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Corrigendum II to: "Characterization of Fitting p-Groups whose Proper Subgroups are Solvable"

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Abstract

Unfortunately "Corrigendum to Characterizations of Fitting p-groups whose proper subgroups are solvable" contains an error in the conclusion part of Lemma 2.1 (c). This forces a minor new adjustment in the statement of Theorem 1.1 and of Corollary 1.2 for $p = 3$ stated in [6]. The new statements of the theorem and its corollary are stated below. The powerfulness property is used throughout the proofs. Therefore a short introduction of this property to infinite nilpotent p-groups is given in Section 2.

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1 Introduction

The following Theorem 1.1 is the same as the one in [6] with a minor change for the case $p = 3$. But it is separated into two parts in order to make the statements simpler.

Theorem 1.1 *Let* G *be a Fitting* p*-group satisfying the normalizer condition, where* $p \neq 2$*. Suppose that in every homomorphic image* H *of* G *each* Λ*-pair* (w_H, V_H) *has a* (w_H, V_H)*-maximal subgroup satisfying the* $(**)$*-property. Then the following hold.*

(a) *If* $p > 3$*, then* G *cannot be perfect.*

(b) *If* $p = 3$*, then* G *either is not perfect or has a homomorphic image* H *having a dominant pair* (w_H, V_H) *with* $W^*(w_H, V_H) = 1$ *such that the following hold.* H *has a normal nilpotent metabelian subgroup* B_H *of class* $c(B_H) = 3$ *and* $\exp(B_H/Z(B_H)) \leqslant 9$ *such that* B *has a normal subgroup* X *with the properties* $Z(B) \leqslant X$*,* B/X *is elementary abelian and* X *is not powerful.*

In the proof one encounters with a normal nilpotent subgroup of class 3. The case $p > 3$ can settled without much difficulty since then this normal subgroup is regular by [9, III.10.2 Satz] but for $p = 3$ there is no similar restriction on B.

Corollary 1.2 *Let* G *be a Fitting* p*-group satisfying the normalizer condition in which every proper subgroup is solvable, where* $p \neq 2$*. Suppose that in every homomorphic image* H *of* G *every dominant pair* (w_H, V_H) *has a* (w_H, V_H)*-maximal subgroup satisfying the* $(**)$*-property. If* G *is not solvable, then* $p = 3$ *and has a homomorphic image* H *having a dominant pair* (w_H, V_H) *with* $W^*(w_H, V_H) = 1$ *such that the following hold.* H *has a normal nilpotent metabelian subgroup* B_H *of class* $c(B_H) = 3$ *and* $\exp(B_H/Z(B_H)) \leqslant 9$ *such that* B *has a subgroup* X *with the properties* $Z(B) \leqslant X$*,* B/X *is elementary abelian and* X *is not powerful.*

By coincidence Theorem 1.1 provides a good example for an application of powerfulness property to infinite groups (see also Corollary 2.7).

The lemmas of this work are new. Lemma 3.1 is parallel to [5, Lemma 2.5] and Lemma 3.2 is extracted from [6, Lemmas 2.1].

$c(X), d(Y)$, denote the class of a nilpotent group X and the derived length of a solvable group Y, respectively. Furthermore for a p-group G

$$\Omega_k(G) = \langle g \in G : g^{p^k} = 1 \rangle \text{ and } \mho_k(G) = \langle g^{p^k} : g \in G \rangle$$

for every $k \geqslant 0$ (for the other notations and definitions see [5,8,9,12]).

2 The Powerfulness Property

We begin by extending the definition of a "powerful group" given by Lubotzky and Mann in [12] for finite p-groups to locally finite p-groups.

Definition 2.1 *Let* G *be locally finite* p*-group and* $L \leqslant G$*, where* $p > 2$*. Then* G *is called powerful if*

$$G' \leqslant \mho_1(G)$$

and L *is called powerfully embedded in* G *if*

$$[L, G] \leqslant \mho_1(L).$$

Thus G is powerful if and only if $\mathrm{Frat}(G) = \mho_1(G)$. Furthermore if L is powerfully embedded in G, then $L \lhd G$ and is powerful. Trivially if L is powerful (powerfully embedded) and $M \lhd L(M \lhd G)$, then L/M (LM/M) is powerful(powerfully embedded).

The following contains just a few of the beginning properties of powerful groups obtained in [12] and the proofs given here (except Lemma 2.6 and Corollary 2.7) are repetitions of the ones given there.

Let $N \leqslant H \leqslant G$ such that N is powerfully embedded in G and H/N is cyclic. Then H is powerful. Indeed

$$\mho_1(H) \geqslant \mho_1(N) \geqslant [N, H] = H'$$

by [7, Lemma 2.1]

Lemma 2.2 *Let* G *be a nilpotent* p*-group and let* K *be a normal subgroup of* G *so that* $K/[K, G, G]$ *is powerfully embedded in* $G/[K, G, G]$*. Then* K *is powerfully embedded in* G*.*

Proof — Assume not. Then $[K, G] \nleqslant \mho_1(K)$. Since

$$[K, G] \leqslant \mho_1(K)[K, G, G]$$

we have $[K, G] = (\mho_1(K) \cap [K, G])[K, G, G]$. Put

$$M = \mho_1(K) \cap [K, G].$$

Then $M < [K, G]$. Put $\overline{G} = G/M$ and let $s = c(\overline{G})$. Then there exists an $r \leqslant s$ so that

$$[\overline{K}, \overline{G}] \nleqslant Z_{r-1}(\overline{G})$$

but $[\overline{K}, \overline{G}] \leqslant Z_r(\overline{G})$. Then

$$[\overline{K}, \overline{G}]Z_{r-1}(\overline{G})/Z_{r-1}(\overline{G}) \leqslant Z(\overline{G}/Z_{r-1}(\overline{G}))$$

and hence $[\overline{K}, \overline{G}, \overline{G}] \leqslant Z_{r-1}(\overline{G})$. Put $Z/M = Z_{r-1}(\overline{G})$. Then $[K, G, G]M \leqslant Z$ but $[K, G] \nleqslant Z$ since $[K, G]M/M \nleqslant Z_{r-1}(\overline{G})$. But since $[K, G] = [K, G, G]M$ this gives a contradiction. □

