjan — Hossein Khass — Behnam Bazigaran

(*Received Sep. 26, 2017; Accepted Feb. 17, 2018 — Communicated by F. de Giovanni*)

Abstract

We investigate the atoms of the lattice $\mathfrak{L}(G)$ of all group topologies on a group G by using the close connection between atoms of $\mathfrak{L}(G)$, and minimal Hausdorff elements of the lattice $\mathfrak{L}(G/N)$ for appropriate quotient groups G/N. We show, among others, that virtually hypercentral groups have only degenerate atoms.

Mathematics Subject Classification (2010): 06B30, 22A05

Keywords: lattice of group topologies; lattice of f-subgroups; atom; minimal group

1 Introduction

1.1 Generalities about posets of topologies

The study of the poset $(\mathfrak{T}(X), \subseteq)$ of all topologies on a set X has been initiated by Birkhoff (see [13]) in the thirties of the last century (see the extended survey of Larson and Andima [51]). This poset is a complete lattice, with bottom element the trivial (or indiscrete) topology ι_X on X and top element the discrete topology δ_X of X.

For a family $\mathfrak{U} \subseteq \mathfrak{T}(X)$, the infimum of $\mathfrak{U}$ is the intersection $\bigcap \mathfrak{U}$; while the supremum of $\mathfrak{U}$ is the least topology on X containing all elements of $\mathfrak{U}$.

* The first author Partially supported by grant PSD-2015-2017-DIMA-PRID-2017-DIKRANJAN PSD-2015-2017-DIMA - progetto PRID TokaDyMA of Udine University; the second and the third named authors were partially supported by the University of Kashan

In case $X = G$ carries also a group structure, one can consider the subset $\mathfrak{L}(G)$ of $\mathfrak{T}(X)$ consisting of all *group* topologies on G and its subset $\mathfrak{L}_H(G)$ of all Hausdorff elements of $\mathfrak{L}(G)$. $(\mathfrak{L}(G), \subseteq)$ is a complete lattice. While the join (supremum) in $\mathfrak{L}(G)$ coincides with the join formed in the bigger lattice $(\mathfrak{T}(G), \subseteq)$, the meet (infimum) $\mathcal{T} \wedge \mathcal{S}$ in $\mathfrak{L}(G)$ is not the intersection (that is, the infimum in $\mathfrak{T}(G)$) of $\mathcal{T}, \mathcal{S} \in \mathfrak{L}(G)$ (see [16]).

The lattice of all group topologies is largely studied (see [8],[6],[18],[17],[50],[58], and [59]).

Long chains in the lattice $\mathfrak{L}_H(G)$ are discussed in [19] and [21], complements – in [12],[39],[40],[29],[50],[64], and [67]. Various subposets of $\mathfrak{L}_H(G)$ are discussed in [10],[25],[26], and [20] (precompact topologies), the pseudocompact ones in [27] and linear module topologies in [43]. The lattice theoretic aspects of functorial topologies on abelian groups are discussed in [30]. Sometimes, whet it is clear from the context, we simply write 0 in place of ι_G.

A *minimal* group topology $\mathcal{T}$ on a group G is a minimal element of the poset $\mathfrak{L}_H(G)$, in such a case one says that $(G, \mathcal{T})$ is a minimal topological group. Historically, much interest has been focused on minimal group topologies (see [28],[33],[34],[35],[44],[45],[56],[57],[61], and [63]).

According to current terminology from order theory, if $\mathcal{S}, \mathcal{T} \in \mathfrak{L}(G)$, one says that $\mathcal{T}$ is a *covering* of $\mathcal{S}$, if $\mathcal{S} \subset \mathcal{T}$ and for every $\mathcal{U} \in \mathfrak{L}(G)$ with $\mathcal{S} \subseteq \mathcal{U} \subseteq \mathcal{T}$, either $\mathcal{U} = \mathcal{S}$ or $\mathcal{U} = \mathcal{T}$ holds (that is, $\mathcal{T}$ contains $\mathcal{S}$ and $\mathcal{T}$ and $\mathcal{S}$ are immediate neighbors, *adjacency* is another term coined to describe this situation; see [1]).

Problems related to this "covering" aspect of the lattice of topologies is studied in [1],[7],[4],[3],[5], and [52].

Now comes the central notion for this paper:

Definition 1.1 *A topological group* $(G, \mathcal{T})$*, is said to be atomic if* $\mathcal{T}$ *is an atom of* $\mathfrak{L}(G)$*. In addition, we may say '*$\mathcal{T}$ *is an atom' instead of '*$\mathcal{T}$ *is an atom of* $\mathfrak{L}(G)$*'.*

In the above terms, the atoms are the elements of the lattice $\mathfrak{L}(G)$ covering its least element ι_G (and dually, the coatoms are the elements covered by the maximum element δ_G of $\mathfrak{L}(G)$).

To the best of our knowledge, the first systematic study of atoms in the lattice of group topologies was carried out by Mutylin in [54]. According to [54], a topological algebra is *completely simple* if any continuous homomorphism of it onto a topological algebra with two or more elements is a topological isomorphism (so that completely simple topological groups are topologically simple in the sense of our Definition 3.5 below). It was explicitly noted by Mutylin (see [54], Definition 1′), that completely simple topological groups are precisely the Hausdorff atomic groups in our terminology. We do not use his term *completely simple group* preferring the term *Hausdorff atomic group*. Here is one of the principal results of [54] most closely related to our paper.

Theorem 1.2 (see [54], Theorem 7) *Let* $(G, \mathcal{T})$ *be a Hausdorff atomic abelian topological group. Then,* G *is isomorphic to* $\mathbb{Z}_p$ *for some prime number* p.

Mutylin in [54] proved also that complete completely simple commutative rings must be either complete fields with minimal ring topology or else the abelian groups $\mathbb{Z}_p$ with zero multiplication.

Atoms and coatoms are investigated in [18], as well as in [58] and [59]. These authors were unaware of the much earlier Mutylin's paper [54]. In particular, Theorem 1.2 was proved independently (yet much later) also by Remus in [59].

1.2 Main results

In this paper, we study the atoms of $\mathfrak{L}(G)$ extending the results from [18],[54], and [59] in several directions: firstly, we extend Mutylin's theorem (Theorem 1.2) to hypercentral groups and we obtain a new immediate proof of Clark and Schneider theorem (see Corollary 4.13). Secondly, we show that large classes of infinite groups admit Hausdorff atoms, and we pay particular attention to their number and size, as well as, the possibility to have Hausdorff atoms with additional compactness properties. We characterize the atoms of $\mathfrak{L}(G)$ in terms of Hausdorff minimal group topologies. Therefore, the chaise for nondegenerate atoms (see Definition 1.3) becomes a chaise for Hausdorff atoms on infinite groups. In order to explain our contribution in more detail, we introduce below some notation and terminology.

For a group topology $\mathcal{T}$ on a group G, we denote by $\mathcal{F}_{\mathcal{T}}$ the filter of all neighborhoods of 1 in $(G, \mathcal{T})$, where 1 is the identity element of G. One can easily recover the topology $\mathcal{T}$ from the filter $\mathcal{F}_{\mathcal{T}}$, this is why in the study of group topologies one studies mainly the filters $\mathcal{F}_{\mathcal{T}}$, which form, on their own, a poset order-isomorphic to $\mathfrak{L}(G)$, and its members are much easier to deal with than those of $\mathfrak{L}(G)$ (see [48] for more details).

Let N be a normal subgroup of a group G. We denote by τ_N the unique group topology $\mathcal{T}$ on G, with

$$\mathcal{F}_{\mathcal{T}} = \left\{ L \subseteq G \,|\, N \subseteq L \right\}.$$

Then τ_N is an Alexandrov topology and all Alexandrov group topologies have this form. Therefore, the assignment $N \mapsto \tau_N$ embeds the lattice $\mathrm{Nor}(G)$ of all normal subgroups of G into $\mathfrak{L}(G)$.

In the opposite direction one has a natural map from $\mathfrak{L}(G)$ to $\mathrm{Nor}(G)$ which is a left inverse to the above map. To define the map, for each $\mathcal{T} \in \mathfrak{L}(G)$ let

$$\operatorname{core} \mathcal{T} := \bigcap \mathcal{F}_{\mathcal{T}}.$$

Actually, $\operatorname{core} \mathcal{T}$ is a $\mathcal{T}$-closed normal subgroup of G (being the $\mathcal{T}$-closure of the trivial subgroup $\{1\}$, see Example 3.2). The map

$$\mathcal{T} \mapsto \operatorname{core} \mathcal{T} \in \mathrm{Nor}(G)$$

is a left inverse to the map

$$\mathrm{Nor}(G) \to \mathfrak{L}(G),\ N \mapsto \tau_N,$$

as mentioned above. This pair of maps witnesses the fact that the subset of all Alexandrov topologies on G is a sublattice of $\mathfrak{L}(G)$, order-isomorphic to the dual of the lattice $\mathrm{Nor}(G)$. In particular, the atoms of $\mathfrak{L}(G)$ that are Alexandrov topologies correspond exactly to the coatoms of $\mathrm{Nor}(G)$ under this correspondence.

If $\mathcal{T} \in \mathfrak{L}(G)$ has core $\mathcal{T}$ of finite prime index, then $\mathcal{T}$ is obviously an atom. More generally, we introduce the following useful notion:

Definition 1.3 *For a group* G, *we call a* degenerate atom *any* $\mathcal{T} \in \mathfrak{L}(G)$ *such that* $G/\mathrm{core}\,\mathcal{T}$ *is a finite simple (not necessarily abelian) group.*

Clearly, degenerate atoms are Alexandrov (however, we will see that Alexandrov atoms $\mathcal{T}$ need not necessarily be degenerate; see Example 4.8).

The paper is organized as follows. In Theorem 2.2, we establish a useful correspondence between the lattices $\mathfrak{L}(G)$ and $\mathfrak{L}(H)$ induced by a surjective homomorphism

$$f : G \to H.$$

In Section 3, we pay special attention to topological simplicity, that turns out to be one of the main ingredients of an atom. More precisely, we show in Theorem 3.9, that a topological group $(G, \mathcal{T})$ is atomic precisely when $(G, \mathcal{T})$ is topologically simple and $\mathcal{T}/\mathrm{core}\,\mathcal{T}$ is a minimal.

Section 4 is entirely dedicated to the main topic of the paper — Hausdorff atoms. Since minimal topologies are essentially involved in the chase of Hausdorff atoms (due to Theorem 3.9), we dedicate the initial part (Paragraph 4.1) of this section for recalling some relevant facts about minimal groups.

In Paragraph 4.2, we find relevant necessary conditions for the existence of a Hausdorff atom on an infinite group G. One of them is: $Z(G) = \{1\}$, the other is: G *has no finite normal subgroups but* $\{1\}$. As a consequence, *all atoms on a virtually hypercentral (in particular, virtually nilpotent, or just nilpotent) group are necessarily degenerate* (see Corollary 5.3). Therefore, these groups, in case they have no subgroups of finite index, carry no atoms at all.

