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Groups in which Every Subgroup of Infinite Rank is Nearly Permutable

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Abstract

In this paper, the structure of locally finite groups of infinite rank whose subgroups of infinite rank have finite index in a permutable subgroup is investigated.

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1 Introduction

A subgroup H of a group G is said to be *permutable* if HK = KH for every subgroup K of G. This concept was introduced by Ore [14]. It is clear that every normal subgroup of a group is permutable, but arbitrary permutable subgroups need not to be normal. It is easy to see that a maximal permutable subgroup is normal, so that a permutable subgroup of a finite group is subnormal. For infinite groups, Stonehewer [17] has proved that a permutable subgroup of an arbitrary group is ascendant. A group G is called *quasihamiltonian* if every subgroup of G is permutable. The structure of quasihamiltonian groups has been completely described by Iwasawa [11], we refer to [16] for a detailed account on this subject.

A subgroup H of a group G is called *nearly normal* if it has finite index in its normal closure H^G. In [13], B. H. Neumann has proved that every subgroup of a group is nearly normal if and only if its commutator subgroup is finite. A corresponding property, where normality is replaced by permutability, has been introduced in [5]. More precisely, a subgroup H of a group G is called *nearly permutable* if it has finite index in a permutable subgroup of G. In [5], the authors proved that a periodic group has all its subgroups nearly permutable if only if it is finite-by-quasihamiltonian.

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements and r is the least positive integer with such property; if such an r does not exist, we will say that the group G has infinite rank. In recent years, many authors have proved that in a (generalized) soluble group of infinite rank the behaviour of subgroups of infinite rank has an influence on the structure of the whole group (for example, see [4] for a survey on this topic). In particular, in [3] the authors have proved that a (generalized) soluble group of infinite rank whose subgroups of infinite rank are nearly normal has finite commutator subgroup.

The aim of this paper is to investigate the structure of a locally finite group of infinite rank in which every subgroup of infinite rank is nearly permutable. It will turn out that the behaviour of the subgroups of finite rank can be neglected.

Theorem *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is finite-by-quasihamiltonian.*

The structure of non-periodic groups of infinite rank in which every subgroup of infinite rank is nearly permutable has been studied in [7] and it has been obtained as a corollary of a more general situation.

In a forthcoming paper [6] a dual concept to nearly permutability will be treated.

Most of our notation is standard and can be found in [15].

2 Primary groups

Our first purpose is to show that a locally finite p-group of infinite rank whose subgroups of infinite rank are nearly permutable is finite-by-quasihamiltonian.

In order to prove the main theorem of this section, we need the following lemma, which shows that, at least in the universe of locally finite groups, under certain conditions a subgroup of finite rank is the intersection of two subgroups of infinite rank.

Lemma 2.1 *Let* G *be a group and let* A *be a periodic normal subgroup of infinite rank of* G. *If* X *is a subnormal Černikov subgroup of* G, *then* A *contains a subgroup of infinite rank* B *such that* $[X, B] = \{1\}$.

Proof — Let

$$X = L_0 \triangleleft L_1 \triangleleft \ldots \triangleleft L_{k-1} \triangleleft L_k = XA$$

be a subnormal series of X in XA and argue by induction on k. If $k = 1$, then X is normal in XA, so the factor group $A/C_A(X)$ is a Černikov group ([15], Theorem 3.29) and we can choose $B = C_A(X)$. Now, let $k > 1$ and put $L = L_{k-1}$. If L has finite rank, then L is a Černikov group and we can choose $B = C_A(L)$. So we can suppose that L has infinite rank. Since $L = X(A \cap L)$, then $A \cap L$ has infinite rank and, by induction, there exists a subgroup B of $A \cap L$ such that $[X, B] = \{1\}$ and the statement is true for any k. □

It is known that a locally finite quasihamiltonian p-group is abelian-by-finite, so that a primary group in which every subgroup is nearly permutable is finite-by-abelian-by-finite. The next proposition shows that it suffices to require only that the subgroups of infinite rank are nearly permutable.