Lemma 2.3 *Let* G *be a nilpotent* p*-group and let* N *be powerfully embedded in* G *such that* $[N, G] \leqslant Z_2(G)$*. Then*

$$\mho_1([N, G]) = [\mho_1(N), G].$$

Proof — First we show that $[\mho_1(N), G] \leqslant \mho_1([N, G])$. Let $a^p \in \mho_1(N)$ and $g \in G$. Then it is easy to see that

$$[a^p, g] = [a, g]^p [a, g, a]^{\binom{p}{2}} = ([a, g][a, g, a]^{(p-1)})^p$$

since $[N, G] \leqslant Z_2(G)$, which verifies the assertion.

Next we show that $[\mho_1(N), G] \geqslant \mho_1([N, G])$. For this it suffices to show that if $[\mho_1(N), G] = 1$, then $\mho_1([N, G]) = 1$. Let $a \in N$ and $g \in G$. Then

$$\begin{aligned}[a, g]^p &= (a^{-1}a^g)^p = (a^{-1})^p (a^g)^p [a^g, a^{-1}]^{\frac{-p(p-1)}{2}} \\ &= [(a^g)^p, a^{-1}]^{-(p-1)/2} = 1\end{aligned}$$

by [8, Lemma 2.2.2 (ii)] since $[a, a^g] = [a, a[a, g]] = [a, [a, g]] \in Z(G)$. Hence it follows that $\mho_1([N, G] \geqslant [\mho_1(N), G]$. □

Lemma 2.4 *Let* G *be a nilpotent* p*-group and let* M,N *be powerfully embedded in* G*. Then* $[N, G]$*,* $\mho_1(N)$*,* $[M, N]$ *and* MN *are powerfully embedded in* G*.*

Proof — First we show that $[N, G]$ is powerfully embedded in G. By Lemma 2.2 we may suppose that $[N, G, G, G] = 1$. Then $[N, G] \leqslant Z_2(G)$. Thus applying Lemma 2.3 gives

$$\mho_1([N, G]) = [\mho_1(N), G] \geqslant [N, G, G]$$

since $[N, G] \leqslant \mho_1(N)$ and so $[N, G]$ is powerfully embedded.

Next we show that $\mho_1(N)$ is powerfully embedded. As above we assume that $[\mho_1(N), G, G] = 1$ and so $\mho_1(N) \leqslant Z_2(G)$. Applying Lema 2.3 gives

$$\mho_1(\mho_1(N)) \geqslant \mho_1([N, G]) = [\mho_1(N), G]$$

which was to be shown.

Next we show that $[M, N]$ is powerfully embedded. Again we assume that $[M, N, G, G] = 1$ and so

$$[M, N] \leqslant Z_2(G).$$

First we show that $[M^p, N] \leqslant [M, N]^p$. Let $m \in M$ and $n \in N$ and put $K = \langle m, [m, n]\rangle$. Then

$$[m^p, n] \equiv [m, n]^p \mod \gamma_2(K)^p \gamma_p(K) \tag{1}$$

by [10, VIII Lemma 1.1 (b)]. Since $[M, N] \leqslant Z_2(G)$, $[m, [m, n]] \in Z(G)$ and so

$$\gamma_2(K) = \langle [m, [m, n]]^K = \langle [m, [m, n]]\rangle \leqslant Z(G).$$

Then $\gamma_p(K) = 1$ since $p > 2$. Thus (1) reduces to

$$[m^p, n] = [m, n]^p ([m, [m, n]]^k)^p \in [M, N]^p \tag{2}$$

for a $k \geqslant 1$. Next

$$[m, n]^p = (m^{-1}m^n)^p \equiv m^{-p}(m^n)^p \gamma_2(K)^p \gamma_p(K)$$

by [10, VIII Lemma 1.1 (a)]. Here

$$\gamma_2(K) = \langle [m, [m, n]]\rangle \leqslant Z(G)$$

and so $\gamma_p(K) = 1$ as above. Therefore

$$[m, n]^p = m^{-p}(m^n)^p([m, [m, n]]^k)^p = [m^p, n][m^{-kp}, [m, n]] \in [M^p, N] \quad (3)$$

Now (2) and (3) gives

$$[M, N]^p = [M^p, N].$$

In the same way it follows that $[N, M]^p = [N^p, M]$. Therefore

$$[M, N]^p = [M^p, N] \geqslant [M, G, N] \text{ and } [N, M]^p = [N^p, M] \geqslant [N, G, M]$$

since M and N are powerfully embedded and hence it follows that

$$[M, N]^p \geqslant [M, N, G]$$

by the Three-Subgroup Lemma, which was to be shown.