In Paragraph 4.3, we discuss the number of Hausdorff atoms, providing an example of a group G that admits infinitely many Hausdorff atoms in $\mathfrak{L}(G)$.

In Paragraph 4.4, we show that a free group admits Hausdorff atoms, if and only if, it is nonabelian. In such a case, the Hausdorff atoms can be chosen with some additional properties (see Theorem 4.20). Let us recall that free groups are hypocentral (to compare with infinite hypercentral groups that admit no Hausdorff atoms at all, see Corollary 4.14).

Section 5 is dedicated to not necessarily Hausdorff atoms. Here we find necessary conditions for the existence of nondegenerate atoms. We show how these results provide a connection to the well-known Schur theorem.

Finally, we discuss the impact of the existence of atoms in $\mathfrak{L}(G)$ on the structure of an abelian group G (see Theorem 5.7 and Theorem 5.8).

Acknowledgments

It is a pleasure to thank U. Bader who offered a hint for Example 4.16 to the first named author and F. de Giovanni, who suggested the Reference [23].

Notation and terminology

Group theory, topological groups and ordered sets concepts and notations are compatible with those in [2],[15] and [14], except those redefined otherwise. The identity element of a group G is denoted by 1 and its center by $Z(G)$. For $A, B \subseteq G$, we denote by $[A, B]$ the subgroup of G generated by

$$\left\{aba^{-1}b^{-1} \;\middle|\; a \in A, b \in B\right\}.$$

The cyclic group of order m is denoted by $\mathbb{Z}_m$. The socle of a group G, denoted Soc(G), is the subgroup generated by the minimal normal subgroups of G (that is, the atoms of Nor(G)). The rank of an abelian group G, is defined by

$$\operatorname{rank}(G) = \sup\left\{\operatorname{rank}(F) \mid F \text{ is a subgroup of } G, F \text{ is a free abelian group}\right\},$$

where rank(F) is the cardinality of a freely generating set for F. For an abelian group G, Soc(G) coincides with the subgroup of G generated by all elements of G with prime order.

The upper central series $(Z_\alpha(G))$ of a group G is defined as follows, where α is an ordinal number. Let $Z_0(G) = Z(G)$, assume that $\alpha \geqslant 1$ and $Z_\beta(G)$ is already defined for all $\beta < \alpha$. If α is a limit ordinal, let

$$Z_\alpha(G) = \bigcup_{\beta<\alpha} Z_\beta(G).$$

If $\alpha = \beta + 1$, consider the quotient map

$$\pi\colon G \to G/Z_\beta(G)$$

and let

$$Z_\alpha(G) = \pi^{-1}\left[Z\left(G/Z_\beta(G)\right)\right].$$

This produces an ascending chain of characteristic subgroups $Z_\alpha(G)$ of G, their least upper bound $Z^*(G) = \bigcup_\alpha Z_\alpha(G)$, is called the *hypercenter* of G. A group G is *hypercentral*, if $G = Z^*(G)$, equivalently if $G = Z_\alpha(G)$ for some α. The group G is *nilpotent* if $Z_n(G) = G$ for some $n < \omega$, in this case, its nilpotency class is the minimum such n.

The lower central series $(\gamma_\alpha(G))$ of a group G is defined by

$$\gamma_1(G) = G, \quad \gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$$

and

$$\gamma_\alpha(G) = \bigcap_{\rho<\alpha} \gamma_\rho(G)$$

for limit ordinal numbers α. This produces a descending chain of characteristic subgroups $\gamma_\alpha(G)$, their greatest lower bound

$$Z_*(G) = \bigcap_\alpha Z_\alpha(G)$$

is called the *hypocenter* of G. A group G is *hypocentral*, if $Z_*(G) = \{1\}$. If $\gamma_\omega(G) = \{1\}$, G is *residually nilpotent*, that is, G is isomorphic to a subgroup of a direct product of nilpotent groups.

A group is *hyperabelian* if it possesses an ascending (possibly transfinite) normal series where all the successive quotients are abelian. For any set X, we denote by Sym(X) the permutation group of X. For a group property P, we say that a group G is *virtually* P, if G has a subgroup with finite index and with the property P. For example, P can be hypercentral, and in this way, the definition of *virtually hypercentral* is known.

A *compact* or *totally bounded* topology need not be Hausdorff, we take *precompact* as a synonym for *totally bounded Hausdroff*. The *weight* of a topological space $(X, \mathcal{T})$ is the cardinality of the smallest base $\mathcal{B}$ for $\mathcal{T}$ and is denoted by $w(X, \mathcal{T})$. The *density character* of $(X, \mathcal{T})$ is defined by

$$d(X, \mathcal{T}) = \min\Big\{|A| \,|\, A \text{ is a dense subset of } (X, \mathcal{T})\Big\}.$$

The *character* of a point $x \in X$ in $(X, \mathcal{T})$ is the cardinality of the smallest local base for x in $(X, \mathcal{T})$. The supremum of the characters of all points of X is called the *character* of $(X, \mathcal{T})$ and is denoted by $\chi(X, \mathcal{T})$. The *pseudocharacter* of $(X, \mathcal{T})$, denoted by $\psi(X, \mathcal{T})$, is the smallest infinite cardinality α such that for each $x \in X$, $\{x\}$ is the intersection of the elements of some $\mathcal{U} \subseteq \mathcal{T}$ with $|\mathcal{U}| \leqslant \alpha$.

We denote by

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\},$$

the circle group, and by G^*, the group of all (algebraic) homomorphisms (called *characters*) of the form

$$\xi : G \to \mathbb{T}$$

for a group G.

For a Hausdorff topological group G, we denote by $\widetilde{G}$, the completion of G with respect to the two-sided uniformity (known also as *Raĭkov com-*

pletion of G). Let us recall that $\widetilde{G}$ is determined up to topological isomorphism. A Hausdroff topological group is called *locally precompact*, if the group $\widetilde{G}$ is locally compact.

$c(G, \mathcal{T})$ or simply $c(G)$ denotes the connected component containing 1 in a topological group $(G, \mathcal{T})$.

For a cardinality $\kappa \geqslant 1$, we denote by F_κ the free group of rank κ.

A topological space $(X, \mathcal{T})$ is *Alexandrov*, when arbitrary intersections of open sets are still open, that is, when $\mathcal{T}$ is a complete sublattice of $\mathcal{P}(X)$.

2 The lattice correspondence theorem

The following Proposition is a special case of a more general result proved in [48] and [50].

Proposition 2.1 *Let* $(G, \mathcal{T})$ *be a topological group and* N *be a normal subgroup of* G*. Then* $\{FN \mid F \in \mathcal{F}\}$ *is a base for the set of all neighborhoods of the identity element in the topological group* $(G, \mathcal{T} \wedge \tau_N)$*.*

Let G and H be groups and

$$f : G \to H$$

be an onto group homomorphism. If $\mathcal{T}$ is a group topology on G, we denote by $\widetilde{\mathcal{T}}$ the topology with base

$$\{f[U] \mid U \in \mathcal{T}\}$$

on H, it coincides with the quotient topology of H (sometimes we write also $\mathcal{T}/\ker f$, when we identify H with $G/\ker f \simeq H$). On the other hand, if $\mathcal{S}$ is a group topology on H, we denote by

$$\widehat{f}(\mathcal{S})$$

the *initial topology* on G with respect to f, namely the family

$$\{f^{-1}[V] \mid V \in \mathcal{S}\}.$$

Letting $\widetilde{f}(\mathcal{T}) = \widetilde{\mathcal{T}}$ for $\mathcal{T} \in \mathfrak{L}(G)$, one obtains, in this way, two maps:

$$\widetilde{f} : \mathfrak{L}(G) \to \mathfrak{L}(H) \quad \text{and} \quad \widehat{f} : \mathfrak{L}(H) \to \mathfrak{L}(G). \tag{2.1}$$

Now we show, among others, that the range of the map $\widehat{f}$ coincides with the interval $[\iota_G, \tau_{\ker f}]$ in the poset $\mathfrak{L}(G)$ (see items (b) and (c)).

Theorem 2.2 *Let* G *and* H *be groups,*

$$f : G \to H$$

be an onto homomorphism and $\widetilde{f}, \widehat{f}$ *the maps from* (2.1).

(a) *For every* $\mathcal{S} \in \mathfrak{L}(H)$*, one has* $\widetilde{f}(\widehat{f}(\mathcal{S})) = \mathcal{S}$*, so* $\widetilde{f}$ *is surjective and* $\widehat{f}$ *is injective.*

(b) *The range of the map* $\widehat{f} : \mathfrak{L}(H) \to \mathfrak{L}(G)$ *is contained in* $[\iota_G, \tau_{\ker f}]$.

(c) *Putting* $\widetilde{\mathcal{T}} = \widetilde{f}(\mathcal{T})$ *for every* $\mathcal{T} \in \mathfrak{L}(G)$*, one has* $\widehat{f}\left(\widetilde{\mathcal{T}}\right) = \mathcal{T} \wedge \tau_{\ker f}$*. Hence,*

$$\widehat{f} : \mathfrak{L}(H) \to [0, \tau_{\ker f}]$$

is surjective and

$$f' := \widetilde{f} \restriction_{[0,\tau_{\ker f}]}$$

is injective.

Consequently,

$$\widehat{f} : \mathfrak{L}(H) \to [0, \tau_{\ker f}] ,$$

as well as its inverse

$$f' = \widetilde{f} \restriction_{[0,\tau_{\ker f}]} : [0, \tau_{\ker f}] \to \mathfrak{L}(H),$$

are order-isomorphisms. Moreover, $\widehat{f}$ *preserves connectedness, compactness, total boundedness, the property of being Alexandrov, as well as various cardinal invariants such as weight, character, density character and cellularity.*

Proof — Let $N = \ker f$.

(a) The first assertion follows from the equality $f\left[f^{-1}[V]\right] = V$ for $V \in \mathcal{S}$. The equality $\widetilde{f} \circ \widehat{f} = id_{\mathfrak{L}(H)}$ implies that $\widetilde{f}$ is surjective and $\widehat{f}$ is injective.

(b) Follows from $\widetilde{f}(\mathcal{S}) \subseteq \tau_N$ for all $\mathcal{S} \in \mathfrak{L}(H)$.

(c) Fix a $\mathcal{T} \in \mathfrak{L}(G)$. Obviously, $\widehat{f}(\widetilde{\mathcal{T}}) \subseteq \mathcal{T}$, as

$$f : (G, \mathcal{T}) \to (H, \widetilde{\mathcal{T}})$$

is continuous. On the other hand, $\widehat{f}(\widetilde{\mathcal{T}}) \subseteq \tau_N$, according to (b). Therefore,

$$\widehat{f}(\widetilde{\mathcal{T}}) \subseteq \mathcal{T} \wedge \tau_N .$$

Conversely, if $W \in \mathcal{F}_{\mathcal{T} \wedge \tau_N}$, then there exists $U \in \mathcal{F}_{\mathcal{T}}$, such that $W \supseteq UN$, by Proposition 2.1. As

$$UN = f^{-1}[f[U]] ,$$

this proves that $W \in \mathcal{F}_{\widehat{f}(\widetilde{\mathcal{T}})}$. Therefore, $\widehat{f}(\widetilde{\mathcal{T}}) = \mathcal{T} \wedge \tau_N$.

