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Finite Groups with H_σ-Permutably Embedded Subgroups

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Abstract

Let G be a finite group. Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$ and n an integer. We write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$, $\sigma(G) = \sigma(|G|)$. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G if every member of $\mathcal{H} \setminus \{1\}$ is a Hall σ_i-subgroup of G for some σ_i and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. A subgroup A of G is called: (i) a *σ-Hall subgroup* of G if $\sigma(|A|) \cap \sigma(|G : A|) = \emptyset$; (ii) *$\sigma$-permutable* in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$. We say that a subgroup A of G is *H_σ-permutably embedded* in G if A is a σ-Hall subgroup of some σ-permutable subgroup of G.

We describe the structure of G assuming that every subgroup of G is H_σ-permutably embedded in G.

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1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, n is an integer, $\mathbb{P}$ is the set of all primes, and if $\pi \subseteq \mathbb{P}$, then $\pi' = \mathbb{P} \setminus \pi$. The symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G. In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; Π is a subset of σ and $\Pi' = \sigma \setminus \Pi$.

Let $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$. Then we say that G is *σ-primary* [14] if G is a σ_i-group for some $\sigma_i \in \sigma$. A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G (see [15],[16]) if every member of $\mathcal{H} \setminus \{1\}$ is a Hall σ_i-subgroup of G for some σ_i and $\mathcal{H}$ contains exact one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. We say that G is *σ-full* if G possesses a complete Hall σ-set. Throughout this paper, G is always supposed to be a σ-full group.

Following [14], a subgroup A of G is called:

(i) a *σ-Hall subgroup* of G if $\sigma(|A|) \cap \sigma(|G : A|) = \emptyset$;

(ii) *σ-subnormal* in G if there is a subgroup chain

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_t = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, t$;

(iii) *σ-quasinormal* or *σ-permutable* in G if there is a complete Hall σ-set $\mathcal{H}$ such that $AH^x = H^xA$ for all $H \in \mathcal{H}$ and all $x \in G$.

In particular, A is called *S-quasinormal* or *S-permutable* in G provided $AP = PA$ for all Sylow subgroups P of G (see [1], [5]).

We say that a subgroup A of G is *H_σ-permutably embedded* in G if A is a σ-Hall subgroup of some σ-permutable subgroup of G. In the special case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, the definition of H_σ-permutably embedded subgroups is equivalent to the concept of Hall S-quasinormally embedded subgroups in [10].

Example For any σ, all σ-Hall subgroups and all σ-permutable subgroups of any group S are H_σ-permutably embedded in S. Now, let $p > q > r$ be primes. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{q, r\}$ and $\sigma_2 = \{q, r\}'$ and let C_p, C_q and C_{r^n} be cyclic groups with $|C_p| = p$, $|C_q| = q$ and $|C_{r^n}| = r^n$ $(n > 1)$. Let

$$H = C_q \wr C_{r^n} = K \rtimes C_{r^n},$$

where K is the base group of the regular wreath product H. Let

$$G = C_p \wr H = P \rtimes H = P \rtimes (K \rtimes C_{r^n}),$$

where P is the base group of the regular wreath product G. Then $C_G(P) \leqslant P$. Let C_r be a subgroup of C_{r^n} of order r. Then the subgroup $V = PC_r$ is σ-permutable in G and C_r is a σ-Hall subgroup of V. Hence C_r is H_σ-permutably embedded in G. Assume C_r is σ-permutable in G, then C_r is σ-subnormal in G (see Lemma 4 (1) below). Hence C_r is σ-subnormal in V by Lemma 5 (1) below. Therefore C_r is normal in V by Lemma 5 (2) below. Then $C_V(P) \leqslant C_r$, a contradiction.

Recall that G is *σ-nilpotent* (see [7]) if $G = H_1 \times \ldots \times H_t$ for some σ-primary groups $H_1, \ldots, H_t$. The *σ-nilpotent* residual $G^{\mathfrak{N}_\sigma}$ of G is the intersection of all normal subgroups N of G with σ-nilpotent quotient G/N, $G^{\mathfrak{N}}$ denotes the nilpotent residual of G. It is clear that every subgroup of a σ-nilpotent group G is σ-permutable and σ-subnormal in G.