Finally

$$\mho_1(MN) \geqslant \mho_1(M)\mho_1(N) \geqslant [M, G][N, G] = [MN, G]$$

and so MN is powerfully embedded. □

Corollary 2.5 *Let G be a powerful nilpotent p-group.*

(a) *The groups $\gamma_i(G)$, $G^{(i)}$, $\mho_i(G)$ and $\Phi(G)$ are powerfully embedded in G.*

(b) *If $G_{i+1} \leqslant H \leqslant G_i$ and $i \geqslant 2$, then H is powerful.*

Proof — (a) is trivial and (b) follows from

$$\begin{aligned} \mho_1(H) &\geqslant \mho_1(\gamma_{i+1}(G)) \geqslant [\gamma_{i+1}(G), G] = \gamma_{i+2}(G) \\ &\geqslant \gamma_{2i}(G) \geqslant [\gamma_i(G), \gamma_i(G)] \geqslant H' \end{aligned}$$

since $i \geqslant 2$. □

Lemma 2.6 *Let B be a nilpotent p-group which is powerful, where $p > 2$. If $B = \Omega_1(B)$, then B is abelian.*

Proof — First we show that $\exp(B) \leqslant p$. We use induction on $c(B)$ and we may assume that $c(B) > 1$. Then

$$\exp(B/\gamma_c(B)) \leqslant p$$

by the induction assumption since $B/\gamma_c(B)$ is powerful. This implies that

$$\mho_1(B) \leqslant \gamma_c(B).$$

Then also $B' \leqslant \gamma_c(B)$ since B is powerful and so $c=2$. Clearly then $\exp(B)=p$ by [8, Lemma 5.3.9 (i)], which completes the induction. Now since

$$B' \leqslant \mho_1(B) = 1$$

it follows that $B' = 1$ and so B is abelian. □

Corollary 2.7 *Let* G *be a locally finite* p*-group whose proper subgroups have finite exponent, were* $p \neq 2$*. If* $\Omega_k(M)$ *is nilpotent and powerful for every proper normal subgroup* M *of* G *and* $k \geqslant 1$*, then* G *cannot be perfect.*

Proof — Assume that G is perfect. Let $a \in G$ and put

$$M_a = \langle a^g : g \in G \rangle.$$

Then $M_a \neq G$ since G is a union of proper normal subgroups by [13, 12.1.6]. Also $M_a = \Omega_k(M_a)$, where $|a| = p^k$. Therefore M_a is nilpotent and powerful by the hypothesis. Clearly it follows from this that G is a Fitting group.

Now let

$$I = \{a \in G : |a| = p\}.$$

Let $a \in I$. Then $M_a \neq G$ since G is a Fitting group by the above paragraph. Thus M_a is nilpotent and powerful by the hypothesis since $M_a = \Omega_1(M_a)$. Therefore M_a is abelian by Lemma 2.6. Next let $b \in I$. Then also M_aM_b is abelian since $M_aM_b = \Omega_1(M_aM_b)$. Hence it follows that $\langle I \rangle = \Omega_1(G)$ is a normal abelian subgroup of G with $\exp(\Omega_1(G)) = p$. Now it follows from this by an easy induction that

$$\exp(\Omega_k(G)) = p^k \quad \text{and} \quad \Omega_k(G)/\Omega_{k-1}(G)$$

is abelian for every $k \geqslant 1$, if $\Omega_k(G) \neq G$ (1).

First suppose that $\exp(G) = p^n$ for an $n \geqslant 1$. Then $\exp(\Omega_{n-1}(G)) \leqslant p^{n-1}$ and so $\exp(G/\Omega_{n-1}(G)) = p$ by (1). Put

$$\overline{G} = G/\Omega_{n-1}(G).$$

Then $\overline{G}$ is perfect and satisfies the hypothesis of the theorem. Let $\overline{a} \in \overline{G}$ and put $\overline{M} = \langle \overline{a}^{\overline{G}} \rangle$. Then $\overline{M} < \overline{G}$ since G is a Fitting group. Thus $\overline{M}$ is nilpotent and powerful by the hypothesis since $\overline{M} = \Omega_1(\overline{M})$. Clearly then $\overline{M}$ is abelian by Lemma 2.6 and so it follows as in the preceding paragraph that $\overline{G}$ is abelian, which is impossible since $\overline{G}$ is perfect. Therefore $\exp(G) = \infty$ and so G is a minimal non-(finite exponent) p-group

Now G is generated by a subset of finite exponent by [1,Theorem 1.1] since $G = G'$. Clearly then without loss of generality we may suppose that G is generated by a subset X of exponent p. Thus now if we consider X instead of I above, then it follows that $\langle X \rangle$ is abelian since G is a Fitting group. But then G is abelian since $G = \langle X \rangle$, which is another contradiction. Therefore the assumption is false and so G cannot be perfect. □

3 Proof of Theorem 1.1

The following lemma is a different version of [5, Lemma 2.5] (see remark below).

Lemma 3.1 *Let* G *be a perfect locally finite* p*-group satisfying the normalizer condition. Let* (w, V) *be a* Λ*-pair for* G *with* $W^*(w, V) = 1$ *and let* $E \in E^*(w, V)$ *such that* $N_G(E) = N_G(E')$*. Let* B *be a normal metabelian subgroup of* G *and* A *be a normal abelian subgroup of* G *contained in* B *such that* B/A *is elementary abelian,* $Z(G) \leqslant Z(B) \leqslant A$ *and* $A \leqslant N_G(E)$*. Put*

$$N = N_G(E),\ R = N \cap B,\ D = R \cap E,\ H = BN \quad \text{and} \quad D_B = \mathrm{Core}_H(D).$$

If N/E *is (locally) cyclic, then the following hold.*

(a) $C_{B/D_B}(R/D_B) = R/D_B$*. Hence* $Z(B/D_B) \leqslant N/D_B$ *and so is (locally) cyclic.*

(b) *If* AD/D *has finite exponent, then* $A = \langle a \rangle (A \cap D)$, $\langle a \rangle \cap Z(G) \neq 1$, $\langle a \rangle \cap D = 1$, $|a| = \exp(A)$, $R = \langle b \rangle D$, $\langle b \rangle \cap D) = 1$ *and* $|b| \leqslant p|a|$*. Furthermore* $|\bar{b}| > |Z(G)|$ *and if* $c(B) \leqslant p$*, then*

$$\mho_1(R/D_B) \leqslant Z(B/D_B) \text{ and } \langle bD_B \rangle \cap Z(G)D_B/D_B \neq 1.$$

(c) *Suppose* R/D_B *is infinite and* B *is nilpotent of class* c*. Then* R/D *is Chernikov and* $R/D_B = (R/D_B)^o \times D/D_B$*, where* $(R/D_B)^o$ *is the infinite locally cyclic subgroup of* R/D_B *and* $\exp(D/D_B)$ *is finite.*

Proof — Clearly B is not contained in N since B is not abelian, N/E is abelian and $\mathrm{Core}_G(E) = 1$ by the hypothesis. Thus $N_B(N) \setminus N \neq \emptyset$ by the hypothesis.

