Advances in Group Theory and Applications

4 (2017), pp. 131–138
ISSN: 2499-1287
DOI: 10.4399/97888255036929

"In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi"

Georg Cantor

ADV Perspectives in Group Theory

– an open space –

ADV – 4A

Let G denote an arbitrary group. If S is a subset of G, then we write

$$S^2 = \{xy \mid x, y \in S\}.$$

If G is an additive group, then we put

$$2S = \{x + y \mid x, y \in S\}.$$

A well-known problem in additive number theory is to find the precise structure of S, if S is a finite subset of G, and $|S^2| \leqslant \alpha|S| + \beta$, with α (the doubling coefficient) and $|\beta|$ small. Problems of this kind are called *inverse problems of small doubling type*. These problems have been first studied in the additive group of the integers. It is very easy to prove that, if S is a finite subset of the integers, $|S| = k$, then $|2S| \geqslant 2|S| - 1$, and $|2S| = 2|S| - 1$ if and only if there exist integers a, q such that

$$S = \{a, a + q, a + 2q, \ldots, a + (k - 1)q\}$$

i.e. S is an arithmetic progression of length k. Small doubling problems with doubling coefficient $\alpha = 3$ in the group of the integers have been studied by G.A. Freiman. He proved that if S is a finite set of integers with $k \geqslant 3$ elements and $|2S| \leqslant 3k-4$, then there exist integers a, q such that $q > 0$ and

$$S \subseteq \{a, a+q, a+2q, \ldots, a+(2k-4)q\}$$

(see [G.A. Freiman: "On the addition of finite sets", *Izv. Vyss. Ucebn. Zaved. Matematika* 6(13) (1959), 202–213]). Freiman obtained similar results if $|2S| \leqslant 3|S|-3$, or $|2S| \leqslant 3|S|-2$. It is quite natural to ask similar questions in any torsion-free group. J.H.B Kemperman showed that if S is a finite subset of any torsion-free group, then $|S^2| \geqslant 2|S|-1$ (see [J.H.B. Kemperman: "On complexes in a semigroup", *Indag. Mat.* 18 (1956), 247–254]), while G.A. Freiman and B.M. Schein proved that, if $S = k$, then $|S^2| = 2|S|-1$ if and only if

$$S = \{a, aq, \ldots, aq^{k-1}\}$$

i.e. S is a geometric progression, and either $aq = qa$ or $aqa^{-1} = q^{-1}$ (see [G.A. Freeman and B.M. Schein: "Interconnections between the structure theory of set addition and rewritability in groups", *Proc. Amer. Math. Soc.* 113 (1991), 899–910]). Therefore it is quite natural to ask the following question.

Question 1 *What is the structure of* S*, if* S *a finite subset of a torsion-free group* G, $|S| = k \geqslant 3$, *and* $|S^2| \leqslant 3|S|-4$*? Is* S *contained in a geometric progression of length at most* $2|S|-3$*?*

Question 2 *What is the structure of* S*, if* S *a finite subset of a torsion-free group* G, $|S| = k \geqslant 3$, *and* $|S^2| \leqslant 3|S|-4$*? Is* S *contained in a geometric progression of length at most* $2|S|-3$*?*

Small doubling problems have been studied in abelian groups by many authors, for example Y.O. Hamidoune, B. Green, M. Kneser, A.S. Lladó, A. Plagne, P.P. Palfy, I.Z. Ruzsa, O. Serra, Y.V. Stanchescu. It could be also interesting to know the answer to the following question.

Question 3 *What is the structure of* $\langle S \rangle$*, if* S *is a finite subset of a torsion-free group and* $|S^2| \leqslant 3|S| - \beta$*, where* $\beta = 2, 3, 4$*?*

We answered to all these questions, if G is an *orderable group*, in a series of papers with G.A. Freiman, Y.V. Stanchescu, A. Plagne and D.J.S. Robinson (see, for example, [Freiman, Herzog, Longobardi, Maj, Plagne, Robinson and Stanchescu: "On the structure of subsets of an orderable group with some small doubling properties", *J. Algebra* 445 (2016), 307-326; Freiman, Herzog, Longobardi, Maj, Plagne and Stanchescu: "Small doubling in ordered groups: generators and structures", *Groups Geom. Dyn.* 11 (2017), 585–612]).