Let $\mathfrak{F}$ be a class of groups. A subgroup H of G is said to be an $\mathfrak{F}$-covering subgroup of G (see [9], VI, Definition 7.8) if $H \in \mathfrak{F}$ and for every subgroup E of G such that $H \leqslant E$ and $E/N \in \mathfrak{F}$ it follows that $E = NH$. We say that a subgroup H of G is a σ-*Carter subgroup* of G if H is an $\mathfrak{N}_\sigma$-covering subgroup of G, where $\mathfrak{N}_\sigma$ is the class of all σ-nilpotent groups.

A group G is said to have a *Sylow tower* if G has a normal series

$$1 = G_0 < G_1 < \ldots < G_{t-1} < G_t = G,$$

where $|G_i/G_{i-1}|$ is the order of some Sylow subgroup of G for each $i \in \{1, \ldots, t\}$. A chief factor of G is said to be *σ-central* in G if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary; otherwise, H/K is called *σ-eccentric* in G (see [14]).

We say that G is a HσE-*group* if the following conditions hold:

(i) $G = D \rtimes M$, where $D = G^{\mathfrak{N}_\sigma}$ is a σ-Hall subgroup of G and $|\sigma(D)| = |\pi(D)|$.

(ii) D has a Sylow tower and every chief factor of G below D is σ-eccentric.

(iii) M acts irreducibly on every M-invariant Sylow subgroup of D.

Our main goal here is to prove the following theorem.

Theorem 1 *Any two of the following conditions are equivalent:*

(i) *Every subgroup of* G *is* H_σ*-permutably embedded in* G*.*

(ii) $G = D \rtimes M$ *is a* HσE*-group, where* $D = G^{\mathfrak{N}_\sigma}$ *is a cyclic group of square-free order.*

(iii) $G = D \rtimes M$, *where* D *is a* σ*-Hall cyclic subgroup of* G *of square-free order with* $|\sigma(D)| = |\pi(D)|$ *and* M *is* σ*-Carter subgroup.*

Groups in which either every subgroup is a Hall S-quasinormally embedded subgroup or every subgroup is a Hall normally embedded subgroup were described in [10],[8], respectively. From Theorem 1 we get the following result in this trend.

Corollary 2 (see [13], Theorem 1) *Every subgroup of* G *is a Hall* S*-quasinormally embedded subgroup of* G *if and only if* $G = D \rtimes M$*, where* $D = G^{\mathfrak{N}}$ *is a cyclic Hall subgroup of* G *of square-free order and* M *is a Carter subgroup of* G*.*

Recall also that a subgroup H of G is said to be a *Hall normally embedded subgroup* of G (see [8]) if H is a Hall subgroup of the normal closure H^G of H in G. From Corollary 2 we also get the following known result.

Corollary 3 (see [11]) *Every subgroup of* G *is a Hall normally embedded subgroup of* G *if and only if* $G = D \rtimes M$*, where* $D = G^{\mathfrak{N}}$ *is a cyclic Hall subgroup of* G *of square-free order and* M *is a Degekind group.*

2 Basic lemmas

An integer n is called a Π*-number* if $\sigma(n) \subseteq \Pi$. A subgroup H of G is called a *Hall* Π*-subgroup* of G [14] if $|H|$ is a Π-number and $|G : H|$ is a Π'-number. A group G is said to be σ*-soluble* [14] if every chief factor of G is σ-primary.

Lemma 4 (see [14], Lemma 2.8 and Theorems B and C) *Let* A*,* K *and* N *be subgroups of* G*, where* A *is* σ*-permutable in* G *and* N *is normal in* G*.*

(1) A *is* σ*-subnormal in* G*.*

(2) *If* $N \leqslant K$, K/N *is* σ*-permutable in* G/N *and* G *is* σ*-soluble, then* K *is* σ*-permutable in* G*.*

Lemma 5 (see [14], Lemma 2.6) *Let* A*,* K *and* N *be subgroups of* G*, where* A *is* σ*-subnormal in* G *and* N *is normal in* G*.*

(1) $A \cap K$ *is* σ*-subnormal in* K*.*

(2) *If* A *is a* σ-Hall subgroup *of* G, *then* A *is normal in* G.

(3) *If* $H \neq 1$ *is a Hall* Π*-subgroup of* G *and* A *is not a* Π'*-group, then* $A \cap H \neq 1$ *is a Hall* Π*-subgroup of* A.