The equality

$$\widehat{f}(\widetilde{\mathcal{T}}) = \mathcal{T} \wedge \tau_N$$

implies $\widehat{f}(\widetilde{\mathcal{T}}) = \mathcal{T}$ for all $\mathcal{T} \in [0, \tau_{\ker f}]$, that is, $\widehat{f} \circ \widetilde{f}$ is the identity of $[0, \tau_{\ker f}]$, so f' is injective. By (a) and (b), f' is surjective as well. Since f' is obviously monotone, this proves that f' is an order-isomorphism.

To prove the last assertion fix a topology $\mathcal{S} \in \mathfrak{L}(H)$. The subgroup N of the group $(G, \widehat{f}(\mathcal{S}))$ is indiscrete (so, connected, totally bounded and compact). Hence, it suffices to note that each of these three properties P satisfies the so called three-space-property, that is,

$$(G, \widehat{f}(\mathcal{S}))$$

has P whenever N and the quotient $(G/N, \mathcal{S})$ have the property P (here we identify H with G/N and we use the equality

$$\widetilde{f}(\widehat{f}(\mathcal{S})) = \mathcal{S},$$

that is, the quotient topology of $\widehat{f}(\mathcal{S})$ is $\mathcal{S}$). As far as the the Alexandrov property is concerned, $\mathcal{S}$ is Alexandrov, precisely when H has an $\mathcal{S}$-open indiscrete normal subgroup K. Again by the three-space-property, applied to the open normal subgroup $f^{-1}[K]$ of $(G, \widehat{f}(\mathcal{S}))$ and $P =$ indiscreteness, we deduce that $f^{-1}[K]$ is indiscrete. Hence, $\widehat{f}(\mathcal{S})$ is Alexandrov.

Now note that $f^{-1}[D]$ is dense in $(G, \widehat{f}(\mathcal{S}))$ for every dense subset D of $(H, \mathcal{S})$. This proves that

$$d((G, \widehat{f}(\mathcal{S})) = d(H, \mathcal{S}).$$

Since $\widehat{f}(\mathcal{S})$ is the initial topology on G with respect to the map f,

$$\chi(G, \widehat{f}(\mathcal{S})) = \chi(H, \mathcal{S}).$$

These two equalities imply $w(G, \widehat{f}(\mathcal{S})) = w(H, \mathcal{S})$.

Finally, the cellularity is preserved since for every pair U, V of disjoint open sets of $(G, \widehat{f}(\mathcal{S}))$ the sets $f(U), f(V)$ remain open and disjoint. □

The pseudocharacter is missing in the above list of properties preserved by $\widehat{f}$ since $\widehat{f}$ fails to have it.

Remark 2.3 The counterpart of this theorem for nonsurjective homomorphisms is not valid even in the case of injective homomorphisms. For simplicity consider the essential case when G is a subgroup of H and f is the inclusion map. This assumption does not bring much loss of generality as

any homomorphism can be written as a composition of such a one and a surjective one, so that the above theorem can be applied.
Let us see now that the whole setting of the theorem needs to be properly re-formulated in this case and the situation radically changes. Indeed, now $\widehat{f}(\mathcal{S})$ is simply the restriction $\mathcal{S}\restriction_G$ of a group topology $\mathcal{S} \in \mathfrak{L}(H)$ to G. Let us see that the map $\widehat{f}$ need not have any right inverse, that is, $\widehat{f}$ need not be surjective in general. Indeed, if such a right inverse of $\widehat{f}$ exists, then it would assign to each $\mathcal{T} \in \mathfrak{L}(G)$ an extension $\mathcal{T}^* \in \mathfrak{L}(H)$, as

$$\widehat{f}(\mathcal{T}^*) = \mathcal{T}^* \restriction_G = \mathcal{T}.$$

It was shown in [38] that this extension is not available in general even when G is a normal subgroup of H, that is, $\widehat{f}$ need not be surjective even in this case. The range of $\widehat{f}$ is described in [38] as the family

$$\mathfrak{L}^{ext,H}(G)$$

of those $\mathcal{T} \in \mathfrak{L}(G)$, such that the restriction to G of every inner automorphism of H is $\mathcal{T}$-continuous. For

$$\mathcal{T} \in \mathfrak{L}^{ext,H}(G),$$

the extension $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ is defined exactly as before, by taking as a base of the filter $\mathcal{F}_{\widetilde{\mathcal{T}}}$ of neighborhoods of 1 in $(H, \widetilde{\mathcal{T}})$ the family

$$\left\{ f\,[U] \mid U \in \mathcal{F}_{\mathcal{T}} \right\}.$$

Clearly, $\widetilde{\mathcal{T}}$ is the finest (although not necessarily unique) topology extending $\mathcal{T}$.

Almost all our uses of Theorem 2.2, in this paper, are through the following immediate corollary.

Corollary 2.4 *Let* N *be a normal subgroup of the group* G*. Then,* $\mathfrak{L}(G/N)$ *is order-isomorphic to the interval* $[0, \tau_N]$ *of the lattice* $\mathfrak{L}(G)$*, where* $0 = \iota_G$*. In particular,* $\mathcal{T} \in [0, \tau_N]$ *is an atom in* $\mathfrak{L}(G)$*, if and only if,* $\widetilde{f}(\mathcal{T})$ *is an atom in* $\mathfrak{L}(G/N)$*.*

So $\mathfrak{L}(G/N)$ can be embedded in $\mathfrak{L}(G)$ as a sublattice.

3 The poset $\mathcal{C}_{\mathcal{T}}$ and topological simplicity

We need another poset related to a topological group G, namely the family of all closed normal subgroups of G:

Note *The set $\mathcal{C}_{\mathcal{T}}$ of all normal $\mathcal{T}$-closed subgroups of a topological group $(G, \mathcal{T})$ will be ordered by $\subseteq$. It is a sublattice of the complete lattice* $\mathrm{Nor}(G)$ *of all normal subgroups of* G*.*

If $N \in \mathcal{C}_{\mathcal{T}}$, then $\mathcal{T}/N$ is a Hausdorff group topology on G/N.

The following definition introduces an important ingredient of $\mathcal{C}_{\mathcal{T}}$.

Definition 3.1 *Let $(G, \mathcal{T})$ be a topological group. The intersection of all elements of $\mathcal{F}_{\mathcal{T}}$ is called the* core *of $\mathcal{T}$ and is denoted by* $\operatorname{core} \mathcal{T}$*.*

Example 3.2 (a) If N is a normal subgroup of group G, then $\operatorname{core} \tau_N = N$. Hence, the map $\mathfrak{L}(G) \to \mathrm{Nor}(G)$, defined by $\mathcal{T} \mapsto \operatorname{core} \mathcal{T}$, is surjective.

(b) For $\mathcal{T} \in \mathfrak{L}(G)$ one has:

$$\operatorname{core}(\mathcal{T} \wedge \tau_N) = \bigcap_{U \in \mathcal{F}_{\mathcal{T}}} UN = \overline{N},$$

where $\overline{N}$ is the $\mathcal{T}$-closure of N, in view of Proposition 2.1. In particular, $\operatorname{core} \mathcal{T}$ coincides with the closure of $\{1\}$, and so it is a closed normal subgroup of G (see [15], chapter on uniform spaces).

The connection of $\operatorname{core} \mathcal{T}$ to $\mathcal{C}_{\mathcal{T}}$ is given by the next lemma.

Lemma 3.3 *For a topological group $(G, \mathcal{T})$ with $N = \operatorname{core} \mathcal{T}$:*

(a) *$\mathcal{T}$ is Hausdorff, if and only if, N is trivial;*

(b) *$\mathcal{C}_{\mathcal{T}}$ is a complete lattice with least element N and top element G.*

(c) *$\mathcal{T} \in [0, \tau_N]$, where $0 = \iota_G$.*

(d) *$\mathcal{T}$ is an atom, if and only if, $\mathcal{T}/N$ is an atom.*

Proof — (a) is a well-known fact in topological group theory.

(b) Clearly G is the top element of $\mathcal{C}_{\mathcal{T}}$. By Example 3.2, $\operatorname{core}(\mathcal{T}) \in \mathcal{C}_{\mathcal{T}}$. Let $K \in \mathcal{C}_{\mathcal{T}}$. Since K is closed, we have:

$$N = \overline{\{1\}} \subseteq \overline{K} = K.$$

So, N is the least element of $\mathcal{C}_{\mathcal{T}}$. Finally, $\mathcal{C}_{\mathcal{T}}$ is a complete lattice since for every family $\mathcal{A} \subseteq \mathcal{C}_{\mathcal{T}}$ the infimum of $\mathcal{A}$ is simply the intersection of all elements of $\mathcal{A}$.

(d) Since $\mathfrak{L}(G/N)$ embeds into $\mathfrak{L}(G)$, for any atom Ξ of $\mathfrak{L}(G/N)$, $\widetilde{\pi}(\Xi)$ is an atom of $\mathfrak{L}(G)$, and, for any atom $\mathcal{T}$ of $\mathfrak{L}(G)$, contained in τ_N, $\mathcal{T}/N$ is an atom of $\mathfrak{L}(G/N)$, and (a) says that this atom is a Hausdorff one. □

Remark 3.4 If $\mathcal{T}$ is a degenerate atom on a group G, then this topology is Alexandrov (equivalently, $\mathcal{F}_{\mathcal{T}}$ is a principal filter), since G/N, where $N = \operatorname{core}(\mathcal{T})$, is finite, so $\mathcal{T}/N$ is Alexandrov and by Theorem 2.2, $\mathcal{T}$ is Alexandrov as well. Moreover, $(G, \mathcal{T})$ is disconnected, as N is a proper nonempty clopen (i.e., both closed and open) subset of $(G, \mathcal{T})$.

3.1 Topological simplicity

Definition 3.5 *A topological group* $(G, \mathcal{T})$ *is said to be topologically simple, and* $\mathcal{T}$ *is said to be simple, if* $|\mathcal{C}_{\mathcal{T}}| = 2$.

This term is motivated by the fact that an abstract group G is simple, if and only if, the discrete topology δ_G is simple.

Remark 3.6 Let $(G, \mathcal{T})$ be a topological group and $N = \operatorname{core} \mathcal{T}$. It is easy to see that the following are equivalent:

(a) $(G, \mathcal{T})$ is topologically simple;

(b) $(G/N, \mathcal{T}/N)$ is topologically simple;

(c) N is a coatom of $\mathcal{C}_{\mathcal{T}}$ (that is, a maximal proper closed normal subgroup of G).