(a) Since $R \lhd N$ it follows that $D \lhd N$. Also $[N, R] \leqslant R \cap E = D$ since N/E is (locally) cyclic by the hypothesis and this implies that $R/D \leqslant Z(N/D)$. Put $\overline{H} = H/D_B$. Clearly $Z(G)$ is finite by [4, Lemma 2.5]. Let $z \in Z(G)$ with $|z| = p$. Then $1 \neq \bar{z} \in \overline{R} \setminus \overline{D}$ since $\mathrm{Core}_G(E) = 1$. Also $\overline{R}$ is abelian since $\overline{R}' \leqslant \overline{E}$ and $\overline{R}' \lhd \overline{H}$. Next assume if possible that $C_{\overline{B}}(\overline{R}) \nleqslant \overline{N}$. Then there exists a $t \in C_{\overline{B}}(\overline{R}) \setminus N$ so that $N^t = N$ and $[\bar{t}, \overline{R}] = 1$ by the normalizer condition. Then since $\bar{t}$ centralizes $[\bar{t}, \overline{N}]$, which is contained in $\overline{R}$, it follows that

$$1 = [\bar{t}^p, \overline{N}] = [\bar{t}, \overline{N}]^p$$

and so $[\bar{t}, \overline{N}]$ is a subgroup of order $\leqslant p$ of $\overline{R}$. Clearly then $[\bar{t}, \overline{N}]\overline{E}/\overline{E} \leqslant \langle \bar{z} \rangle \overline{E}/\overline{E}$ since $\overline{RE}/\overline{R}$ is (locally) cyclic. This implies that $[t, E] \leqslant \langle z \rangle E$ and then t normalizes $\langle z \rangle E$. Then since t normalizes $(\langle z \rangle E)' = E'$ it follows that $t \in N$ by the hypothesis, which is a contradiction. Therefore $C_{\overline{B}}(\overline{R}) = \overline{R}$. In particular then $Z(\overline{B}) \leqslant \overline{N}$ and so $Z(\overline{B})$ is (locally) cyclic by [4, Lemma 2.2].

(b) Now suppose that $\exp(A/(A \cap D))$ is finite. Also $Z(G)$ is finite and $Z(G) \leqslant A$. Clearly $A \nleqslant E$ since $\mathrm{Core}_G(E) = 1$. In particular $Z(G) \cap E = 1$. Also AE/E is cyclic by [4, Lemma 2.2]. Let $z \in Z(G)$ with $|z| = p$. Then z has finite height, say h in A, since $\langle z \rangle \cap E = 1$ and $\exp(A/(A \cap D))$ is finite. Therefore there exists an $a \in A$ so that

$$a^{p^h} = z \quad \text{and} \quad A = \langle a \rangle \times A_1$$

for a subgroup A_1 of A by [11, p.180, Lemma] or [13, 4.3.8]. Then also $\langle a \rangle \cap D = 1$ since $\langle z \rangle \cap E = 1$. If $|a| < \exp(A)$, then

$$A^* = \langle y^{|a|} : y \in A \rangle$$

is a non-trivial normal subgroup of G with $Z(G) \cap A^* = 1$. Then A^* contains a normal subgroup $L \neq 1$ of G so that $w \notin VL$ by [3, Lemma 3.5]. But then $L \leqslant E_1$ for an $E_1 \in E^*(w, V)$, which contradicts the hypothesis

$$W^*(w, V) = 1$$

(this also holds if (w, V) is a distinguished pair by [5, Lemma 2.1 (d)]). Therefore $|a| = \exp(A)$ and then $A = \langle a \rangle \times (A \cap D)$ since $A/(A \cap D)$ is cyclic and $\langle a \rangle \cap D = 1$. Also $\exp(B) \leqslant p\exp(A)$. Hence $R = \langle b \rangle D$ for a $b \in R$ since RE/E is cyclic and $|b| \leqslant p|a|$. Next we show that $\langle b \rangle \cap D = 1$. If $|bD| = |aD|$, then we may let $b = a$. Then $R = \langle a \rangle D$ and $\langle a \rangle \cap D = 1$. Thus suppose that $|bD| > |aD|$. Then $|bD| > |a|$ since $|aD| = |a|$, which implies that $|bD| = |b|$ since $\exp(B) \leqslant p|a|$. Clearly then $\langle b \rangle \cap D = 1$.

Clearly $|\overline{b}| > |Z(G)|$ by [4, Corollary 2.6]. Assume that $c(B) \leqslant p$. Then for every $\overline{r} \in \overline{R}$ and for every $\overline{y} \in \overline{B}$ it follows that

$$1 = [\overline{r}, \overline{y}]^p = [\overline{r}^p, \overline{y}]$$

since $\exp([\overline{R}, \overline{B}]) = p$ by [5, Lemma 2.6] and $\overline{R}$ is abelian, which means that $\mho_1(\overline{R}) \leqslant Z(\overline{B})$.