Lemma 6 *Let* H *be a normal subgroup of* G. *If* $H/H \cap \Phi(G)$ *is a* Π*-group, then* H *has a Hall* Π*-subgroup, say* E, *and* E *is normal in* G. *Hence, if* $H/H \cap \Phi(G)$ *is* σ*-nilpotent, then* H *is* σ*-nilpotent.*

Proof — Let $D = O_{\Pi'}(H)$. Then, since $H \cap \Phi(G)$ is nilpotent, D is a Hall Π'-subgroup of H. Hence by the Schur-Zassenhaus theorem, H has a Hall Π-subgroup, say E. It is clear that H is π'-soluble where $\pi' = \cup_{\sigma_i \in \Pi'} \sigma_i$, so any two Hall Π-subgroups of H are conjugate. By the Frattini argument,

$$G = HN_G(E) = (E(H \cap \Phi(G)))N_G(E) = N_G(E).$$

Therefore E is normal in G. □

Lemma 7 *If every chief factor of* G *below* $D = G^{\mathfrak{N}_\sigma}$ *is cyclic, then* D *is nilpotent.*

Proof — Assume that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G. Then from the G-isomorphism $D/D \cap R \simeq DR/R = (G/R)^{\mathfrak{N}_\sigma}$ we know that every chief factor of G/R below DR/R is cyclic, so the choice of G implies that $D/D \cap R \simeq DR/R$ is nilpotent. Hence $R \leqslant D$ and R is the unique minimal normal subgroup of G. In view of Lemma 6, $R \nleqslant \Phi(G)$ and so $R = C_R(R)$ by [3], Chapter A, Theorem 15.2. But by hypothesis, $|R|$ is a prime, hence $G/R = G/C_G(R)$ is cyclic, so G is supersoluble and so $G^{\mathfrak{N}_\sigma}$ is nilpotent since $G^{\mathfrak{N}_\sigma} \leqslant G^{\mathfrak{N}}$. □

The following lemma is evident.

Lemma 8 *The class of all* σ*-soluble groups is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the* σ*-soluble group by a* σ*-soluble group is a* σ*-soluble group as well.*

Let A, B and R be subgroups of G. Then A is said to R-*permute* with B [6] if for some $x \in R$ we have $AB^x = B^xA$. If G has a complete Hall σ-set $\mathcal{H} = \{1, H_1, \ldots, H_t\}$ such that $H_iH_j = H_jH_i$ for all i, j, then we say that $\{H_1, \ldots, H_t\}$ is a σ-*basis* of G.

Lemma 9 (see [15], Theorems A and B) *Assume that* G *is* σ*-soluble.*

(i) G *has a* σ*-basis* $\{H_1, \dots, H_t\}$ *such that for each* $i \neq j$ *every Sylow subgroup of* H_i G*-permutes with every Sylow subgroup of* H_j.

(ii) *For any* Π*, the following hold:* G *has a Hall* Π*-subgroup* E*, every* Π*-subgroup of* G *is contained in some conjugate of* E *and* E G*-permutes with every Sylow subgroup of* G.

Lemma 10 *Let* H*,* E *and* R *be subgroups of* G*. Suppose that* H *is* H_σ*-permutably embedded in* G *and* R *is normal in* G*.*

(1) *If* $H \leqslant E$*, then* H *is* H_σ*-permutably embedded in* E*.*

(2) HR/R *is* H_σ*-permutably embedded in* G/R*.*

(3) *If* $|G : H|$ *is* σ*-primary, then* H *is either a* σ*-Hall subgroup of* G *or* σ*-permutable in* G*.*

Proof — Let V be a σ-permutable subgroup of G such that H is a σ-Hall subgroup of V.

(1) Since H is a σ-Hall subgroup of V and $V \cap E$ is σ-permutable in E, H is a σ-Hall subgroup of $V \cap E$. Hence H is H_σ-permutably embedded in E.

(2) Let H be a π-group. Since $|V : H|$ is a π'-number,

$$|VR : HR| = |V : H|/|V \cap R : H \cap R|$$

is a π'-number. Hence, HR/R is a σ-Hall subgroup of VR/R and, therefore, HR/R is H_σ-permutably embedded in G/R.