Now we provide two prominent examples of topologically simple groups.

Example 3.7 We give here two examples of nonsimple topologically simple groups.

(a) Let $\alpha \in \mathbb{T}$ be a nontorsion element and let C be the cyclic subgroup of $\mathbb{T}$ generated by α. Then C, as well as all its nontrivial subgroups, are dense subgroups of $\mathbb{T}$, since $\mathbb{T}$ has no proper infinite closed subgroups. Therefore, C is topologically simple. Using the unique isomorphism

$$\mathbb{Z} \to C$$

sending 1 to α, we obtain a topologically simple precompact topology on $\mathbb{Z}$ that we denote by $\mathcal{T}_\alpha$. Note that for $\alpha, \beta \in \mathbb{T}$ the equality $\mathcal{T}_\alpha = \mathcal{T}_\beta$ holds if and only if $\beta = \alpha^{\pm 1}$.
Actually, $\mathbb{Z}$ admits much more simple topologies, obtained in a similar way, replacing $\mathbb{T}$ by any monothetic group (that is, a group having a dense infinite cyclic subgroup C). For this purpose, one can use the fact that there are $2^{\mathfrak{c}}$ many pairwise nonhomeomorphic compact monothetic groups. This gives the maximum number possible of such topologies, namely $2^{\mathfrak{c}}$.

(b) For every infinite set X the symmetric group Sym(X) carries the so called pointwise convergence topology $\mathcal{T}_p$, that is the topology induced on Sym(X) by the natural embedding into the product X^X equipped with the product topology letting X carry the discrete topology. Then $(\mathrm{Sym}(X), \mathcal{T}_p)$ is topologically simple.

Lemma 3.8 *If* N *is a normal subgroup of a dense subgroup* H *of a topological group* $(G, \mathcal{T})$, *then* $M = \overline{N}$, *the* $\mathcal{T}$*-closure of* N, *is a normal subgroup of* G. *Consequently,* H *is topologically simple whenever* G *is topologically simple.*

Proof — Let g be any an arbitrary element of G. To prove the first assertion, it suffices to check that $g^{-1}Mg \subseteq M$. There is a net (h_α) in H with $h_\alpha \longrightarrow g$. Pick $n \in N$, then

$$h_\alpha^{-1} n h_\alpha \longrightarrow g^{-1} n g$$

and
$$h_\alpha^{-1} n h_\alpha \in h_\alpha^{-1} N h_\alpha = N.$$
So $g^{-1}ng \in \overline{N} = M$. This proves that $g^{-1}Mg \subseteq M$.

To prove the last assertion take a nontrivial closed normal subgroup N_0 of H. Then
$$M_0 = \overline{N_0}$$
is a nontrivial normal subgroup of G, so, $M_0 = G$. This yields $N_0 = H$ in view of the equality $N_0 = M_0 \cap G$ due to the closedness of N_0 in H. Hence, H is topologically simple. □

Example 3.7 above shows that the implication of Lemma 3.8 cannot be inverted. Nevertheless, under some additional restraint on H, this implication becomes invertible (see Lemma 4.5 below).

Next comes a simple and natural criterion for a group topology to be an atom involving the well-studied class of minimal topologies.

Theorem 3.9 *Let $(G, \mathcal{T})$ be a topological group and $N = \operatorname{core} \mathcal{T}$. Then, $\mathcal{T}$ is an atom of $\mathfrak{L}(G)$, if and only if, $\mathcal{T}$ is simple, and $\mathcal{T}/N$ is minimal.*

Proof — Suppose $\mathcal{T}$ is an atom of $\mathfrak{L}(G)$. According to Remark 3.6, to see that $\mathcal{T}$ is simple, it suffices to check that $N = \operatorname{core}(\mathcal{T})$ is a coatom. Toward a contradiction, suppose that
$$N \subset K \subset G$$
for some $K \in \mathcal{C}_{\mathcal{T}}$. Then
$$\mathcal{T}' := \mathcal{T} \wedge \tau_K \in \mathfrak{L}(G)$$
obviously satisfies
$$\mathcal{O} \subseteq \mathcal{T}' \subseteq \mathcal{T},$$
where $\mathcal{O} = \iota_G$. Since K is $\mathcal{T}$-closed, we deduce $\operatorname{core} \mathcal{T}' = K$, by Example 3.2. So:
$$\operatorname{core} \mathcal{T} = N \subset \operatorname{core} \mathcal{T}' = K \subset G = \operatorname{core} \mathcal{O}.$$
This gives $\mathcal{O} \subset \mathcal{T}' \subset \mathcal{T}$, a contradiction, because $\mathcal{T}$ is an atom of $\mathfrak{L}(G)$.

By Theorem 2.4, $[\mathcal{O}, \tau_N]$ is order-isomorphic to $\mathfrak{L}(G/N)$ and since $\mathcal{T}$ is an atom of $[\mathcal{O}, \tau_N]$, $\mathcal{T}/N$ is an atom of $\mathfrak{L}(G/N)$. By Lemma 3.3, $\mathcal{T}/N$ is Hausdorff, and so,
$$\mathcal{T}/N \in \mathfrak{L}_H(G/N).$$
Clearly, $\mathcal{T}/N$ is a minimal element of $\mathfrak{L}_H(G/N)$. This proves the necessity.

Conversely, suppose $\mathcal{T}$ is simple, and $\mathcal{T}/N$ is minimal. Let $\mathcal{S} \in \mathfrak{L}(G)$ and $\mathcal{O} \subset \mathcal{S} \subseteq \mathcal{T}$. We need to show that $\mathcal{S} = \mathcal{T}$. Clearly,
$$\operatorname{core}(\mathcal{T}) \subseteq \operatorname{core}(\mathcal{S}) \subset G.$$
Since $\mathcal{T}$ is simple and $\mathcal{S}$ is not trivial,
$$\operatorname{core}(\mathcal{S}) = \operatorname{core}(\mathcal{T}) = N.$$

By Theorem 2.4, $\mathcal{S}/N \subseteq \mathcal{T}/N$ and by Lemma 3.3, $\mathcal{S}/N$ is Hausdorff. By the minimality of $\mathcal{T}/N$, $\mathcal{S}/N = \mathcal{T}/N$ and by Theorem 2.4, $\mathcal{S} = \mathcal{T}$; as required. □

Here we give an immediate corollary of Theorem 3.9:

Corollary 3.10 *If* G *has a nontrivial normal abelian subgroup* N *and allows a Hausdorff atom* $\mathcal{T}$, *then* G *is abelian.*

Proof — The closure $\overline{N}$ of N is a closed non-trivial normal subgroup of G, so $\overline{N} = G$, by Theorem 3.9. Thus, N is dense in G. Since N is abelian and G is Hausdorff, this implies that G is abelian too. □

Many groups have nontrivial normal abelian subgroups, among them hyperabelian (in particular, solvable) groups.

In the next remark, we anticipate some first elementary results which will significantly be strengthened later on in Theorem 4.12. We prefer to anticipate them here, since the sharper results make use of some nontrivial facts (for example, Theorem 1.2 or Prodanov's precompactness, Theorem 4.4),

Remark 3.11 Let G be a group and $\mathcal{T}$ be an atom of $\mathfrak{L}(G)$.

(a) If G is abelian, then $\mathcal{T}$ is pseudometrizable. In particular, if $(G, \mathcal{T})$ is abelian and Hausdorff (and atomic as assumed), then it is metrizable.

Indeed, let $B_1 \neq G$ be any symmetric element of $\mathcal{F}_{\mathcal{T}}$. If B_n is defined for some positive integer n, pick a symmetric element B_{n+1} of $\mathcal{F}_{\mathcal{T}}$ with

$$B_{n+1}B_{n+1} \subseteq B_n.$$

This gives is a sequence (B_n) satisfying

$$B_{n+1}B_{n+1} \subseteq B_n$$

and

$$\mathcal{B} = \{B_n \mid n \in \mathbb{N}\}$$

is a base for the neighborhood-filter $\mathcal{F}_{\mathcal{T}^*}$ of a first countable group topology $\mathcal{T}^*$ with $\mathcal{T}^* \subseteq \mathcal{T}$. Hence, $\mathcal{T}^* = \mathcal{T}$, because $\mathcal{T}$ is an atom of $\mathfrak{L}(G)$. This means that $\mathcal{T}$ is first-countable, so pseudometrizable, by Birkhoff-Kakutani theorem; see [2].

(b) If the group G is hyperabelian (in particular, solvable) and $\mathcal{T}$ is Hausdorff, then $|G| \leqslant \mathfrak{c}$.

If G is finite, there is nothing to prove, so assume that G is infinite. According to Corollary 3.10 (see also the comment after the corollary), G is abelian. Pick a nontrivial element $a \in G$ and let $C = \langle a \rangle$. Then, by Theorem 3.9, a is not torsion, and so, $C \simeq \mathbb{Z}$ and $N = \overline{C}$ is a nontrivial closed subgroup of G. By Theorem 3.9, $G = \overline{C}$. By item (a), $(G, \mathcal{T})$ is metrizable. Hence, $|G| \leqslant \mathfrak{c}$ as a metrizable separable space.

(c) It is easy to see that the assumption of Hausdorffness cannot be relaxed in item (b). Similarly the assumption of abelianness in item (a) cannot be removed either. Indeed, let X be an infinite set of size κ. Then the group $G = \mathrm{Sym}(X)$ equipped with the pointwise convergence topology $\mathcal{T}_p$ is Hausdorff and atomic, by Example 4.7 below. It has size 2^κ and local weight $\chi(G) = \kappa$, so it is not metrizable whenever κ is uncountable.

(d) $|G| \leqslant |\mathrm{core}\,\mathcal{T}|\mathfrak{c}$, if G is hyperabelian.

Indeed, if $N = \mathrm{core}\,\mathcal{T}$, $\mathcal{T}/N$ is a Hausdorff atom of $\mathfrak{L}(G/N)$. By (b), $|G/N| \leqslant \mathfrak{c}$ and so $|G| \leqslant |N|\mathfrak{c} = |\mathrm{core}\,\mathcal{T}|\mathfrak{c}$.

We will see in the sequel that one can replace the hypothesis "hyperabelian" by "virtually hypercentral" with a much stronger conclusion in item (b) of Remark 3.11.

One can trade "hyperabelian" for "precompact" to obtain an extension of Remark 3.11 (b) (see Corollary 4.18 below).

4 Hausdorff atoms

4.1 Background on minimal groups

A subgroup H of a topological group $(G, \mathcal{T})$ is said to be *essential* (in G) if H nontrivially meets every closed nontrivial normal subgroup of G.

Proposition 4.1 (Stephenson-Prodanov-Banaschewski theorem; see p.56 of [28]) *Let G be a Hausdorff topological group. Then a dense topological subgroup H of G is minimal, if and only if, G is minimal and H is essential in G.*

We state Prodanov-Stoyanov theorem which plays a fundamental role in this area.