Marcel Herzog
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ADV – 4B

In [A. Caranti and F. Dalla Volta: "Groups that have the same holomorph as a finite perfect group", *ArXiv* (2016); 1612.03573] examples are constructed of a finite group Q (actually, an infinite family of groups) which is *perfect* and centrally indecomposable, such that

1. $Z(Q)$ is not elementary abelian (in particular, $Z(Q)$ is non-trivial), and
2. the automorphism group $\mathrm{Aut}(Q)$ acts trivially on $Z(Q)$.

Question *Does there exist a finite quasisimple group with these two properties?*

Andrea Caranti

ADV – 4C

Recall that a left brace $(B, +, \cdot)$ is a set B with two operations $+$ and $\cdot$ such that $(B, +)$ is an abelian group, $(B, \cdot)$ is a group and the following condition is satisfied for all $a, b, c \in B$:

$$a \cdot (b + c) + a = a \cdot b + a \cdot c.$$

Algebraic structures of this type yield an important tool in the study of set theoretic solutions of the Yang-Baxter equation (more precisely, they correspond to the so called non-degenerate involutive solutions; see [D. Bachiller, F. Cedó and E. Jespers: "Solutions of the Yang-Baxter equation associated with a left brace", *J. Algebra* 463 (2016), 80–102]), but they also show up in a variety of other mathematical contexts. The main challenging problem, perhaps too difficult at the moment, is to describe all finite left braces. It is known that, in every finite left brace B, $(B, \cdot)$ is a solvable group (this is an easy consequence of Theorem 2.15 of [P. Etingof, T. Schedler and A. Soloviev: "Set-theoretical solutions to the quantum Yang-Baxter equation", *Duke Math. J.* 100 (1999), 169–209], see also [W. Rump: "Braces, radical rings, and the quantum Yang-Baxter equation", *J. Algebra* 307 (2007), 153–170]). However, there exists a finite nilpotent group (in fact, a p-group for some prime p) of nilpotency class 9 which is not isomorphic to the multiplicative group of any left brace (see [D. Bachiller: "Counterexample to a conjecture about braces", *J. Algebra* 453 (2016), 160–176]). A finite group isomorphic to the multiplicative group of a left brace is called an involutive Yang-Baxter (IYB, for short) group. Another important problem is to describe the class of IYB groups. This also seems too difficult in full generality. But we can begin with the following question.

Question 1 *Find the greatest positive integer n such that every finite nilpotent group of class $< n$ is an IYB group. Is $n > 3$?*

It is known that every finite nilpotent group of class 2 is the circle group of a nilpotent ring of index 3 (Theorem 2 of [J.C. Ault and J.F. Watters: "Circle groups of nilpotent rings", *Amer. Math. Monthly* 80 (1973), 48–52]). Thus every finite nilpotent group of class 2 is an IYB group. It is not known whether every finite nilpotent group of class 3 is an IYB group. Although, if G is a nilpotent group with centre $Z(G)$ and with a presentation of the following form:

$$G = \langle x_1, \dots, x_r \mid [x_k, [x_j, x_i]] \in Z(G),\ [x_k, [x_j, x_i]]^{n_{k,j,i}} = [x_j, x_i]^{n_{j,i}} = 1,$$
$$x_i^{n_i} = 1,\ 1 \leqslant i, j, k \leqslant r\rangle,$$

for some $r > 1$ and non-negative odd integers $n_{k,j,i}, n_{j,i}, n_i$, then G is an IYB group (see [F. Cedó, E. Jespers and J. Okniński: "Nilpotent groups of class three and braces", *Publ. Mat.* 60 (2016), 55–79]).

Question 2 *Let G be a group with a presentation of the above form with $r > 1$ and $n_{k,j,i}, n_{j,i}, n_i$ powers of 2. Is G an IYB group?*

A natural intermediate step towards a classification of all finite left braces is the following question.

Question 3 *Find new classes of finite simple left braces.*

Here, $(B, +, \cdot)$ is simple if it has no proper ideals. An ideal I of a left brace B is a normal subgroup of $(B, \cdot)$ such that $ba - b \in I$ for every $b \in B, a \in I$ (every ideal leads to a natural left brace structure on the group B/I).