(3) Assume that H is not σ-permutable in G. Then $H < V$. By hypothesis, $|G : H|$ is σ-primary, say $|G : H|$ is a σ_i-number. Then $|V : H|$ is a σ_i-number. But H is a σ-Hall subgroup of V. Hence H is a σ-Hall subgroup of G. □

Lemma 11 *Let* H *be a* σ*-subnormal subgroup of a* σ*-soluble group* G*. If* $|G : H|$ *is a* σ_i*-number and* B *is a* σ_i*-complement of* H*, then* $G = HN_G(B)$*.*

Proof — Assume that this lemma is false and let G be a counterexample of minimal order. Then $H < G$, so G has a proper subgroup M such that $H \leqslant M$, $|G : M_G|$ is a σ_i-number and H is σ-subnormal in M. The choice of G implies that $M = HN_M(B)$. On the other hand, clearly that B is a σ_i-complement of M_G. Since G is σ-soluble, Lemma 9 and the Frattini argument imply that

$$G = M_G N_G(B) = MN_G(B) = HN_M(B)N_G(B) = HN_G(B).$$

The statement is proved. □

The following lemma is well-known (see for example [12], Lemma 3.29, or [4], 1.10.10).

Lemma 12 *Let H/K be an abelian chief factor of G and V a maximal subgroup of G such that $K \leqslant V$ and $HV = G$. Then G/V_G is isomorphic to $(H/K) \rtimes (G/C_G(H/K))$.*

Recall that the intersection of all such S-quasinormal subgroups of G which contain a subgroup H of G is called the *S-quasinormal closure of* H *in* G and denoted by H^{sG} (see [11]).

Lemma 13 *If H is a Hall normally embedded subgroup of G, then H is a Hall S-quasinormally embedded subgroup of G.*

Proof — Since every normal subgroup of G is a S-quasinormal subgroup of G, $H^{sG} \leqslant H^G$. Moreover, H is a Hall subgroup of H^G by hypothesis, so H is a Hall subgroup of H^{sG}. □

3 Proofs of the results

Proof of Theorem 1 — (i) ⇒ (ii) Assume that this is false and let G be a counterexample of minimal order. Moreover, $D = G^{\mathfrak{N}_\sigma} \neq 1$, so $|\sigma(G)| > 1$.

(1) *Condition (ii) is true on every proper section H/K of G, that is, $K \neq 1$ or $H \neq G$.*

This directly follows from Lemma 10 and the choice of G.

(2) *D is a cyclic group of square-free order.*

Let $p \in \sigma_i \cap \pi(D)$ and let P be a Sylow p-subgroup of D. Since G possesses a σ-permutable subgroup E such that $|E| = |G|_{\sigma_i'}p$. Lemma 4 (1) implies that E is σ-subnormal in G, so Lemma 5 (3) shows that G/E_G is a σ_i-group. Hence $D \leqslant E_G \leqslant E$, so $|P| = p$. Therefore G is supersoluble by [9], Kapitel IV, Satz 2.9, and so every chief factor of G below D is cyclic. Hence D is nilpotent by Lemma 7, so D is cyclic of square-free order.

(3) G *is* σ*-soluble.*

In view of Claim (1) and Lemma 8, it is enough to show that G is not simple. Assume that this is false. Then 1 is the only proper σ-permutable subgroup of G since $|\sigma(G)| > 1$. Hence every subgroup of G is a σ-Hall subgroup of G. Therefore for a Sylow p-subgroup P of G, where p is the smallest prime divisor of $|G|$, we have $|P| = p$ and so $|G| = p$ by [9], Kapitel IV, Satz 2.8. This contradiction shows that we have (3).

(4) *If* $|G : H|$ *is a* σ_i*-number and* H *is not a* σ*-Hall subgroup of* G*, then* H *is* σ*-permutable in* G *and a* σ_i*-complement* E *of* H *is normal in* G*.*

This follows from Lemmas 10(3) and 11.

(5) D *is a Hall subgroup of* G*. Hence* D *has a complement* M *in* G*.*

Suppose that this is false and let P be a Sylow p-subgroup of D such that $1 < P < G_p$, where $G_p \in \mathrm{Syl}_p(G)$. We can assume without loss of generality that $G_p \leqslant H_1$. Let R be a minimal normal subgroup of G contained in D.

Since D is soluble by Claim (2), R is a q-group for some prime q. Moreover, $D/R = (G/R)^{\mathfrak{N}_\sigma}$ is a Hall subgroup of G/R by Claim (1) and Proposition 2.2.8 in [2]. Suppose that $PR/R \neq 1$. Then PR/R belongs to $\mathrm{Syl}_p(G/R)$. If $q \neq p$, then $P \in \mathrm{Syl}_p(G)$. This contradicts the fact that $P < G_p$. Hence $q = p$, so $R \leqslant P$ and therefore P/R is a Sylow p-subgroup of G/R. It follows that $P \in \mathrm{Syl}_p(G)$. This contradiction shows that $PR/R = 1$, which implies that $R = P$ is a Sylow p-subgroup of D. Therefore R is the unique minimal normal subgroup of G contained in D. It is also clear that a p-complement of D is a Hall subgroup of G.