Proposition 2.2 *Let* G *be a locally finite* p*-group of infinite rank whose subgroups of infinite rank are nearly permutable. Then* G *is finite-by-abelian-by-finite.*

Proof — Suppose by contradiction that G is not finite-by-abelian-by-finite and put $A = \Omega_1(G)$. Every subgroup of infinite rank of A is nearly normal in A and so A' is finite (see [3], Theorem B). Moreover, G/A is finite-by-quasihamiltonian (see [5], Theorem). In particular, G/A is finite-by-abelian-by-finite. Let H be a normal subgroup of finite index of G such that H/A is finite-by-abelian. Thus H/A' is still a counterexample to the proposition and we may assume that A is abelian and G/A is finite-by-abelian. Let N/A be a finite normal subgroup of G/A such that G/N is abelian, and let K be a permutable subgroup of G. Then K is normal in KA, the index $|KN : KA|$ is finite and KN is normal in G; it follows that there exists a positive integer n such that every permutable subgroup of G is subnormal of defect at most n. Hence every subgroup of infinite rank of G has finite index in a subnormal subgroup of defect at most n. In particular, every subgroup of infinite rank of G is subnormal. Therefore every subgroup of G is subnormal in G (see [12], Theorem 5). Let X be any subgroup of finite rank of G, then X is a Černikov group (see [15], Corollary 1, p.38, Part 2) and X is subnormal in XA, so, by Lemma 2.1, A contains a subgroup $C = C_1 \times C_2$, with C_1 and C_2 of infinite rank and $X \cap C = \{1\}$, such that $X = XC_1 \cap XC_2$. As XC_i has infinite rank for $i = 1, 2$, X has finite index in a subnormal subgroup of G of defect at most n. Thus there exists a finite normal subgroup K of G such that G/K is nilpotent (see [8], Theorem 1) and G is nilpotent. Among all counterexamples to the proposition obtained in this way, choose a nilpotent group G with minimal nilpotency class $c > 1$.

If the centre $Z(G)$ of G has infinite rank, then $Z(G)$ contains a subgroup $Z_1 \times Z_2$, with Z_1 and Z_2 of infinite rank. Then G/Z_i is finite-by-abelian-by-finite, for $i = 1, 2$, and so the same holds for G, a contradiction. It follows that $Z(G)$ has finite rank and, by the minimality of c, $G/Z(G)$ is finite-by-abelian-by-finite. Thus $Z(G) \cap \Omega_1(G)$ is finite and $G/(Z(G) \cap \Omega_1(G))$ is finite-by-abelian-by-finite, so that G is finite-by-abelian-by-finite and this last contradiction completes the proof of the proposition. □

Lemma 2.3 *Let G be a group of infinite rank whose subgroups of infinite rank are nearly permutable. If G contains an elementary abelian normal p-subgroup A of finite index, then the commutator subgroup G' of G is finite.*

Proof — Let H be any subgroup of infinite rank of G and let K be a permutable subgroup of G such that H has finite index in K. Then K is normal in KA, further KA has finite index in G and it follows from Proposition 3.3 of [7] that G' is finite. □

Theorem 2.4 *Let G be a locally finite p-group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is finite-by-quasihamiltonian.*

Proof — By Proposition 2.2, G contains a finite normal sub-group N such that G/N is abelian-by-finite. Without loss of generality it can be assumed that $N = \{1\}$, so that G is abelian-by-finite. Let A be an abelian normal subgroup of finite index of G. First, suppose that G has infinite exponent. By Lemma 6 of [3], $\Omega_1(A)$ contains a direct product $Y_1 \times Y_2$ of G-invariant subgroups of infinite rank Y_1 and Y_2 and G/Y_i is finite-by-quasihamiltonian, for $i = 1, 2$. Since Y_1 and Y_2