Proposition 4.2 (Prodanov-Stoyanov theorem; see [57]) *Let $\mathcal{T}$ be a minimal Hausdorff group topology on an abelian group G. Then $\mathcal{T}$ is totally bounded.*

Combining Proposition 4.2 and Proposition 4.1, we conclude that the minimal abelian groups are precisely the dense essential subgroups of the compact abelian groups.

Definition 4.3 (see [33]) *A topological group $(G, \mathcal{T})$, is said to be totally minimal if $\mathcal{T}/N$ is minimal for any closed normal subgroup N of G.*

However, in this paper, all we need is the following weaker form of Proposition 4.2.

Theorem 4.4 (Prodanov theorem; see [55]) *Let $\mathcal{T}$ be a totally minimal group topology on an abelian group G. Then $\mathcal{T}$ is totally bounded.*

In [56] Prodanov studied the coatoms of $\mathfrak{L}(G)$ for an abelian group G and the infimum $\mathcal{M}_G$ in $\mathfrak{L}(G)$ of all coatoms, called *submaximal topology* of G, proving that $\mathcal{M}_G$ is finer than any minimal topology on G, so finer than the supremum of all minimal topologies. In order to discuss in the sequel some of the properties of $\mathcal{M}_G$ we adopt additive notation, in particular

$$nG = \{nx : x \in G\}.$$

In the same paper Prodanov found an explicit description of $\mathcal{M}_G$ and noticed that $\mathcal{M}_G$ it is totally bounded if the index $[G : (nG + \mathrm{Soc}(G))] < \infty$ for all integers $n > 0$. This obviously holds true in divisible groups as well as in finitely generated groups. This implies that all minimal topologies on such groups, in particular on $\mathbb{Z}$, are precompact. Since all minimal topologies on $\mathbb{Z}$ are also totally minimal, this follows also from Theorem 4.4. The first proof (although a somewhat implicit one) of the fact that the minimal group topologies on $\mathbb{Z}$ are precompact can be found in Lemma 9 of [54]. Minimal topological groups were not introduced yet by that time, so the question of their precompactness was not explicitly discussed in Lemma 9 of [54].

Lemma 4.5 *If* H *is a dense essential subgroup of a Hausdorff topological group* G*, then* H *is topologically simple, if and only if,* G *is topologically simple.*

Proof — Due to Lemma 3.8, we only have to check that if H is topologically simple, then so is G. Let N be any nontrivial normal closed subgroup of G. By the essentiality of H, $M = N \cap H$ is a nontrivial subgroup of H. Clearly, M is a normal closed subgroup of H. However, since $\mathcal{T}_H$ is simple, and M is nontrivial, we have $M = H$, and so, $H \subseteq N$. Thus,

$$G = \overline{H} \subseteq \overline{N} = N,$$

that is, $N = G$. This proves that G is topologically simple. □

Note that essentiality is necessary in the above lemma, as a dense nonessential subgroup H of a topological group G witnesses the existence of a nontrivial closed normal subgroup N of G which trivially meets H, so $N \neq G$ and G cannot be topologically simple.

4.2 Applications of Theorem 3.9

Here we intend to apply the criterion of Theorem 3.9 to Hausdorff atoms, that's why we explicitly give the following immediate corollary:

Corollary 4.6 *A Hausdorff topological group* $(G, \mathcal{T})$ *is atomic, if and only if, it is topologically simple and minimal.*

Example 4.7 The above corollary provides the first example of a nondegenerate Hausdorff atom. For every infinite set X, the symmetric group Sym(X) was proved to be minimal (see [46]) when equipped with the pointwise convergence topology $\mathcal{T}_p$. Since this group is also topologically simple as mentioned in Example 3.7, the above theorem allows us to claim that $\mathcal{T}_p$ is an atom.

For a group G, consider the topology

$$\mathfrak{M}_G := \bigcap \{\mathcal{T} \mid \mathcal{T} \in \mathfrak{L}_H(G)\},$$

named *Markov topology* after [37], implicitly present in Markov's work (see [53] and the surveys [41] and [42] for more detail on this issue). This topology is the infimum taken in the larger poset $\mathfrak{T}(G)$ of all topologies on G and need not be a group topology in general, although inversion and left and right shifts are continuous and $\mathfrak{M}_G$ is a T_1-topology. The $\mathfrak{M}_G$-closed sets are called *unconditionally closed* sets, following Markov.

Example 4.8 (a) For every group G the discrete topology δ_G is both Alexandrov and Hausdorff (actually, the unique Alexandrov and Hausdorff topology). By Corollary 4.6, δ_G is an atom precisely when G is simple and δ_G is minimal, in other words the group G is not topologizable (in the above terms, this means $\mathfrak{M}_G = \delta_G$). Such an infinite group was built first by Shelah in [62] under the assumption of Continuum Hypothesis.

(b) For every $\mathcal{T} \in \mathfrak{L}_H(G)$, the center

$$Z(G) = \bigcap_{a \in G} \{x \in G \mid xa = ax\}$$

is closed in $(G, \mathcal{T})$, because every centralizer

$$\{x \in G \mid xa = xa\}$$

is closed in $(G, \mathcal{T})$. Hence, $Z(G)$ is $\mathfrak{M}_G$-closed. One can prove by induction that for every positive integer n the subgroup $Z_n(G)$ is $\mathfrak{M}_G$-closed (that is, closed in any Hausdorff group topology on G).

Proposition 4.9 *Let $(G, \mathcal{T})$ be a Hausdorff topological group and let* H *be dense subgroup of* G*. Then* H *is atomic, as a topological subgroup, if and only if,* G *is atomic.*

Proof — Assume that H is atomic, so $(H, \mathcal{T}_H)$ is topologically simple and minimal. According to Proposition 4.1, G is minimal and H is essential in G. By Lemma 4.5, G is topologically simple, as well. By Corollary 4.6, G is atomic.

Now, assume that G is Hausdorff and atomic. Then G is minimal and topologically simple, by Corollary 4.6. This yields that H is essential in G, so H is minimal, by Proposition 4.1. On the other hand, H is topologically simple, by Lemma 3.8. Now, Corollary 4.6 applies again to conclude that H is a Hausdorff atom. □

Since this section is focused on Hausdorff atoms we formulated and proved the above proposition for Hausdorff atoms. Nevertheless, the conclusion remains true without the assumption that $(G, \mathcal{T})$ is Hausdorff (see Proposition 5.1 below).

Remark 4.10 One cannot omit density in the above proposition. Indeed, $\mathrm{Sym}(X)$ contains plenty of infinite discrete abelian subgroups H, which cannot be atoms by Theorem 1.2.

The next corollary shows that the completion of a Hausdorff atomic group is still a Hausdorff atomic group.

Corollary 4.11 *A Hausdorff group* $(G, \mathcal{T})$ *is atomic, if and only if, its Raĭkov completion is atomic.*

This fact was proved by Mutylin (see [54], Theorem 1) for abelian groups (and commutative rings).

Now we see that atomic Hausdorff topological groups are subject to very rigid algebraic restraints: they are either finite abelian of prime order or center-free.

Theorem 4.12 *Let* $(G, \mathcal{T})$ *be an atomic Hausdorff topological group, then either* G *has a trivial center or* G *is isomorphic to* $\mathbb{Z}_p$ *for some prime number* p*.*

Proof — Assume that G has a nontrivial center. Since $Z(G)$ is a nontrivial normal abelian subgroup of G, we deduce that G is abelian, by Corollary 3.10. Therefore, $G \simeq \mathbb{Z}_p$, by Theorem 1.2. □

We substantially used Theorem 1.2 in the above proof. Let us recall that an alternative proof of Theorem 1.2 was given in [59], based on the fact that the minimal topologies on $\mathbb{Z}$ are precompact by Prodanov's precompactness theorem (see [55]).

Now, we obtain a short proof of Clark-Schneider Theorem:

Corollary 4.13 (Clark-Schneider theorem) *Let* G *be a group and suppose* $Z(G)$ *is nontrivial and is not isomorphic to* $\mathbb{Z}_p$ *for any prime number* p*. Then* $\mathfrak{L}(G)$ *has no Hausdorff atoms.*

Proof — Suppose $\mathfrak{L}(G)$ has a Hausdorff atom. Then, $G \simeq \mathbb{Z}_p$ for some prime number p, in view of Theorem 4.12 and our hypothesis $Z(G) \neq \{1\}$. Hence,

$$Z(G) = G \simeq \mathbb{Z}_p,$$

a contradiction. □

One can replace "abelian" by the much weaker assumption "hypercentral" or "hyperabelian" in Theorem 1.2.

Corollary 4.14 *Let* $(G, \mathcal{T})$ *be a Hausdorff atomic topological group. If* G *is either hypercentral or hyperabelian then* G *is isomorphic to* $\mathbb{Z}_p$ *for some prime number* p*.*

Proof — In case G is hypercentral, we observe that hypercentral groups have nontrivial center. So, if $\mathcal{T}$ is a Hausdorff atom of $\mathfrak{L}(G)$, then, by Theorem 4.12, G is isomorphic to $\mathbb{Z}_p$ for some prime number p. In case G is hyperabelian Corollary 3.10 applies. □

Here we resume briefly what was obtained so far, as necessary conditions for an infinite Hausdorff atomic group:

Corollary 4.15 *Let* $(G, \mathcal{T})$ *be an infinite Hausdorff atomic topological group. Then every nontrivial normal subgroup of* G *is infinite and nonabelian. In particular,* G *has a trivial center and* G *is not hyperabelian (in particular, not solvable).*

4.3 On the number of Hausdorff atoms

In the case considered in Example 4.7, the Hausdorff atom is unique. The uniqueness is due to the deep fact that $\mathcal{T}_p$ is not only a minimal topology on $G = \mathrm{Sym}(X)$, it is actually the least element of $\mathfrak{L}_H(G)$. In other words, it coincides with

$$\mathfrak{M}_G = \inf\{\mathcal{T} \mid \mathcal{T} \in \mathfrak{L}_H(G)\}.$$

One can state the general result as follows: if $\mathfrak{M}_G \in \mathfrak{L}(G)$ for an infinite group G and $(G, \mathfrak{M}_G)$ is topologically simple, then $\mathfrak{M}_G$ is the unique Hausdorff atom of $\mathfrak{L}_H(G)$. Note that $\mathfrak{M}_G \in \mathfrak{L}(G)$ implies that $\mathfrak{M}_G$ is the least element of $\mathfrak{L}_H(G)$.

Now we give an example of a group G that admits infinitely many Hausdorff atoms in $\mathfrak{L}(G)$.