(c) Suppose that $\overline{R}/\overline{D}$ is infinite and B is nilpotent of class c. Then $\overline{R}/\overline{D}$ is infinite locally cyclic by the hypothesis and so

$$\overline{R}/\overline{D} \leqslant Z(\overline{B}/\overline{D}).$$

Let $\overline{x} \in \overline{B}$ and put $\overline{L} = \langle \overline{x} \rangle \overline{R}$. Here $\overline{x}^p \in \overline{R}$, which is abelian and the subnormal index of $\overline{x}$ in $\overline{B}$ is $\leqslant c$. Therefore applying [2, Lemma 2.1] we get

$$\overline{R}^{p^{c-1}} \leqslant C_{\overline{R}}(\overline{x}).$$

Now since $x \in B$ is arbitrary it follows that

$$\overline{R}^{p^{c-1}} \leqslant Z(\overline{B}).$$

Furthermore

$$\overline{R}^{p^{c-1}} \cap \overline{E} = 1$$

by definition of D_B. Therefore $\overline{R}^{p^{c-1}}$ is infinite locally cyclic since $\overline{R}/\overline{R}^{p^{c-1}}$ has finite exponent. It follows that

$$\overline{R}^{p^{c-1}} = (\overline{R}^{p^{c-1}})^{o}$$

is the unique divisible abelian subgroup of $\overline{R}$. Now since $\overline{N} = (\overline{R}^{p^{c-1}})^{o}\overline{E}$, we have

$$\overline{R} = (\overline{R}^{p^{c-1}})^{o}(\overline{R} \cap \overline{E}) = (\overline{R}^{p^{c-1}})^{o}\overline{D} = (\overline{R}^{p^{c-1}})^{o} \times \overline{D}.$$

Also $\exp(\overline{D}) \leqslant p^{c-1}$. □

Lemma 3.2 *Let* G *be a perfect locally finite* p*-group satisfying the normalizer condition, where* $p > 2$*. Suppose that the notations and the hypothesis are those of Lemma 3.1 but here* B *is a normal nilpotent metabelian subgroup of* G *with* $c(B) \leqslant p$*. Then the following hold.*

(a) $\overline{D} \neq 1$.

(b) $\exp(\overline{D}) = p$.

(c) $\mho_1(\overline{R}) \leqslant Z(\overline{B})$.

(d) $\exp(B/Z(B)) \leqslant p^2$. *In particular* $\exp(B/A) = \exp(A/Z(B)) = p$.

Proof — Thus $D_B = \mathrm{Core}_H(D)$ and $\overline{H} = H/D_B$, where

$$D = R \cap E \quad \text{and} \quad R = B \cap N.$$

Also N/E is (locally) cyclic. By the hypothesis $B \nleqslant N$ since B is metabelian. If $c(\overline{B}) < p$, then $\overline{B}$ is abelian by [5, Lemma 2.7] since $p > 2$, which is impossible since $C_{\overline{B}}(\overline{R}) = \overline{R}$ by Lemma 3.1 (a). Therefore $c(\overline{B}) = p$. Furthermore $\exp([\overline{R}, \overline{t}]) \leqslant p$ by [5, Lemma 2.6].

(a) Assume if possible that $\overline{D} = 1$. First suppose that $\overline{R}$ is infinite. Then

$$\overline{R} = \overline{R}^{o} \times \overline{D} = \overline{R}^{o}$$

by Lemma 3.1 (c). Also $\overline{R}^{o} \leqslant Z(\overline{B})$ since $\overline{B}$ is nilpotent, which is impossible since $C_{\overline{B}}(\overline{R}) \leqslant \overline{R}$ and $\overline{B} \nleqslant \overline{R}$. Therefore $\overline{D} \neq 1$ in this case. Next suppose that $\overline{R}$ is finite. Then

$$\overline{R} = \langle \overline{b} \rangle \quad \text{and} \quad \exp(\overline{B}/\langle \overline{b} \rangle) = 3.$$

Let $|\overline{b}| = p^n$. Then $n > 1$ by Lemma 3.1 (b). Since $\overline{B}/C_{\overline{B}}(\overline{b})$ is cyclic by [8, Lemma 5.4.1 (iii)] and since $C_{\overline{B}}(\overline{b}) = \langle \overline{b} \rangle$, it follows that $|\overline{B}/\langle \overline{b} \rangle| \leqslant p$. This gives

$$[\overline{B}, \langle \overline{b} \rangle] \leqslant \langle \overline{b}^{p^{n-1}} \rangle$$

by [8, Corollary 5.4.2 (ii)]. But also $\overline{b}^p \leqslant Z(\overline{B})$ by Lemma 3.1 (b) and hence it follows that $\overline{B}' = [\overline{B}, \langle \overline{b} \rangle] \leqslant Z(\overline{B})$, which is impossible since $c(\overline{B}) = p > 2$. Therefore the assumption is false and so $\overline{D} \neq 1$.

(b) Let $d \in D$ and $t \in B$. Put

$$\overline{y} = \overline{d}\overline{d}^{\overline{t}} \ldots \overline{d}^{\overline{t}^{p-1}}.$$

Then $\overline{y} \in C_{\overline{B}}(\overline{t})$. Also it is easy o see that

$$\overline{y} = \prod_{i=1}^{p} [\overline{d},_{i-1} \overline{t}]^{\binom{p}{i}} = \overline{d}^p [\overline{d},_{p-1} \overline{t}]$$

since $\exp([\overline{R}, \overline{t}]) \leqslant p$. Moreover

$$[\overline{d},_{p-1} \overline{t}] \in Z(\overline{B})$$

since $c = p$. Therefore $\overline{d}^p \in C_{\overline{B}}(\overline{t})$. Since t is any element of B it follows that $\overline{d}^p \in Z(\overline{B})$ and then $\overline{d}^p = 1$ by definition of D_B. Therefore $\exp(\overline{D}) = p$.