Now we show that $R \nleqslant \Phi(G)$. Indeed, assume that $R \leqslant \Phi(G)$. Then $D \neq R$ by Lemma 6 since $D = G^{\mathfrak{N}_\sigma}$. On the other hand, D/R is a p'-group. Hence $O_{p'}(D) \neq 1$ by Lemma 6. But $O_{p'}(D)$ is characteristic in D and so it is normal G. Therefore G has a minimal normal subgroup L such that $L \neq R$ and $L \leqslant D$. This contradiction shows that $R \nleqslant \Phi(G)$.

Let S be a maximal subgroup of the group G such that $RS = G$. Then $|G : S|$ is a p-number. Hence, since R is not a Sylow p-subgroup of G, p divides $|S|$. Then S is not a Hall subgroup of G and so S is not a σ-Hall subgroup of G. Therefore S is σ-permutable in G by Claim (4) and so G/S_G is a σ_i-group, which implies that

$$R \leqslant D \leqslant S_G \leqslant S$$

and and so $G = RS = S$. This contradiction completes the proof of (5).

(6) *If* $M \leqslant E < G$, *then* E *is not* σ*-permutable in* G *and so* E *a* σ*-Hall subgroup of* G.

Assume that E is σ-permutable in G. Then E is σ-subnormal in G by Lemma 4 (1). Then there is a subgroup chain

$$E = E_0 \leqslant E_1 \leqslant \ldots \leqslant E_r = G$$

such that either E_{i-1} is normal in E_i or $E_i/(E_{i-1})_{E_i}$ is σ-primary for all $i = 1, \ldots, r$. Let $V = E_{r-1}$. We can assume without loss of generality that $V \neq G$. Therefore, since G is σ-soluble by Claim (2), for some σ-primary chief factor G/W of G we have $E \leqslant V \leqslant W$. Also we have $D \leqslant W$ and so $G = DE \leqslant W$, a contradiction. Hence E is not σ-permutable in G.

By hypothesis, G has a σ-permutable subgroup S such that E is a σ-Hall subgroup of S. But then $S = G$, by the above argument, so E is a σ-Hall subgroup of G. In particular, M is a σ-Hall subgroup of G and so D is a σ-Hall subgroup of G.

(7) D *is soluble*, $|\sigma(D)| = |\pi(D)|$ *and* M *acts irreducibly on every* M*-invariant Sylow subgroup of* D.

Let $p \in \sigma_i \in \sigma(D)$. Lemma 9 and Claims (3) and (5) imply that for some Sylow p-subgroup P of G we have $PM = MP$. Moreover, MP is a σ-Hall subgroup of G by Claim (6). Hence $|\sigma_i \cap \pi(G)| = 1$ for all i such that $\sigma_i \cap \pi(D) \neq \emptyset$ and so $|\sigma(D)| = |\pi(D)|$. Therefore, since D is soluble by Claim (2), M acts irreducibly on every M-invariant Sylow subgroup of D by Claim (6).

(8) D *possesses a Sylow tower.*

Let R be a minimal normal subgroup of G contained in D. Then R is a p-group for some prime p by Claim (7). Then $R \leqslant P$, where P is a Sylow p-subgroup of D. But M acts irreducible on P by Claim (7), so $R = P$ and D/R possesses a Sylow tower by Claim (1). Hence D possesses a Sylow tower.

(9) *Every chief factor of* G *below* D *is* σ*-eccentric.*

Let H/K be a chief factor of G below D. Then H/K is a p-group for some prime p since D is soluble by Claim (7). By the Frattini argument, there exist a Sylow p-subgroup P and a p-complement E

of D such that $M \leqslant N_G(P)$ and $M \leqslant N_G(E)$. Then $M \leqslant N_G(P \cap K)$ and $M \leqslant N_G(P \cap H)$. Hence $P \cap K = 1$ and $P \cap H = P$ by Claim (7), so $H = K \rtimes P$. Let $V = EM$. Then $K \leqslant V$ and $HV = G$, so V is a maximal subgroup of G. Hence

$$G/V_G \simeq (H/K) \rtimes G/C_G(H/K)$$

by Lemma 12. Therefore, if H/K is σ-central in G, then $D \leqslant V_G$, which is impossible since evidently p does not divide $|V|$. Thus we have (9).