Example 4.16 Let $G = PSL_2(\mathbb{Q})$ and recall that this group is simple (see [24]). Next we consider an infinite family of groups G_p, where p is either 0 or a prime number, more precisely,

$$G_0 = PSL_2(\mathbb{R}) \quad \text{and} \quad G_p = PSL_2(\mathbb{Q}_p)$$

when p is a prime and $\mathbb{Q}_p$ is the field of p-adic integers, equipped with the p-adic topology. Again by [24], all these groups are simple. We provide them with their natural (quotient of the) matrix group topology. These topologies are locally compact. According to Theorem 7.4.1 of [35], G_0 is minimal and $G_p = PSL_2(\mathbb{Q}_p)$ is minimal by Theorem 5.3 of [9].
For every p consider the obvious embedding f_p of G into G_p and let $\mathcal{T}_p$ denote the topology induced on G by the embedding f_p. Since all these embeddings are dense and each G_p is a Hausdorff atom (being simple and minimal), we deduce from Proposition 4.9 that each $\mathcal{T}_p$ is a Hausdorff atom. To see that the topologies $\mathcal{T}_0$ and $\mathcal{T}_p$ are different take a nonidentical unitriangular matrix $A \in SL_2(\mathbb{Q})$ and let α be its image in the quotient group G. Then for $p > 0$ and arbitrary q,

$$\alpha^{p^n} \longrightarrow 1$$

with respect to $\mathcal{T}_q$, if and only if, $q = p$. This shows that $\mathcal{T}_p \neq \mathcal{T}_q$ whenever $0 \neq p \neq q$.

It is possible to conclude that $\mathcal{T}_q \neq \mathcal{T}_p$, for $p \neq q$, by arguing that they have different completions with respect to the two sided uniform structure.

It is worth mentioning that the infinite family $\{\mathcal{T}_p\}$ of pairwise distinct Hausdorff atoms on G provided by the above example satisfy on one hand the condition

$$\inf\{\mathcal{T}_p, \mathcal{T}_q\} = \iota_G$$

for distinct p, q. On the other hand, these topologies are not pairwise *independent* in the sense of [64] (let us recall that independence of $\mathcal{T}_p$ and $\mathcal{T}_q$

means that $\mathcal{T}_p \cap \mathcal{T}_q$ is the cofinite topology on G). Actually, even the intersection $\bigcap_p \mathcal{T}_p$ is not the cofinite topology. To see this, for every $n \in \mathbb{N}$ denote by m_n the product of the n-th powers of the first n prime numbers and consider the infinite set

$$K = \{1\} \cup \{\alpha^{m_n} \mid n \in \mathbb{N}\}$$

with α as in Example 4.16. Then K is closed in $(G, \mathcal{T}_p)$ when p is prime, because

$$\alpha^{m_n} \longrightarrow 1$$

in $(G, \mathcal{T}_p)$, and so K is compact, while $\mathcal{T}_p$ is Hausdorff. On the other hand, K is obviously closed (and discrete) in $(G, \mathcal{T}_0)$ as well. Therefore,

$$K \in \bigcap \mathcal{T}_p.$$

For more on this "uniqueness line" see also Paragraph 4.4 and in particular Remark 4.21).

4.4 Compact-like Hausdorff atoms

The Hausdorff atoms built in Example 4.16 were locally precompact (as the groups G_p are locally compact). This suggests to examine further the impact of imposing compactness or similar properties on atoms. For this purpose, the next theorem offers a complete description of topologically simple compact Hausdorff groups.

Theorem 4.17 *A compact Hausdorff topological group $(G, \mathcal{T})$ is topologically simple, if and only if, it is either a finite simple group, or a simple compact connected Lie group. In such a case, G is a Hausdorff atomic group.*

Proof — If G is finite, there is nothing to prove. Assume that $(G, \mathcal{T})$ is infinite, compact, Hausdorff and topologically simple. Since $c(G)$, the connected component containing 1, is a closed normal subgroup of G, $c(G)$ is either trivial or $G = c(G)$ is connected. In the former case, G is a profinite topological group, thus, G has a set of open (so, closed as well) normal subgroups that form a local base at 1, according to a well-known theorem of van Dantzig (see [65]). Since G is topologically simple, this is possible only when G is finite, a contradiction. Therefore, $G = c(G)$ is connected.

To prove that G is a Lie group pick a neighborhood $U \neq G$ of 1. Then there exists a closed normal subgroup N of G contained in U such that G/N is a Lie group. As $N \neq G$ and G is topologically simple, we deduce that $N = \{1\}$, that is, the group G itself is a connected Lie group. Since G is infinite, $Z(G)$ is trivial, by Theorem 4.12. Then

$$G = \prod_{i=1}^{n} L_i,$$

where each L_i is a simple compact connected Lie group (see [47]). As G is topologically simple and each L_i is a closed normal subgroup, we deduce that $n = 1$. Therefore, G itself is a simple compact connected Lie group. The last assertion follows from the minimality of compact Hausdorff groups and Corollary 4.6. □

Theorem 4.17 implies that a compact Hausdorff topological group is atomic, if and only if, either it is a finite simple group, or a simple compact connected Lie group. In particular, $\mathfrak{L}(SO(3))$, has a Hausdorff atom. We do not know if this Hausdorff atom is unique. Nevertheless, it is known that the natural compact topology on SO(3) is its unique precompact topology (due to the fact that every homomorphism $SO(3) \to K$, where K is a compact Hausdorff group, is continuous; see [22]).

Corollary 4.18 *An infinite precompact Hausdorff topological group* $(G, \mathcal{T})$ *is atomic, if and only if, the Raĭkov completion of* $(G, \mathcal{T})$ *is a compact connected simple Lie group. In particular, precompact Hausdorff atomic groups are metrizable, have size* $\leqslant \mathfrak{c}$ *and have the NSS property (defined below).*

A topological group $(G, \mathcal{T})$ is said to have *no small subgroups* (shortly, is *NSS* or has *NSS property*), if there exists $U \in \mathcal{F}_{\mathcal{T}}$ such that U contains no nontrivial subgroup of G. It is known that Lie groups are NSS. Using this fact, one can easily see that precompact Hausdorff atomic groups have also the NSS property.

Corollary 4.19 *An infinite pseudocompact Hausdorff topological group* $(G, \mathcal{T})$ *is atomic, if and only if, it a compact connected simple Lie group.*

Proof — Follows from Corollary 4.18 and Theorem 4.17, using the well-known facts that pseudocompact Hausdorff topological groups are precompact and metrizable pseudocompact groups are compact (see for example [36]). □

Theorem 4.20 *For a cardinality* $\kappa > 0$ *the free group* F_κ *admits a Hausdorff atom, if and only if,* $\kappa > 1$*. More precisely:*

(a) *if* κ *is infinite, then* F_κ *admits a nonarchimedean Hausdorff atom of countable pseudocharacter and character* κ*;*

(b) *if* $1 < \kappa \leqslant \mathfrak{c}$*, then* F_κ *admits a Hausdorff precompact atom with the NSS property.*

Proof — The necessity in the first assertion follows from Theorem 1.2.

(a) Assume that κ is infinite. Then the assertion follows immediately from a construction of Shakhmatov (see [61]), who built a dense embedding of F_κ in the group $\mathrm{Sym}(\kappa)$, such that the induced topology $\mathcal{T}$ on F_κ has $\psi(F_\kappa, \mathcal{T}) = \omega$ and $\chi(F_\kappa, \mathcal{T}) = \kappa$. According to Proposition 4.9 and Example 4.7, $(F_\kappa, \mathcal{T})$ is a Hausdorff atom.

(b) According to [11], every compact connected simply Lie group L has a dense free subgroup G that can be chosen isomorphic to F_κ for any $1 < \kappa \leqslant \mathfrak{c}$. Now Proposition 4.9, along with Theorem 4.17, applies. One can see as above, that again the NSS property is available here. □

Remark 4.21 From Theorem 4.20, one can also deduce that for $\omega \leqslant \kappa \leqslant \mathfrak{c}$ the group F_κ admits more than just one Hausdorff atom.

Let us recall that free groups are hypocentral, so this theorem shows that arbitrarily large hypocentral groups may admit Hausdorff atoms (we will see below that hypercentral groups do not admit any nondegenerate atoms). Since free groups F_κ are residually nilpotent (that is, $\gamma_\omega(F_\kappa) = \{1\}$), this leaves open the following general problem:

Question 4.22 *Which hypocentral groups admit Hausdorff atoms?*

This should be compared with Corollary 5.3, where we prove that (even) virtually hypercentral infinite groups admit no Hausdorff atoms.

In view of item (b) of Theorem 4.20, one may ask whether free groups admit pseudocompact Hausdorff atoms:

Theorem 4.23 *No free group admits a pseudocompact Hausdorff atom.*

Proof — Assume $G = F_\kappa$ admits a pseudocompact Hausdorff atom $\mathcal{T}$. By Corollary 4.19, $(G, \mathcal{T})$ is a compact Lie group. This contradicts the fact that free groups do not admit even countably compact group topologies (see Corollary 5.14 of [36]). □

5 Atoms that are not necessarily Hausdorff

It follows from Lemma 3.3, that a topological group $(G, \mathcal{T})$ is atomic, if and only if, $\mathcal{T}/\operatorname{core}\mathcal{T}$ is a Hausdorff atom in $\mathfrak{L}(G/\operatorname{core}\mathcal{T})$. Therefore, atomic topological groups can be built as *extensions of topological groups by Hausdorff atomic groups*. Namely, these are all topological groups $(G, \mathcal{T})$ containing a normal subgroup N such that the quotient $(G/N, \mathcal{T}/N)$ has a Hausdorff atom $\mathfrak{S}$. Let

$$\pi : G \to G/N$$

be the quotient map. Then the topology

$$\mathcal{T} = \widehat{\pi}(\mathfrak{S})$$

on G is an atom and $N = \operatorname{core}\mathcal{T}$ is a closed indiscrete subgroup of $(G, \mathcal{T})$. This construction explains the relevance of Hausdorff atoms in the realm of all atoms.

Proposition 5.1 *Let* $(G, \mathcal{T})$ *be a topological group and let* H *be dense subgroup of* G*. Then* H *is atomic, as a topological subgroup, if and only if,* G *is atomic.*

Proof — Let $G_1 = G/\operatorname{core}\mathcal{T}$ and

$$\mathcal{T}_1 = \mathcal{T}/\operatorname{core}\mathcal{T}.$$

By Lemma 3.3 (a), $(G_1, \mathcal{T}_1)$ is Hausdorff. It is easy to see that $H_0 = H \cap \operatorname{core}\mathcal{T}$ coincides with $\operatorname{core}(\mathcal{T}\restriction_H)$ and $H_1 = H/H_0$ equipped with $\mathcal{T}\restriction_H/\operatorname{core}(\mathcal{T}\restriction_H)$ is topologically isomorphic to the subgroup

$$H \cdot \operatorname{core}\mathcal{T}/\operatorname{core}\mathcal{T}$$

of G_1. Applying the above proposition to G_1 and its dense subgroup H_1, we deduce that H_1 is atomic, if and only if, G_1 is atomic. On the other hand, H (respectively, G) is atomic, if and only if, H_1 (respectively, G_1) is atomic, by Lemma 3.3 (d). This proves that H is atomic, if and only if, G is atomic. □

The dichotomy of Theorem 4.12 and its corollaries now give:

Theorem 5.2 *Let* $(G, \mathcal{T})$ *be an atomic topological group and* $N = \operatorname{core}\mathcal{T}$. *Then:*

(a) *either* $Z(G/N) = \{N\}$, *or* $[G : N] = p$.