(c) If $\exp(A/(A \cap D))$ is finite, then $\mho_1(\overline{R}) \leqslant Z(\overline{B})$ by Lemma 3.1 (b). So suppose that $A/(A \cap D)$ is infinite. Then $\overline{R} = \overline{R}^o \times \overline{D}$. Hence

$$\mho_1(\overline{R}) = \overline{R}^p = (\overline{R}^o)^p \times \overline{D}^p = \overline{R}^o \leqslant Z(\overline{B}).$$

(d) Since $\exp(B/R) = p$ and $\mho_1(\overline{R}) \leqslant Z(\overline{B})$ by (c), it follows that

$$\mho_2(\overline{B}) = \langle \overline{y}^{p^2} : \overline{y} \in \overline{B} \rangle \leqslant Z(\overline{B})$$

Put $K = \mho_2(B)$. Then $K \lhd G$. Now since $\mho_2(\overline{B}) = \overline{K}$, it follows that $[K, B] \leqslant D_B$. But since $D_B \leqslant E$ and $\mathrm{Core}_G(E) = 1$, this gives $[K, B] = 1$ and so $K \leqslant Z(B)$. Clearly then

$$\exp(B/Z(B)) \leqslant p^2.$$

Moreover as $\overline{A}^p \leqslant Z(\overline{B}$ by (c), it follows that $[\overline{A}^p, \overline{B}] = 1$ and so $A^p \leqslant Z(B)$. Thus $\exp(B/A) = \exp(A/Z(B)) = p$ since $c(\overline{B}) = p > 2$. □

Remark In Lemma 3.1 we let $H = BN$ and consider $\overline{H} = H/D_B$, where $D_B = \mathrm{Core}_H(D)$. In [5, Lemma 2.5] we let $T = \langle t\rangle R$, $H = TN$ and consider $\overline{H} = H/D_t$, where $t \in B \setminus N$, $N^t = N$ and $D_t = \mathrm{Core}_H(D)$. In the remaining part of this work both lemmas are used but no confusion arises. Note that $D_B \leqslant D_t$.

Lemma 3.3 *Let* G *be a perfect locally finite* p*-group, where* $p > 2$*. Let the notations and the hypothesis be those of* [5, Lemma 2.5]*. Let* $c(B) \leqslant p$ *and let* $\overline{t} \in \overline{B} \setminus \overline{N}$ *with* $\overline{N}^{\overline{t}} = \overline{N}$ *and put* $\overline{T} = \langle \overline{t}\rangle\overline{R}$*. Then* $(\langle\overline{t}\rangle\overline{R})' \cap \overline{D} \neq 1$.

Proof — If $c(\overline{T}) < p$, then $\overline{T}$ is abelian by [5, Lemma 2.7] since $\overline{R} \leqslant Z(\overline{N})$ which is impossible since

$$C_{\overline{T}}(\overline{R}) = \overline{R}$$

by [5, Lemma 2.5 (a)]. Therefore $c(\overline{T}) = p$. Since $\overline{T}' = [\overline{R}, \overline{t}]$ is elementary abelian by [5, Lemma 2.6], $\overline{T}'$ is contained in $\Omega_1(\overline{R})$. Also

$$\Omega_1(\overline{R}) = \langle \overline{z_1}, \overline{D}\rangle$$

by [5, Lemma 2.5 (a)], where

$$1 \neq z_1 \in \Omega_1(Z(G)),$$

since $\exp(D/D_B) = p$ by Lemma 3.2 (b) and $D_t \geqslant D_B$. In addition $\overline{z_1} \in \overline{T}'$ since $Z(\overline{T})$ is cyclic. So if $\overline{T}' \cap \overline{D} = 1$, then $\overline{T}' \leqslant \langle\overline{z_1}\rangle$ since $\overline{z_1} \in \overline{T}'$ but then $c(\overline{T}) = 2$ which is impossible since $c(\overline{T}) = p > 2$. Therefore $\overline{T}' \cap \overline{D} \neq 1$.□

Lemma 3.4 *Let* G *be a perfect locally finite* 3*-group. Let the notations and the hypothesis be those of* [5, Lemma 2.5]*. Let* $c(B) = 3$ *and let* $\overline{t} \in \overline{B} \setminus \overline{N}$ *with* $\overline{N}^{\overline{t}} = \overline{N}$ *and put* $\overline{T} = \langle\overline{t}\rangle\overline{R}$*. Then* $\mho_1(\overline{T})$ *is (locally) cyclic.*

Proof — By the hypothesis B is a normal nilpotent metabelian subgroup of class 3 of G and has a characteristic abelian subgroup A with $\exp(B/A) = 3$. Then $\mho_1(\overline{T}) \leqslant \overline{A} \leqslant \overline{R}$. Assume that $\mho_1(\overline{T})$ is not (locally) cyclic. Now it follows from by [5, Lemma 2.5] that

$$\overline{R} = \langle\overline{b}\rangle \times \overline{D} \text{ or } \overline{R} = \overline{R}^o \times \overline{D} \qquad (1)$$

according to whether $\overline{R}$ is finite or infinite, where $\overline{R}^o$ is infinite locally cyclic and contained in $Z(\overline{B})$ and $\exp(\overline{D}) = 3$ by Lemma 3.2 (b) since $D_t \geqslant D_B$. Also $\overline{z} \in \mho_1(\overline{T})$ since $Z(\overline{B})$ is (locally) cyclic, where $1 \neq z \in \Omega_1(Z(G))$. Therefore there exists a

$$1 \neq \overline{d} \in \mho_1(\overline{T}) \cap \overline{D}$$

since $\mho_1(\overline{T})$ is not (locally) cyclic by the assumption and $\Omega_1(\overline{R}) = \langle \overline{z}, \overline{D} \rangle$. Also $[\overline{d}, \overline{t}] \neq 1$ by definition of D_t. We have $\exp(\overline{T}') = 3$ by [5, Lemma 2.6]. Hence $\exp(\gamma_3(\overline{T})) = 3$ and, since $Z(\overline{T})$ is (locally) cyclic by [5, Lemma 2.5], it follows that $\gamma_3(\overline{T}) = \langle \overline{z} \rangle$. Thus $c(\overline{T}/\langle \overline{z} \rangle) = 2$. This implies that $\overline{d}\langle \overline{z} \rangle = \overline{y}^3 \langle \overline{z} \rangle$ for a $\overline{y} \in \overline{T}$ and hence