From Claims (5)–(9) it follows that G is a HσE-group. Hence (i) implies (ii).

(ii) ⇒ (iii) It is enough to show that M is a σ-Carter subgroup of G. Let R be a minimal normal subgroup of G contained in D and E a subgroup of G containing M. We need to show that $E = E^{\mathfrak{N}_\sigma}M$. The choice of G implies that RM/R is a σ-Carter subgroup of G/R, so

$$ER/R = (ER/R)^{\mathfrak{N}_\sigma}(RM/R).$$

Hence $ER = E^{\mathfrak{N}_\sigma}MR$ since $(ER/R)^{\mathfrak{N}_\sigma} = E^{\mathfrak{N}_\sigma}R/R$. Moreover, R is a p-group for some prime p and R, E and $E^{\mathfrak{N}_\sigma}M$ are σ-Hall subgroups of G by hypothesis. Therefore, if $R \nleqslant E$, then E and $E^{\mathfrak{N}_\sigma}M$ are Hall p'-subgroups of $ER = E^{\mathfrak{N}_\sigma}MR$, so $E = E^{\mathfrak{N}_\sigma}M$. Finally, assume that $R \leqslant E$ but $R \nleqslant E^{\mathfrak{N}_\sigma}M$. Then $R \cap E^{\mathfrak{N}_\sigma} = 1$. On the other hand, since $DE/D \simeq E/D \cap E$ is σ-nilpotent, $E^{\mathfrak{N}_\sigma} \leqslant D$ and so $M \cap E^{\mathfrak{N}_\sigma} = 1$. Therefore

$$E^{\mathfrak{N}_\sigma} \cap RM = (E^{\mathfrak{N}_\sigma} \cap R)(E^{\mathfrak{N}_\sigma} \cap M) = 1.$$

Then $E/E^{\mathfrak{N}_\sigma} = E^{\mathfrak{N}_\sigma}MR/E^{\mathfrak{N}_\sigma} \simeq MR$ is σ-nilpotent. Hence $M \leqslant C_G(R)$. Suppose that $C_G(R) < G$ and let $C_G(R) \leqslant W < G$, where G/W is a chief factor of G. Since G is σ-soluble, G/W is σ-primary and so $D \leqslant W$. But then $G = DM \leqslant W < G$, a contradiction. Therefore $C_G(R) = G$, that is, $R \leqslant Z(G)$. Let V be a complement to R in D. Then V is a Hall normal subgroup of D, so it is characteristic in D. Hence V is normal in G and $G/V \simeq RM$ is σ-nilpotent, so $D \leqslant V < D$. This contradiction completes the proof of the implication (ii) ⇒ (iii).

(iii) ⇒ (i) Let A be any subgroup of G. Then DA is σ-permutable in G by Lemma 4 (2) since G is σ-soluble. On the other hand, since $|\sigma(D)| = |\pi(D)|$ and D is a cyclic σ-Hall subgroup of G of square-free order, A is a σ-Hall subgroup of DA. Hence A is H_σ-permutably embedded in G. Therefore the implication (iii) ⇒ (i) is true.

The theorem is proved. □

PROOF OF COROLLARY 3 — *Necessity* Let R be a Hall normally embedded subgroup of G. Then R is a Hall S-quasinormally embedded subgroup of G by Lemma 13, so in view of Corollary 2 and [1], Theorem 1.4, it is enough to show that G is a T-group. Let H be a subnormal subgroup of G. Then H is subnormal in H^G by [3], Chapter A, Theorem 14.8. Then, since H is a Hall subgroup of H^G by hypothesis, H is characteristic in H^G. Hence H is a normal subgroup of G, so G is a T-group.

Sufficiency Let H be a subgroup of G. Let $D_1 = H \cap D$. Clearly, D_1 is a Hall subgroup of D and D_1 has a complement D_2 in D.

Since $M \simeq G/D$ is Dedekind, all subgroups of G/D are normal in G/D. Then DH/D is normal in G/D. Hence DH is normal in G. Therefore $H \leqslant H^G \leqslant DH$. It is clear also that H is a Hall subgroup of DH, therefore H is a Hall subgroup of H^G. Hence H is Hall normally embedded in G.

The corollary is proved. □

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