(b) *if* $\mathcal{T}$ *is a nondegenerate atom, then* $Z^*(G) \subseteq N$.

(c) *if* G *is either hypercentral or hyperabelian, then* $[G : N] = p$, *so* $\mathcal{T}$ *is degenerate.*

Proof — (a) By Lemma 3.3, $\mathcal{T}/N$ is a Hausdorff atom of $\mathfrak{L}(G/N)$. Now Theorem 4.12 applies.

(b) Let $\pi : G \to G/N$ be the quotient map. We first prove that:

$$\pi[Z^*(G)] \subseteq Z^*(G/N). \tag{5.1}$$

For this purpose, it suffices to prove by transfinite induction that:

$$\pi[Z_\lambda(G)] \subseteq Z_\lambda(G/N) \tag{5.2}$$

for every ordinal number λ. This is clear for $\lambda = 0$. Assume that $\lambda > 0$ and the assertion is true for all $\mu < \lambda$ and let

$$\pi_1 : G \to G/Z_\mu(G)$$

and

$$\pi_2 : G/N \to (G/N)/Z_\mu(G/N)$$

be the quotient maps.

Our assumptiom immediately gives (5.2) in case λ is a limit ordinal number. In case $\lambda = \mu + 1$, for an ordinal number μ, one has

$$\pi[Z_\mu(G)] \subseteq Z_\mu(G/N).$$

So there exists a (necessarily) surjective homomorphism

$$\overline{\pi} : G/Z_\mu(G) \to (G/N)/Z_\mu(G/N)$$

such that $\overline{\pi} \circ \pi_1 = \pi_2 \circ \pi$. This gives the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\pi_1} & G/Z_\mu(G) \\ \pi \downarrow & & \downarrow \overline{\pi} \\ G/N & \xrightarrow[\pi_2]{} & (G/N)/Z_\mu(G/N) \end{array}$$

Since

$$\pi_1 \left[Z_{\mu+1}(G)\right] \subseteq Z\left(G/Z_\mu(G)\right) \quad \text{and} \quad \overline{\pi}\,[Z(G/Z_\mu(G)] \subseteq Z\left((G/N)/Z_\mu(G/N)\right),$$

we deduce from the commutative diagram above that:

$$\pi\left[Z_{\mu+1}(G)\right] \subseteq \pi_2^{-1}\left[Z\left((G/N)/Z_\mu(G/N)\right)\right] = Z_{\mu+1}(G/N).$$

If $\mathcal{T}$ is a nondegenerate atom, then $\mathcal{T}/N$ is a Hausdorff atom on the infinite group G/N, by Lemma 3.3. By item (a), we conclude that $Z(G/N) = \{N\}$, and consequently, $Z^*(G/N) = \{N\}$. This yields $Z^*(G) \subseteq N$ by (5.1).

(c) As G is hypercentral, (5.1) implies that $G/N = Z^*(G/N)$ is hypercentral as well. Therefore, the alternative $Z(G/N) = \{N\}$ in item (a) is ruled out. Then (a) implies $[G : N] = p$, and $\mathcal{T}$ is degenerate.

Assume now that G is hyperabelian. Then G/N is a Hausdorff atom, by Lemma 3.3. Since G/N is hyperabelian as well, we deduce from Corollary 3.10 that G/N is abelian. Hence, the alternative

$$Z(G/N) = \{N\}$$

in item (a) is again ruled out. Then (a) implies $[G : N] = p$, and $\mathcal{T}$ is degenerate. □

It can be helpful to note that

$$Z^*(G) \subseteq \operatorname{core} \mathcal{T}$$

in item (b) of the previous theorem is equivalent to $\overline{Z^*(G)} = \operatorname{core} \mathcal{T}$ (see Example 5.6 (b) showing when these conditions fail).

Corollary 5.3 *Let G be a group that satisfies one of the following conditions:*

(a) G *is virtually hypercentral;*

(b) G *is hyperabelian;*

(c) *there exists a finite normal subgroup* N *of* G *such that* G/N *is hypercentral.*

Then, every atom of $\mathfrak{L}(G)$ *is degenerate.*

Proof — Assume first that G is virtually hypercentral or hyperabelian (i.e., cases (a) and (b)). Let $\mathcal{T}$ be an atom of $\mathfrak{L}(G)$. By Theorem 5.2 (c), $\mathcal{T}$ is degenerate.

(c) Now assume that there exists a finite normal subgroup N of G such that G/N is hypercentral. It was proved in [23] that this property is equivalent to hypercentrality of G. Hence, the above argument works. □

Corollary 5.4 *Let* $(G, \mathcal{T})$ *be an atomic topological group. Then, either*

$$Z^*(G) \subseteq \operatorname{core}\mathcal{T} \quad \text{or} \quad G/\operatorname{core}\mathcal{T} \simeq \mathbb{Z}_p,$$

so $\mathcal{T}$ *is degenerate and* G' *is indiscrete, equivalently,* $\overline{G'} = \operatorname{core}\mathcal{T}$.

Proof — Suppose $Z^*(G) \nsubseteq \operatorname{core}\mathcal{T}$. So $Z^*(G) \neq \{1\}$. Hence, $G/\operatorname{core}\mathcal{T} \simeq \mathbb{Z}_p$, by Theorem 5.2. Since $G/\operatorname{core}\mathcal{T}$ is abelian, we deduce that $G' \subseteq \operatorname{core}\mathcal{T}$. This proves the last assertion. □

Corollary 5.5 *Let* G *be a hypercentral group and* $\mathcal{T}$ *be an atom of* $\mathfrak{L}(G)$. *Then* $\mathcal{T}$ *is degenerate and* $\operatorname{core}\mathcal{T}$ *has index* p *for some prime number* p.

Proof — As $\mathcal{T} \neq \iota_G$, we deduce that $Z^*(G) = G \nsubseteq \operatorname{core}\mathcal{T}$. Hence, Corollary 5.4 applies. □

Corollary 5.4 offers a curious topological counterpart of the celebrated Schur theorem. Indeed, according to Schur's theorem, if

$$[G : Z(G)] < \infty$$

for a group G, then

$$|G'| < \infty,$$

that is, if the center is "large", then the derived group is "small". Here, the criterion of being large or small is simply measuring the index or size of the relevant subgroup. When a group is equipped with a topology, one can change the method of measurement and consider as "large" the dense subgroups, and, as "small" the indiscrete subgroups (those with an indiscrete subspace topology). Through this looking glass on "small/large", things become quite transparent in the case of a group G equipped with a simple group topology since in this case one has a dichotomy as far as normal subgroups N of G are concerned: either N is "small" (indiscrete) or "large" (dense). It is this point of view that makes Shur's paradigm work in an atomic group G even in a somewhat relaxed form: if $Z^*(G)$ is dense ("large") in G, then G' is indiscrete ("small"). Finally, it must be underlined that Schur theorem cannot be deduced from Corollary 5.4.

Now we provide an easy example with $Z^*(G) \subseteq \operatorname{core}\mathcal{T}$, so the conclusion of Corollary 5.4 fails.

Example 5.6 (a) Let A be any nontrivial abelian group and

$$G = A \times \mathrm{Sym}(\mathbb{N}),$$

equipped with

$$\mathcal{T} = \iota_A \times \mathcal{T}_p.$$

Then

$$\mathrm{core}\,\mathcal{T} = Z(G) = A \times \{1\}$$

and G is atomic by Theorem 3.9, as

$$G/\mathrm{core}\,\mathcal{T} = \mathrm{Sym}(X)$$

is a Hausdorff atomic group (see Example 4.7). If we take $G = A \times L$, equipped with $\mathcal{T} = \iota_A \times \mathcal{T}'$, wher $(L, \mathcal{T}')$ is a compact simple connected Lie group, then we have similar properties. In both cases, the subgroup G' (that is $\mathrm{Sym}(X)$, and L, respectively) is Hausdorff.

(b) Let G_1 be a center-free group and let p be a prime. Equip $G = \mathbb{Z}_p \times G_1$ with the product topology, so that $\mathbb{Z}_p$ is discrete and G_1 is indiscrete. Then

$$Z^*(G) = \mathbb{Z}_p \times \{1\} \not\subseteq \mathrm{core}\,\mathcal{T} = \{0\} \times G_1.$$

The next two theorems show the strong impact of lattice theoretic properties of $\mathfrak{L}(G)$ (in terms of existence of atoms) on the structure of the group G.

Theorem 5.7 *For an abelian group G, the following conditions are equivalent:*

(a) $\mathfrak{L}(G)$ *has no atoms;*

(b) G *is divisible.*

Theorem 5.8 *For an abelian group G, the following conditions are equivalent:*

(a) $\mathfrak{L}(G/H)$ *has atoms for every proper subgroup H of G;*

(b) G/H *is divisible for no proper subgroup H of G;*

(c) $\mathrm{rank}(G) = n < \infty$ *and for every free subgroup H of rank n of G the quotient group G/H has finite p-primary components for every prime number p.*

The implication (a) $\to$ (b) clearly follows from Theorem 5.7. The implication (b) $\to$ (c) is due to [34], a proof can be found also in Theorem 3.5 of [31].

Theorem 5.9 *Let $(G, \mathcal{T})$ be a compact topological group. Then following conditions are equivalent:*

(a) $(G, \mathcal{T})$ *is atomic;*

(b) $(G, \mathcal{T})$ *is topologically simple;*

(c) $G/\mathrm{core}\,\mathcal{T}$ *is either simple and bounded, or a simple compact connected Lie group.*

Proof — (a) → (b) If $\mathcal{T}$ is an atom of $\mathfrak{L}(G)$, by Theorem 3.9, $\mathcal{T}$ is simple.

(b) → (c) Suppose $\mathcal{T}$ is simple and let $N = \mathrm{core}\,\mathcal{T}$. By Theorem 2.4 and Lemma 3.3, $\mathcal{T}/N \in \mathfrak{L}_H(G/N)$. So, $(G/N, \mathcal{T}/N)$ is a compact Hausdorff topological group. By Remark 3.6, $(G/N, \mathcal{T}/N)$ is also topologically simple. Now Theorem 4.17 applies.

(c) → (a) The compact Hausdorff group $(G/N, \mathcal{T}/N)$ is minimal. Moreover, it is also topologically simple, by Remark 3.6. Hence, $\mathcal{T}$ is an atom of $\mathfrak{L}(G)$, by Theorem 3.9. □

Because the structure of any compact topological group $(G, \mathcal{T})$ with $\mathcal{T}$ simple, is known, Theorem 4.17, characterizes the structure of all compact atoms of $\mathfrak{L}(G)$.