have finite exponent, it follows that G/Y_i is finite-by-abelian, for $i = 1, 2$. Hence G is finite-by-abelian. So we can suppose that G has finite exponent. Put $G = AE$, where E is a finite subgroup of G and let H be any subgroup of infinite rank of G. Then there exists a permutable subgroup K_1 of G such that $|K_1 : H|$ is finite. Let K be a permutable subgroup of G such that K_1E has finite index in K. It follows that $|K : H|$ is finite and $G = AK$. As a consequence $|K : H_K|$ is finite and $K \cap A$ is a G-invariant subgroup of finite index of K. Hence, $H_K \cap A$ has finite index in K and, being a normal subgroup of H_KA, it is also normal in G. In particular every subgroup of infinite rank of G is normal-by-finite, so that every subgroup of G is normal-by-finite (see [3], Theorem C). Since A is a bounded abelian group, it is the direct product of cyclic subgroups and so it is clearly residually finite. Application of Lemma 2.1 of [2] yields that A contains a subgroup B of finite index such that every subgroup of B is G-invariant. Then B has finite index in G and, replacing A by B, we may assume that every subgroup of A is G-invariant.

Let $Y = Y_1 \times Y_2$ be a subgroup of A with Y_1 and Y_2 of infinite rank such that $E \cap Y = \{1\}$ and let K_i be a permutable subgroup of G such that $|K_i : EY_i|$ is finite, for $i = 1, 2$. Then $E = EY_1 \cap EY_2$ has finite index in $F = K_1 \cap K_2$, so that F is finite and $G = AF$. Without loss of generality we may assume that $E = K_1 \cap K_2$. Moreover, we can replace G with G/E_G, so that E is a core-free subgroup of G. In particular $A \cap E = C_E(A) = \{1\}$, and E acts on A as a group of power automorphisms. If $p > 2$, then E is cyclic and by Lemma 2.3.4 of [16] we have that G is locally quasihamiltonian and hence even quasihamiltonian. So we can assume $p = 2$. If A has exponent 2, then G is finite-by-abelian by Lemma 2.3. So, we can suppose that the exponent of A is at least 4. Let U be a cyclic subgroup of order 4 of A, then $UK_i/C_{K_i}(U)$ has order at most 8 and, as K_i is permutable, it follows that $[U, K_i]$ is contained in K_i. Thus $[U, E] = \{1\}$ and $U \leqslant Z(G)$. Hence, E is cyclic and, applying Lemma 2.3.4 of [16] again, we obtain that G is quasihamiltonian and so the theorem is completely proved. □

3 Periodic Groups

Since a quasihamiltonian group is locally nilpotent, a periodic group G in whose subgroups are nearly permutable is finite-by-(locally nilpotent) and hence G is also (locally nilpotent)-by-finite. In order to prove the main theorem of this paper, the first step is to show that a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable is (locally nilpotent)-by-finite.

Recall that the *Hirsch-Plotkin radical* of a group G is the largest locally nilpotent normal subgroup of G and it contains every locally nilpotent ascendant subgroup of G.

Lemma 3.1 *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G contains a nilpotent normal subgroup A of infinite rank such that the commutator subgroup A' of A is finite and for every prime p the p-component of A is generated by elements of order p.*

Proof — f Let B be an abelian subgroup of infinite rank of G and let K be a permutable subgroup of G such that $|K : B|$ is finite. Then $|K : B_K|$ is finite and B_K is an abelian ascendant subgroup of infinite rank of G. Thus, B_K is contained in the Hirsch-Plotkin radical R of G and, in particular, R has infinite rank. Let

$$A = Dr_p \Omega_1(R_p),$$

where R_p is the p-component of R. Then A has infinite rank and every permutable subgroup of A is normal in A. It follows that every subgroup of infinite rank of A is nearly normal in A and A' is finite (see [3], Theorem B) and the lemma is proved. □

We say that a subgroup X of a group G is *finite-permutable-finite* if there exist subgroups H and K of G such that the indeces $|H : X|$ and $|G : K|$ are finite and H is permutable in K. This definition has been introduced in [7] and clearly every nearly permutable subgroup is also finite-permutable-finite.