$$\overline{d} = \overline{y}^3 \overline{s} \tag{2}$$

for an $\overline{s} \in \langle \overline{z} \rangle$ since $\overline{d} \in \mho_1(\overline{T})$. Here if $\overline{y} \in \overline{R}$, then $\overline{y}^3 \in Z(\overline{T})$ by Lemma 3.2 (c), which is impossible since $\overline{d} \notin Z(\overline{T})$. Therefore $\overline{y} \in \overline{T} \setminus \overline{N}$. Thus $\overline{y} = \overline{t}^i \overline{r}^j \overline{f}$ for an $\overline{f} \in \overline{D}$ and $i, j \geqslant 1$ by (1), where $\langle \overline{r} \rangle \cap \overline{D} = 1$. Since $c(\overline{T}/\langle \overline{z} \rangle) = 2$

$$\overline{y}^3 = (\overline{t}^i (\overline{r}^j \overline{f}))^3 = \overline{t}^{3i} \overline{r}^{3j} \overline{c} \tag{3}$$

for a $\overline{c} \in \langle \overline{z} \rangle$ since $\overline{R}$ is abelian. Clearly (3) implies that $[\overline{y}^3, \overline{t}] = 1$ but since $[\overline{d}, \overline{t}] \neq 1$, this is impossible by (2). Therefore $\mho_1(\overline{T}) \cap \overline{D} = 1$ and since $\overline{R}/\overline{D}$ is (locally) cyclic, $\mho_1(\overline{T})$ must be (locally) cyclic, which completes the proof. □

Proof of Theorem 1.1 — Let G be a Fitting p-group satisfying the normalizer condition and $p \neq 2$. Suppose that in each homomorphic image H of G every Λ-pair (w_H, V_H) has a maximal element satisfying the $(**)$-property and in addition if $p = 3$ and $W(w_H, V_H) = 1$, then the converse of (b) is satisfied. This means that every normal nilpotent metabelian 3-subgroup B_H of H with $\exp(B/Z(B)) \leqslant 9$ and of class 3 satisfies the following: if X is any subgroup of B_H with $Z(B) \leqslant X$ and $\exp(B/X)$ is elementary abelian, then X is powerful. Obviously then B and every homomorphic image of B is powerful. Assume that G is perfect. First we show the following. G has a homomorphic image H with the following property. H has a Λ-pair (w_H, V_H) satisfying $(**)$ and the condition $W^*(w_H, V_H) = 1$ such that every normal nilpotent subgroup of H which is abelian- by-elementary abelian is abelian. Assume that there exists no such H. For each homomorphic image X of G satisfying the above properties let $n(X)$ be the minimum of the classes of all the normal nilpotent abelian-by-elementary abelian subgroups of X which are not abelian. Among all the homomorphic images X of G having a Λ-pair (w_X, V_X), satisfying $(**)$ and the condition $W^*(w_X, V_X) = 1$ there is a homomorphic image H such that $n(H) \leqslant n(X)$ for all such X. Without loss of generality we may suppose that $H = G$. Thus G has a Λ-pair (w, V) such that $(**)$ and the condition $W^*(w, V) = 1$ are satisfied. Also $n(G)$ is minimal in the above sense and $n(G) > 1$ by the assumption. Let B be a normal nilpotent abelian-by-elementary abelian subgroup of G so that $c(B) = n(G)$. Let A be the largest normal abelian subgroup of G contained in B such that $\exp(B/A) = p$ and $B' \leqslant A$. By the hypothesis there exists an $E \in E^*(w, V)$ satisfying $(**)$. Put $N = N_G(E)$. Then N/E is (locally) cyclic by [4, Lemma 2.2] since $p \neq 2$. Also $A \leqslant N$ by [5, Lemma 2.4]. Furthermore $B \not\leqslant N$ as in [5, Lemma 2.7] since B is not abelian. Let $R = B \cap N$, $D = R \cap E$ as in Lemma 3.1.

If $c(B) < 3$, then B is abelian by [5, Lemma 2.7] since $p \neq 2$. Therefore $c(B) \geqslant 3$.

Assume first if possible that $[B', G] \leqslant \gamma_c(B)$. Then

$$[B, B, B, B] = 1$$

and so $c \leqslant 3$. Then $c = 3$ since $c \geqslant 3$ and now a second application of [5, Lemma 2.7] shows that if $p > 3$, then B is abelian and so we get a contradiction. Therefore we may suppose that $p = 3$.

By the hypothesis G satisfies the normalizer condition. By the assumption G has a Λ-pair (w, V) such that

$$W^*(w, V) = 1$$

and there exists an $E \in E^*(w, V)$ such that $N_G(E) = N_G(E')$. Furthermore $\exp(B/Z(B)) \leqslant 9$ by Lemma 3.2 (d) and B is powerful by the assumption.

Next there exists a $t \in B \setminus N$ with $N^t = N$. Put $T = \langle t \rangle R$, $H = TN$, $D_t = \mathrm{Core}_H(D)$ and $\overline{H} = H/D_t$. Then $C_{\overline{B}}(\overline{R}) = \overline{R}$, $\overline{R} \leqslant Z(\overline{N})$, $Z(\overline{T})$ is (locally) cyclic and

$$\overline{R} = \langle \overline{b} \rangle \times \overline{D} \text{ or } \overline{R} = \overline{R}^o \times \overline{D}$$

according as $\overline{R}/\overline{D}$ is finite or infinite by [5, Lemma 2.5]. Also $\overline{D}$ is elementary abelian by Lemma 3.2 (b).