REFERENCES

[1] O. Alas – S. Hernández – M. Sanchis – M. Tkachenko – R. Wilson: "Adjacency in subposets of the lattice of T_1-topologies on a set", *Acta Math. Hungar.* 112 (2006), 199–219.

[2] A. Arhangel'skii – M. Tkachenko: "Topological Groups and Related Structures", *World Scientific*, New Jersey (2008).

[3] V. Arnautov: "On coverings in the lattice of all group topologies of arbitrary abelian groups", *Sibirsk. Mat. Zh.* 47 (2006), 961–973; translated in *Siberian Math. J* 47 (2006), 787–796.

[4] V. Arnautov: "Properties of final unrefinable chains of groups topologies", *Bul. Acad. Ştiinţe Repub. Mold. Mat.* 2010 no. 2, 3–19.

[5] V. Arnautov: "Properties of covers in the lattice of group topologies for nilpotent groups", *Bul. Acad. Ştiinţe Repub. Mold. Mat.* 2012 no. 3, 38–44.

[6] V. Arnautov – G. Ermakova: "On the number of group topologies on countable groups", *Bul. Acad. Ştiinţe Repub. Mold. Mat.* 2014 no. 1, 101–112.

[7] V. Arnautov – K. Filippov: "On coverings in the lattice of topologies on a group of finite period", *Bul. Acad. Ştiinţe Repub. Mold. Mat.* 2002 no. 2 , 77–87.

[8] V.I. Arnautov – A. G. Topalè: "On the nonmodularity of the lattice of group topologies", *Izv. Akad. Nauk Respub. Moldova Mat.* 1997 no. 1, 84–92,132,134.

[9] U. Bader and T. Gelander: "Equicontinuous actions of semisimple groups"; *Groups Geom. Dyn.* 11 (2017), 1003–1039.

[10] A. Berarducci – D. Dikranjan – M. Forti – S. Watson: "Cardinal invariants and independence results in the lattice of precompact group topologies", *J. Pure Appl. Algebra* 126 (1998), 19–49.

[11] S. Balcerzyk – J. Mycielski: "On the existence of free subgroups of topological groups", *Fund. Math.* 44 (1957), 303–308.

[12] M. Berri: "The complement of a topology for some topological groups", *Fund. Math.* 58 (1966), 159–162.

[13] G. Birkhoff: "On the combination of topologies", *Fund. Math.* 26 (1936), 156–166.

[14] T.S. Blyth: "Lattices and Ordered Algebraic Structures", *Springer*, Berlin (2005).

[15] N. Bourbaki: "General Topology", *Springer*, Berlin (1998).

[16] B. Clark – V. Schneider: "Relationship between the meet and join operators in the lattice of group topologies", *Proc. Amer. Math. Soc.* 98 (1986), 681–682.

[17] B. Clark – V. Schneider: "Normal decompositions of the lattice of group topologies", *Arch. Math. (Basel)* 46 (1986), 304–306.

[18] B. Clark – V. Schneider: "Atoms and antiatoms in the lattice of group topologies", *Arch. Math. (Basel)* 52 (1989), 173–174.

[19] W. Comfort – D. Remus: "Long chains of Hausdorff topological group topologies", *J. Pure Appl. Algebra* 70 (1991), 53–72.

[20] W. Comfort – D. Remus: "Intervals of totally bounded group topologies", in Papers on General Topology and Applications, *Ann. New York Acad. Sci.*, New York (1995), 121–129.

[21] W. Comfort – D. Remus: "Long chains of topological group topologies—A continuation", *Topology Appl.* 75 (1997), 51–79.

[22] W. Comfort – D. Remus: "Counting compact group topologies", *Topology Appl.* 213 (2016), 92–109.

[23] M. De Falco – F. de Giovanni – C. Musella – Y.P. Sysak: "On the upper central series of infinite groups", *Proc. Amer. Math. Soc.* 139 (2011), 385–389

[24] L.E. Dickson: "Theory of linear groups in an arbitrary field", *Trans. Amer. Math. Soc.* 2 (1901), 363–394.

[25] D. Dikranjan: "The lattice of compact representations of an infinite group", in Groups 93, *London Math. Soc. Lecture Notes* 211 (1995), 138–155.

[26] D. Dikranjan: "On the poset of precompact group topologies", in Topology with Applications (Szekszàrd, 1993), *Elsevier*, Amsterdam (1995), 135–149.

[27] D. Dikranjan: "Chains of pseudocompact group topologies", *J. Pure Appl. Algebra* 124 (1998) 65–100.

[28] D. Dikranjan: "Recent advances in minimal topological groups", *Topology Appl.* 85 (1998), 53–91.

[29] D. Dikranjan: "The lattice of group topologies and compact representations", *Proc. Workshop on Lie groups and topological groups*, Madrid (1999), 105–126.

[30] D. Dikranjan – A. Giordano Bruno: "Functorial topologies and finite-index subgroups of abelian groups", *Topology Appl.* 158 (2011), 2391–2407.

[31] D. Dikranjan – A. Giordano Bruno – L. Salce – S. Virili: "Fully inert subgroups of divisible Abelian groups", *J. Group Theory* 16 (2013), 915-939.

[32] D. Dikranjan – M. Megrelishvili: "Minimality conditions in topological groups", in Recent Progress in General Topology III, *Atlantis*, Paris (2014), 229–327.

[33] D. Dikranjan – Iv. Prodanov: "Totally minimal topological groups", *Annuaire Univ. Sofia Fac. Math. Méc.* 69 (1974/75), 5–11.

[34] D. Dikranjan – Iv. Prodanov: "A class of compact abelian groups", *Annuaire Univ. Sofia Fac. Math. Méc.* 70 (1975/76), 191–206.

[35] D. Dikranjan – I. Prodanov – L. Stoyanov: "Topological Groups: Characters, Dualities and Minimal Group Topologies", *Dekker*, New York (1989).

[36] D. Dikranjan – D. Shakhmatov: "Algebraic Structure of the Pseudocompact Groups", *Memoirs Amer. Math. Soc.* 133/633 (1998).

[37] D. Dikranjan – D. Shakhmatov: "Selected topics from the structure theory of topological groups", in Open Problems in Topology II, *Elsevier* (2007), 389–406.

[38] D. Dikranjan – D. Shakhmatov: "Reflection principle characterizing groups in which unconditionally closed sets are algebraic", *J. Group Theory* 11 (2008), 421–442.

[39] D. Dikranjan – M. Tkačenko – I. Yaschenko: "On transversal group topologies", *Topology Appl.* 153 (2005), 786–817.

[40] D. Dikranjan – M. Tkačenko – I. Yaschenko: "Transversal group topologies on nonabelian groups", *Topology Appl.* 153 (2006), 3338–3354.

[41] D. Dikranjan – D. Toller: "Markov's problems through the looking glass of Zariski and Markov topologies", in Ischia Group Theory 2010, *World Sci.*, New Jersey (2012), 87–130.

[42] D. Dikranjan – D. Toller: "Zariski topology and Markov topology on groups", *Topology Appl.* 241 (2018), 115–144.

[43] D. Dikranjan – A. Tonolo: "On the lattice of linear module topologies", *Comm. Algebra* 21 (1993), 275-298.

[44] D. Doïtchinov: "Produits de groupes topologiques minimaux", *Bull. Sci. Math.* 97 (1972), 59–64.

[45] D. Doïtchinov: "Minimal topological groups", *Colloq. Math. Soc. Janos Bolyai* 8 (1972), 213–214.

[46] E. Gaughan: "Topological group structures of infinite symmetric groups", *Proc. Nat. Acad. Sci. U.S.A.* 58 (1967) 907–910.

[47] K.H. Hofmann – S. A. Morris: "The Structure of Compact Groups", *de Gruyter*, Berlin (1998).

[48] H. Khass – A.R. Ashrafi – B. Bazigaran: "On the lattice of f-subgroups", *Siberian Electr. Math. Reports* 14 (2017), 600–613.

[49] H. Khass: "f-Subgroups and Their Lattice Structures", PhD Thesis, Kashan, Iran.

[50] M. Lamper: "Complements in the lattice of all topologies of topological groups", *Arch. Math. (Brno)* 10 (1974), 221–230.

[51] R. Larson – S. Andima: "The lattice of topologies: a survey", *Rocky Mountain J. Math.* 5 (1975), 177–198.

[52] R. Larson – W.J. Thron: "Covering relations in the lattice of T_1-topologies", *Trans. Amer. Math. Soc.* 168 (1972), 101–111.

[53] A.A. Markov: "On unconditionally closed sets", *C.R. Dokl. Acad. Sci. URSS (N.S.)* 44 (1944), 180–181.

[54] M. Mutylin: "Completely simple commutative topological rings", *Mat. Zametki* 5 (1969), 161–171.

[55] Iv. Prodanov: "Some minimal group topologies are precompact", *Math. Ann.* 227 (1977), 117–125.

[56] I. Prodanov: "Maximal and minimal topologies on abelian groups", *Colloq. Math. Soc. János Bolyai* 23 (1980), 985–987.

[57] I.R. Prodanov – L.N. Stojanov: "Every minimal abelian group is precompact", *C.R. Acad. Bulgare Sci.* 37 (1984), 23–26.

[58] D. Remus: "Zur Struktur des Verbandes der Gruppentopologien", Dissertation, Hannover, Germany (1983).

[59] D. Remus: "On the structure of the lattice of group topologies", *Results Math.* 6 (1983), 251–252.

[60] I. Schur: "Neuer Beweis eines Satzes über endliche Gruppen", *Proc. Sympos. Appl. Math. Sitzber. Akad. Wiss. Berlin* (1902), 1013–1019.

[61] D. Shakhmatov: "Character and pseudocharacter in minimal topological groups", *Mat. Zametki* 38 (1985), 908–914; translated in *Math. Notes* 38 (1985), 1003–1006.

[62] S. Shelah: "On a problem of Kurosh, Jonsson groups, and applications", in Word Problems II, *North-Holland*, Amsterdam (1980), 373–394.

[63] R. Jr. Stephenson: "Minimal topological groups", *Math. Ann.*, 192 (1971), 193–195.

[64] M. Tkachenko – I. Yaschenko: "Independent group topologies on abelian groups", *Topology Appl.* 122 (2002), 425–451.

[65] D. van Dantzig: "Studien over Topologische Algebra", Dissertation, Amsterdam, Netherland (1931).

[66] S. Willard: "General Topology", *Dover*, New York (2004).

[67] E. Zelenyuk – I. Protasov: "Complemented topologies on abelian groups", *Sibirsk. Mat. Zh.* 42 (2001), 550–560; translated in *Siberian Math. J.* 42 (2001), 465–472.

Dikran Dikranjan
Department of Mathematics and Computer Science
University of Udine (Italy)
e-mail: dikranjan@dimi.uniud.it

Behnam Bazigaran, Hossein Khass
Department of Mathematics
University of Kashan (Iran)
e-mail: bazigaran@kashanu.ac.ir; hosseinkhass@gmail.com