Lemma 3.2 *Let G be a locally finite group, and let S be a Sylow p-subgroup of G. If S is finite-permutable-finite, then $S/O_p(G)$ is finite.*

Proof — Let H and K be subgroups of G such that the indexes $|H : S|$ and $|G : K|$ are finite and H is permutable in K. The core S_H of S in H is an ascendant p-subgroup of K and so it is contained in $O_p(K)$. Since S/S_H is finite, it follows that $S/O_p(K)$ is finite. Clearly

$$O_p(G) = O_p(K) \cap K_G$$

and, since G/K_G is finite, we have that also the factor group $S/O_p(G)$ is finite. □

We put here a technical lemma that will be needed in the following.

Lemma 3.3 *Let* G *be a locally finite group of infinite rank whose subgroups of infinite rank are finite-permutable-finite and such that every section* H/K *of* G *is finite-by-quasihamiltonian, when* K *has infinite rank. If* G *contains an abelian normal subgroup of infinite rank* A *such that for every prime* p *the* p*-component of* A *is elementary abelian, then one of the following conditions holds:*

1. G *is (locally nilpotent)-by-finite,*
2. G *contains a non-(locally nilpotent)-by-finite subgroup* $M = QB$*, where* B *is a normal elementary abelian* p*-subgroup of infinite rank of* M *and* Q *is a locally nilpotent* p'*-group of finite rank, for some prime* p*.*

Proof — Assume that G is not (locally nilpotent)-by-finite. If for every prime p the p-component A_p of A has finite rank, then A contains a direct product $B_1 \times B_2$ of G-invariant subgroups of infinite rank B_1 and B_2. Then G/B_i is (locally nilpotent)-by-finite, for $i = 1, 2$, and also G is (locally nilpotent)-by-finite, a contradiction. It follows that for some prime p the rank of $B = A_p$ is infinite. Then there exists a normal subgroup H of finite index of G such that H/B is locally nilpotent. Therefore, H is not (locally nilpotent)-by-finite and we can replace G by H, so that we can assume that G/B is locally nilpotent. Moreover, as G/B is finite-by-quasihamiltonian, its primary components are nilpotent and so every finite subgroup of G/B is subnormal in G/B. Let x be any element of G of order p^n, for some positive integer n, then the p-subgroup $\langle x\rangle B$ is subnormal in G and so it is contained in $O_p(G)$. In particular, x belongs to $O_p(G)$ and, as a consequence, $P = O_p(G)$ is the unique Sylow p-subgroup of G. Clearly, $B \leqslant P$ and so G/P is a locally nilpotent p'-group. By contradiction, suppose that there exists a Sylow q-subgroup Q of infinite rank of G, with $q \neq p$. Then Q is finite-permutable-finite and $Q/O_q(G)$ is finite by Lemma 3.2. Thus, $O_q(G)$ has infinite rank and, as $P \cap O_q(G) = \{1\}$, the group G is (locally nilpotent)-by-finite, a contradiction. It follows that for every $q \neq p$, every Sylow q-subgroup of G has finite rank and G satisfies the minimal condition on q-subgroups. Therefore, by Lemma 2.5.10 of [9] every q-component of G/P is a Černikov group and so, in particular, G/P is countable. Hence, there exists a locally nilpotent p'-subgroup Q of G such that $G = QP$ (see [9], Theorem 2.4.5). Since G/B is locally nilpotent, QB is normal in G, so that QB is not (locally nilpotent)-by-finite. If Q has infinite rank, then there exist subgroups H and K of QB such that the indexes $|H : Q|$ and $|QB : K|$ are finite and H is permutable in K. In particular

$$K = H(K \cap B)$$

and H is normal in K. It follows that $H \cap B$ is a finite normal subgroup of K and $K/(H \cap B)$ is the product of two (locally nilpotent)-by-finite normal subgroups. Hence, K is finite-by-(locally nilpotent)-by-finite and this implies that K is also (locally nilpotent)-by-finite. Since K has finite index in G, also G is (locally nilpotent)-by-finite, a contradiction. Thus, Q has finite rank and $M = QB$ is the required subgroup. □

Proposition 3.4 *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is (locally nilpotent)-by-finite.*