Since $B' \leqslant A \leqslant R$, $Z(B) \leqslant A$ by [5, Lemma 2.4] and $R \leqslant T$, it follows that T is powerful by the assumption. Then also $\overline{T}$ is powerful and so $\overline{T}' \leqslant \mho_1(\overline{T})$. Thus $\overline{T}'$ is cyclic by Lemma 3.4. Furthermore $\overline{z} \in \overline{T}'$ since $Z(\overline{T})$ is (locally) cyclic. But since

$$\overline{T}' \cap \overline{D} \neq 1$$

by Lemma 3.3 and $\overline{D} \cap Z(\overline{T}) = 1$, this gives a contradiction. Therefore the assumption is false and so $[B', G] \nleqslant \gamma_3(B)$. Therefore there exists an $s \in B'$ with $s\gamma_3(B) \notin Z(G/\gamma_3(B))$.

Now let $\overline{G} = G/\gamma_3(B)$. Then $c(\overline{B}) < c$. Let F be a finite subgroup of G so that $\overline{s} \notin \overline{F}$ (for example $\overline{F} = 1$). Then $(\overline{s}, \overline{F})$ is a Λ-pair for $\overline{G}$. Let $\overline{M}$ be a maximal element of $W^*(\overline{s}, \overline{F})$. If $\overline{M} = 1$, then $\overline{B}$ is abelian by the induction hypothesis since $c(\overline{B}) < c$. But then $c(B) = 2$, which is impossible. Therefore $\overline{M} \neq 1$. Now consider $\overline{G}/\overline{M}$. By [5, Lemma 2.1 (c)] there exists a finite subgroup U of G so that $\overline{s} \notin \overline{U}$, $\overline{F} \leqslant \overline{U} \nleqslant \overline{M}$ and there exists a $u \in U \setminus M$ so that $(\overline{su}\overline{M}, \overline{U}\overline{M}/\overline{M})$ is a Λ-pair for $\overline{G}/\overline{M}$. Also $(\overline{su}\overline{M}, \overline{U}\overline{M}/\overline{M})$ satisfies the hypothesis and

$$W^*(\overline{su}\overline{M}, \overline{U}\overline{M}/\overline{M}) = 1$$

But in this case $\overline{B}\overline{M}/\overline{M}$ is abelian by the induction hypothesis since $c(\overline{B}) < c$ and this implies that $\overline{B}' \leqslant \overline{M}$. However since $\overline{s} \in \overline{B}'$ but $\overline{s} \notin \overline{M}$ this gives another contradiction. Therefore the assumption is false and so it follows that B is abelian.

Thus we have shown that every normal nilpotent abelian-by-elementary abelian subgroup of G is in fact abelian. Now let A be a maximal normal abelian subgroup of G. Let $g \in G \setminus A$ and put $H = \langle g^G \rangle A$. Then H is nilpotent since G is a Fitting group. Put $B/A = \Omega_1(Z(H/A))$. Then B/A is elementary abelian and $B \neq A$ since H is nilpotent. But since B must be abelian by the first part of the proof this contradicts the maximality of A. Therefore the assumption is false and so G is not perfect. This completes the proof of the theorem. □

Proof of Corollary 1.2 — Let G be a Fitting p-group satisfying the normalizer condition whose proper subgroups are solvable, where $p \neq 2$. Assume that G is perfect. By [3, Theorem 1.4 (b)] we may suppose that G has no homomorphic images having $(*)$-triples for non-central elements. Then in every homomorphic image of G there exist distinguished pairs and dominant pairs by [3, Lemmas 3.1 and Lemma 4.1 (b)]. Assume that in every homomorphic image H of G every dominant pair (w_H, V_H) has a maximal element satisfying $(**)$ and in addition if $p = 3$ and $W(w_H, V_H) = 1$, then the converse of (b) is satisfied. This means that every normal nilpotent metabelian 3-subgroup B_H of H with $\exp(B/Z(B)) \leqslant 9$ and of class 3 satisfies the following: if X is any subgroup of B_H with $Z(B) \leqslant X$ and $\exp(B/X)$ is elementary abelian, then X is powerful. Thus every homomorphic image of B_H is powerful.

First we show that in every proper homomorphic of G every Λ-pair has a maximal element satisfying $(**)$. Thus let $H \neq 1$ be a homomorphic image of G and let (w_H, V_H) be a Λ-pair for H. Clearly without loss of generality we may let $H = G$ and let (w, V) be a Λ-pair for G. We must show that there exists an

$$E \in E^*(w, V)$$

satisfying $(**)$. Since $w \notin V$, applying [3, Lemma 3.1] to (w, V) we obtain a finite subgroup T of G containing V and excluding w so that (w, T) is a distinguished pair for G. Next applying [3, Lemma 4.1 (a)] to (w, T) we obtain a finite subgroup U of G containing T and excluding w so that (w, U) is a dominant pair for G. Also

$$E^*(w, U) \subseteq E^*(w, V)$$

since $V \leqslant U$ by [3, Lemma 3.2]. Now by the hypothesis there exists an

$$E \in E^*(w, U)$$

satisfying $(**)$. Since $E \in E^*(w, V)$, the assertion is verified. Thus we have shown that in every homomorphic image of G every Λ-pair has a maximal element satisfying $(**)$. But then G cannot be perfect if $p > 3$ by Theorem 1.1 (a), which is impossible since G is perfect by the assumption. Therefore $p = 3$ since $p \neq 2$.

In this case Theorem 1.1 (b) can be applied. This gives a homomorphic image H of G having a Λ-pair (w_H, V_H) with $W^*(w_H, V_H) = 1$ such that the following holds. H contains a normal nilpotent and metabelian subgroup B_H with such that $\exp(B/Z(B) \leqslant 9$, $c(B) = 3$ and there exists a normal subgroup K of B with the property that $Z(B) \leqslant K$ and B/K is elementary abelian such that K is not powerful. But this is also impossible since K must be powerful by the assumption. Therefore G cannot be perfect and so it must be solvable. □

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