In order to approach Question 3, we propose three other questions.

Question 4 *Construct new classes of finite left braces* $(B, +, \cdot)$ *of order* p^n *for a prime number* p.

Question 5 *Describe the automorphism group* $\mathrm{Aut}(B, +, \cdot)$ *of all such braces.*

Recall that every finite left brace B can be written in the form of the matched product $B = B_1 \cdots B_k$ of left braces, where $B_1, \dots, B_k$ are the Sylow subgroups of $(B, +)$. Therefore, the left braces of Question 4 are the building blocks, while the collection of compatible actions of B_i on B_j, for all i, j is an essential factor in the definition of a matched product of braces; whence the significance of Question 5 (see Proposition 2.7 of [D. Bachiller, F. Cedó, E. Jespers and J. Okniński: "Iterated matched products of finite braces and simplicity; new solutions of the Yang-Baxter equation", *Trans. Amer. Math. Soc*, to appear; *arXiv*:1610.00477]).

Question 6 *Describe all simple left braces of orders* $p^n q^m$*, for two different primes* p, q *and positive integers* n, m.

Notice that, according to a result of W. Rump (see ["Braces, radical rings, and the quantum Yang-Baxter equation", *J. Algebra* 307 (2007), 153–170]), every simple left brace of order p^n is the trivial brace of order p, meaning that $(B, +)$ is a cyclic group of order p and the operation $\cdot$ coincides with $+$.

More constructions of finite simple left braces can be found in [D. Bachiller, F. Cedó, E. Jespers and J. Okniński: "Asymmetric product of left braces and simplicity; new solutions of the Yang-Baxter equation", preprint on *arXiv*:1705.08493].

Ferran Cedó
Jan Okniński

ADV – 4D

A group G is said to admit a complete resolution if there exists a doubly infinite exact complex of projective **Z**G-modules $\underline{\mathcal{E}}$, which coincides with a projective resolution $\underline{\mathcal{P}}$ of the trivial **Z**G-module **Z** in sufficiently high dimensions, i.e

$$\begin{array}{l} \underline{\mathcal{E}} : \ldots \to E_{n+1} \to E_n \to E_{n-1} \to \ldots \to E_0 \to E_{-1} \to \ldots \\ \qquad\qquad\quad \| \qquad\quad \| \\ \underline{\mathcal{P}} : \ldots \to P_{n+1} \to P_n \to P_{n-1} \to \ldots \to P_0 \to \mathbb{Z} \to 0 \end{array}$$

A group G is said to admit a complete resolution in the strong sense if it admits a complete resolution such that $\mathrm{Hom}_{\mathbb{Z}G}(\underline{\mathcal{E}}, P)$ is exact for any projective $\mathbb{Z}$G-module P.

Admitting a complete resolution is a subgroup closed property and if a group G admits a complete resolution then $H^i(G, P) \neq 0$ for some i and some projective $\mathbb{Z}$G-module P (see [G. Mislin and O. Talelli: "On groups which act freely and properly on finite dimensional homotopy spheres", *London Math. Soc. Lecture Note Ser.* 275, 208–228]). As there are groups with $H^i(G, P) = 0$ for all i and all projective $\mathbb{Z}$G-modules P, e.g. the free abelian group of infinite rank, $GL(n, K)$, with $n \geqslant 1$ and K a subfield of the algebraic closure of the rational numbers [G. Mislin: "Tate cohomology for arbitrary groups via satellites", *Topology Appl.* 56 (1994), 293–300], it follows that there are groups which do not admit a complete resolution. Any finite group or more generally every group with finite Gorenstein cohomological dimension admits a complete resolution in the strong sense.

If a group G admits a complete resolution in the strong sense, then its complete cohomology $\widehat{H}^i(G, A)$ (generalized Tate cohomology, defined for any group G) can be calculated via a complete resolution in the strong sense, i.e.

$$\widehat{H}^i(G, A) = H^i(\mathrm{Hom}_{\mathbf{Z}G}(\underline{\mathcal{E}}, A))$$

for all $i \in \mathbb{Z}$ and $\mathbb{Z}$G-modules A (see [J. Cornick and P. Kropholler: "On complete resolutions", *Topology Appl.* 78 (1997), 235–250]). If one wants the complete cohomology to have nice properties, e.g. to satisfy the Eckmann-Schapiro lemma, which relates the cohomology of a subgroup to the cohomology of the group, then the complete cohomology has to be calculated via complete resolutions in the strong sense (see [F. Dembegioti and O. Talelli: "A note on complete resolutions", *Proc. Amer. Math. Soc.* 138 (2010), 3815–3820]).