Proof — Suppose that G is not (locally nilpotent)-by-finite. By Lemma 3.1, G contains a normal subgroup of infinite rank A such that A' is finite and for every prime p the p-component of A is generated by elements of order p. Thus, G/A' is still a counterexample and so, replacing G by G/A', we can suppose that A is abelian. Then it follows from Lemma 3.3 that G contains a non-(locally nilpotent)-by-finite subgroup $M = QB$, where B is a normal elementary abelian p-subgroup of infinite rank of M and Q is a locally nilpotent p'-group of finite rank, for some prime p. Without loss of generality we can replace G by M. Put $\pi = \pi(Q)$ and first suppose that π is a finite set. Then Q is a Černikov group. Let J be any quasicyclic subgroup of Q and let x be any element of J. It follows from Lemma 2.9 of [5] that JB and $X = \langle x\rangle B$ are normal subgroups of G. Moreover, by Lemma 2.3, $X' = [x, B]$ is finite. As a consequence, J is normal in $J[x, B]$ for every $x \in J$ and, hence, J is normal also in $J[J, B] = J^B$. Therefore the finite residual of Q is subnormal in G and $Q/O_\pi(G)$ is finite. As $O_\pi(G)B$ is contained in the Hirsch-Plotkin radical R of G, we have that G/R is finite. By this contradiction, the set π is infinite. Let $C = C_1 \times C_2$ be a subgroup of B, with C_1 and C_2 of infinite rank, and let K_i be a permutable subgroup of G such that $|K_i : C_i|$ is finite, for $i = 1, 2$. Then $K_1 \cap K_2$ is finite and, by Lemma 1.2.5 of [1], C has finite index in K_1K_2 and, it follows that the set $\sigma = \pi(K_1K_2)$ is finite. Put $Q = Q_\sigma \times Q_{\sigma'}$, then $Q_{\sigma'} \cap K_1K_2 = \{1\}$, so that

$$Q_{\sigma'}K_1 \cap Q_{\sigma'}K_2 = Q_{\sigma'}(K_1 \cap K_2).$$

Let L_i be a permutable subgroup of G such that $Q_{\sigma'}K_i$ has finite index in L_i, for $i = 1, 2$. Then $Q_{\sigma'}$ has finite index in $L = L_1 \cap L_2$. As L is normal in LB, there exists a normal subgroup N of $Q_{\sigma'}B$ such that $|N : Q_{\sigma'}|$ is finite. Hence, $N \cap B$ is finite and $Q_{\sigma'}B/(N \cap B)$ is locally nilpotent, so that $Q_{\sigma'}B$ is (locally nilpotent)-by-finite. On the other hand, as σ is a finite set, the previous argument shows that also $Q_\sigma B$ is (locally nilpotent)-by-finite. Thus, G is the product of its (locally nilpotent)-by-finite normal subgroups $Q_\sigma B$ and $Q_{\sigma'}B$ and this last contradiction completes the proof. □

Lemma 3.5 *Let G be a locally finite group of infinite rank whose subgroups of infinite rank are nearly permutable. If X is a subgroup of finite rank of G, then X is finite-by-quasihamiltonian.*

Proof — Let A be an abelian subgroup of infinite rank of G such that $A \cap X = \{1\}$ and let L be any subgroup of X. As A has finite index in a permutable subgroup H of G, $H \cap X$ is finite and L has finite index $HL \cap X = L(H \cap X)$. Let K be a permutable subgroup of G such that $|K : HL|$ is finite, then $K \cap X$ is permutable in X and L has finite index in $K \cap X$. It follows that every subgroup of X is nearly permutable and X is finite-by-quasihamiltonian (see [5], Theorem). □

The next lemma is a generalization of Lemma 3.3 of [5]. We omit the proof since it is analogous to the proof contained in [5].