If a group G has periodic cohomology after some steps, i.e. there are natural numbers q, k such that the functors

$$H^i(G, -) \quad \text{and} \quad H^{i+q}(G, -)$$

are naturally equivalent for all $i > k$, then G admits a complete resolution (see [O. Talelli: "Periodicity in group cohomology and complete resolutions", *Bull. London Math. Soc.* 37 (2005), 547–554]).

Question *If G is a group admitting a complete resolution, does it admits a complete resolution in the strong sense?*

There are a few algebraic invariants of a group whose finiteness is equivalent to the existence of a complete resolution in the strong sense, e.g. a group G admits a complete resolution in the strong sense iff the Gorenstein cohomological dimension of G is finite (see [A. Bahlekeh, F. Dembegioti and O. Talelli, *Bull. London Math. Soc.* 41 (2009), 859–871]). The conjecture holds for all groups in Kropholler's class of groups $\mathbf{LH}\mathfrak{F}$ which contains among others all soluble and all linear groups.

If the conjecture were true, it would in particular imply that if a group G has periodic cohomology after some steps then the periodicity natural equivalences are induced by cup product with an element in $H^q(G, Z)$. It would then follow from a Theorem of A. Adem and J.H. Smith (see ["Periodic complexes and group actions", *Ann. of Math.* 154(2) (2001), 407–435]) that periodicity in cohomology after some steps is the algebraic characterization of those countable groups which act freely and properly on $\mathbb{R}^n \times \mathbb{S}^m$ for some n, m.

Olympia Talelli

ADV – 4E

Any finite group G contains a subgroup H that has the same exponent as G and can be generated by three elements (see [E. Detomi and A. Lucchini: "Probabilistic generation of finite groups with a unique minimal normal subgroup", *J. Lond. Math. Soc.* 87 (2013), 689–706]). In several relevant situations a stronger result holds. For example any finite soluble group G contains a 2-generated subgroup H with the same exponent (see [A. Lucchini, M. Morigi and P. Shumyatsky: "Boundedly generated subgroups of finite groups", *Forum Math.* 24 (2012), 875–887]).

Question *Is it true that any finite group* G *contains a 2-generated proper subgroup with the same exponent?*

Andrea Lucchini

ADV – 4F

Let G be a group, let $\zeta_\alpha(G)$ denote the αth term of the upper central series and let $\gamma_\alpha(G)$ denote the αth term of the lower central series. Let $\zeta_\infty(G)$ denote the upper hypercentre of G. It is a well-known theorem attributed to I. Schur that if G is a group and $G/\zeta_1(G)$ is finite, then $\gamma_2(G)$ is finite. A theorem attributed to R. Baer asserts that if $G/\zeta_n(G)$ is finite, then $\gamma_{n+1}(G)$ is finite for all natural numbers n. In [M. De Falco, F. de Giovanni, C. Musella, and Y.P. Sysak: "On the upper central series of infinite groups", *Proc. Amer. Math. Soc.* 139 (2011), 385–389] it was shown that if $G/\zeta_\infty(G)$ is finite, then the locally nilpotent residual L of G is finite and G/L is hypercentral. There have been various other generalizations of this latter result concerned with hypotheses restricting the various ranks of $G/\zeta_\infty(G)$ and some of these results are collected together in [M.R. Dixon, L.A. Kurdachenko, I.Ya. Subbotin: "Ranks of Groups: The Tools, Characteristics, and Restrictions", *J. Wiley and Sons*, Hoboken (2017)]. The following question arises.

Question *Let* G *be a group such that* $G/\zeta_\infty(G)$ *is a generalized radical group of finite* 0*-rank. Is there a normal subgroup* H *of* G *such that* H *has finite* 0*-rank and* G/H *is hypercentral?*

Martyn R. Dixon