Lemma 3.6 *Let G be a periodic group and let $(E_n)_{n \in \mathbb{N}}$ be a sequence of subgroups of G such that $\pi(E_n)$ is finite for every n, $\pi(E_n) \cap \pi(E_m) = \emptyset$ for $n \neq m$ and all subgroups of E_{n+1} are normalized by $\langle E_1, \dots, E_n \rangle$ for each positive integer n. If every E_n contains a non-permutable subgroup H_n, then the subgroup $H = \langle H_n \mid n \in \mathbb{N} \rangle$ is not nearly-permutable in G.*

We are now in the position to prove the main theorem of the paper. First, we consider the locally nilpotent case.

Proposition 3.7 *Let G be a periodic locally nilpotent group of infinite rank whose subgroups of infinite rank are nearly permutable. Then G is finite-by-quasihamiltonian.*

PROOF — Assume by contradiction that G is not finite-by-quasihamiltonian. Let n be a positive integer for which there exist n subgroups $E_1, \dots, E_n$ of G such that $\pi(E_i)$ is finite for every $i \leqslant n$, $\pi(E_i) \cap \pi(E_j) = \emptyset$ for $i \neq j$, every E_i contains a non-permutable subgroup H_i of rank r_i and $r_i < r_{i+1}$ for every $i < n$. By Theorem 2.4 and Lemma 3.5, every primary component G_p of G is finite-by-quasihamiltonian. As the set

$$\pi = \pi(E_1) \cup \cdots \cup \pi(E_n)$$

is finite, it follows that G_π is finite-by-quasihamiltonian and so $G_{\pi'}$ contains a finite subgroup $\overline{E}_{n+1}$ and a subgroup $\overline{H}_{n+1}$ of $\overline{E}_{n+1}$ such that $\overline{H}_{n+1}$ is not permutable in $\overline{E}_{n+1}$. Let r_{n+1} be the rank of $\overline{H}_{n+1}$. If $r_n < r_{n+1}$, put $E_{n+1} = \overline{E}_{n+1}$ and $H_{n+1} = \overline{H}_{n+1}$. So, suppose that $r_{n+1} \leqslant r_n$ and put $\pi_{n+1} = \pi \cup \pi(\overline{E}_{n+1})$. As π_{n+1} is finite, $G_{\pi'_{n+1}}$ is not finite-by-quasihamiltonian and hence it has infinite rank, by Lemma 3.5. It follows that there exists a prime $p \notin \pi_{n+1}$ such that r_n is strictly less than the rank of G_p. In this case, put $E_{n+1} = \overline{E}_{n+1} \times G_p$ and $H_{n+1} = \overline{H}_{n+1} \times G_p$.

In both cases, we have that $\pi(E_{n+1})$ is finite, $\pi(E_i) \cap \pi(E_{n+1}) = \emptyset$ for $i \leqslant n$, H_{n+1} is not permutable in E_{n+1} and $r_n < r_{n+1}$. It follows from Lemma 3.6 that $H = \langle H_n \mid n \in \mathbb{N} \rangle$ is not nearly permutable in G and this is a contradiction, since H has infinite rank. □

PROOF OF THE THEOREM — By Proposition 3.4, G contains a locally nilpotent normal subgroup Q such that the index $|G : Q|$ is finite, so there exists a finite subgroup E of G such that $G = QE$. It follows from Theorem 1 of [10] that Q contains an abelian subgroup $A = A_1 \times A_2$ such that A_1 and A_2 are E-invariant subgroups of infinite rank and $A \cap E = \{1\}$. Let K_i be a permutable subgroup of G such that EA_i has finite index in K_i, for $i = 1, 2$. Then E has finite index in $K_1 \cap K_2$ and $K = K_1 \cap K_2$ is a finite subgroup of G such that $G = QK$. Replacing G with G/K_G, it can be assumed without loss of generality that K is core-free. In particular, $(K_1)_G \cap (K_2)_G = \{1\}$. Since $(K_i)^G/(K_i)_G$ is locally nilpotent, for $i = 1, 2$ (see [16], Theorem 6.3.1), K^G is locally nilpotent. Then $G = QK^G$ is locally nilpotent and, by Proposition 3.7, G is finite-by-quasihamiltonian. □

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