Advances in Group Theory and Applications

4 (2017), pp. 65–82
ISSN: 2499-1287
DOI: 10.4399/97888255086975

Relationships between the Factors of the Central Series and the Nilpotent Residual in Some Infinite Groups

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(*Received Apr. 02, 2017; Accepted May 10, 2017 — Communicated by I. Subbotin*)

Abstract

We consider some natural relationships between the factors of the central series in groups. It was proved that if G is a locally generalized radical group and $G/\zeta_k(G)$ has finite section p-rank r (for some positive integer k), then G includes a normal subgroup L such that G/L is nilpotent. Moreover, there exists a function g such that $sr_p(L) \leqslant g(r)$.

Mathematics Subject Classification (2010): 20F14, 20F18, 20F19, 20K15

Keywords: finite section p-rank; hypercenter; nilpotent residual

1 Introduction

Let W be a set of words and let G be a group. As usual, denote by $W(G)$ the verbal subgroup of G determined by W, and denote by $W^*(G)$ the marginal subgroup of G determined by W. Recall that a group G belongs to the variety $\mathfrak{V}(W)$, defined by W, if and only if $W(G) = \langle 1 \rangle$ (respectively $G = W^*(G)$). Thus the "size" of $W(G)$ (respectively $G/W^*(G)$) shows how far or how near is a group G from the variety $\mathfrak{V}(W)$. The problem about the relationships between the subgroup $W(G)$ and the factor-group $G/W^*(G)$ naturally rose here. The first step here is the case when $W(G)$ is finite (respectively $G/W^*(G)$ is finite). We note at once that the finiteness of $W(G)$

does not imply that $G/W^*(G)$ is finite even for the case when $W = \{\theta\}$. Indeed, for example, if $\theta = x_1^{-1}x_2^{-1}x_1x_2$, then

$$W(G) = [G, G] \quad \text{and} \quad W^*(G) = \zeta(G).$$

However, there are many examples of groups, having finite derived subgroup, whose factor-group by the center is infinite. We have a better situation when the factor-group $G/W^*(G)$ is finite, in particular, if $G/\theta^*(G)$ is finite. In the papers [11] and [12] P. Hall found some types of words θ such that the finiteness of $G/\theta^*(G)$ always implies that $\theta(G)$ is finite. Also P. Hall proved that if G is polycyclic-by-finite and $G/\theta^*(G)$ is finite, then $\theta(G)$ is finite for arbitrary word θ. In particular, the word $[\dots[[x_1, x_2], \dots, x_n], x_{n+1}]$ satisfies the conditions of Hall's theorem, so we obtain that if $G/\zeta_n(G)$ is finite, then $\gamma_{n+1}(G)$ is also finite. This last result is derived from the basic results of the paper [1]. The following natural problem rose in this connection:

Suppose that $G/\theta^*(G)$ and $\theta(G)$ are finite. Is there a function f such that $|\theta(G)| \leqslant f(|G/\theta^*(G)|)$?

For the word $\theta = x_1^{-1}x_2^{-1}x_1x_2$ the best function has been obtained by J. Wiegold [22]. For the word $\theta = [\dots[[x_1, x_2], \dots, x_n], x_{n+1}]$ such function also exists, it has been obtained in the paper [17]. Of course, this function depends not only of $|G/\zeta_n(G)|$, it depends also of n. In addition, the paper [17] contains the following result:

If $G/\zeta_n(G)$ is finite and has order t, then G includes a finite normal subgroup L such that G/L is nilpotent and $|L| \leqslant k(t)$ where k depends only of t.

There are many natural extensions of the class of finite groups. Recall that a group G has *finite special rank* r if every finitely generated subgroup of G has at most r generators (A.I. Maltsev, [18]). Recall also that a group G is called *generalized radical* if it has an ascending series, whose factors are either locally nilpotent or locally finite. Recently, in the paper [14] it was proved that if G is a locally generalized radical group such that $G/\zeta_n(G)$ has finite special rank, then $\gamma_{n+1}(G)$ has finite special rank. Moreover, there exists a function k_1 such that $r(\gamma_{n+1}(G)) \leqslant k_1(r(G/\zeta_n(G), n)$. For the case when $G/\zeta_n(G)$ is a locally finite group having finite special rank, it was proved that G includes a normal locally finite subgroup L

such that G/L is hypercentral and there exists a function k_2 such that $r(L) \leqslant k_2(r(G/\zeta_n(G))$. We emphasize that the function k_2 depends only of $r(G/\zeta_n(G))$. We can see that in this case a locally nilpotent residual $G^{L\mathfrak{N}}$ of G is locally finite, has finite special rank, and the factor-group $G/G^{L\mathfrak{N}}$ is hypercentral. Note also that a restriction on G is essential, since, in general, this result is not true (see, for example, the survey [5]).

Let p be a prime. We say that a group G has *finite section p-rank* $sr_p(G) = r$ if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$ (see [8]). Obviously, the concept of section p-rank generalizes the concept of special rank. Actually, if a group G has finite special rank r, then G has finite section p-rank $sr_p(G)$ for every prime p and $sr_p(G) \leqslant r$.

We say that a group G has a finite section rank if $sr_p(G)$ is finite for each prime number p. We can make this definition more concretely. Let σ be a function from the set $\mathbb{P}$ of all primes into $\mathbb{N}_0$. We say that a group G has a *finite section rank* σ if $sr_p(G) = \sigma(p)$ for every prime p.

In the paper [4] it has been obtained a generalization of a result of the paper [14]. It was proved that if G is a locally generalized radical group such that $G/\zeta_n(G)$ has finite section p-rank for some prime p, then $\gamma_{n+1}(G)$ has finite section p-rank. Moreover, there exists a function k_3 such that $sr_p(\gamma_{n+1}(G)) \leqslant k_3(sr_p(G/\zeta_n(G), n)$. Here the function k_3 depends also of n. In this connection the following problem appears:

Does the group G contain a normal subgroup L such that G/L is nilpotent and the section p-rank of L is bounded by a function depending only of $sr_p(G/\zeta_n(G))$?

In the present paper we obtain a positive answer on this question. More concretely, we proved the following theorem.

Theorem *Let G be a locally generalized radical group and p be a prime. Suppose that there is a positive integer k such that $G/\zeta_k(G)$ has finite section p-rank r. Then G includes a normal subgroup L such that G/L is nilpotent and $sr_p(L) \leqslant g(r)$ for some function g depending only of r.*

We remark that $sr_p(L) \leqslant f_2(r) + 5r = g(r)$ where $f_2(r)$ is a function from Theorem A of the paper [2]. Note that in this case we can not say that L coincides with the locally nilpotent residual. Moreover, the locally nilpotent residual can be trivial, as the example of an infinite dihedral group shows.

2 Preliminary results

Lemma 2.1 *Let* G *be a group such that* $G/\zeta_k(G)$ *is locally finite. Then* $\gamma_{k+1}(G)$ *is locally finite.*

Proof — Let F be an arbitrary finitely generated subgroup of $\gamma_{k+1}(G)$. Then we can choose a finitely generated subgroup K such that $F \leqslant \gamma_{k+1}(K)$. Put $Z = \zeta_k(G)$ and $L/Z = KZ/Z$. Since G/Z is locally finite, L/Z is finite. Clearly $\zeta_k(G) \leqslant \zeta_k(L)$, so that $L/\zeta_k(L)$ is finite. It follows that $\gamma_{k+1}(L)$ is finite (see, for example, [19], 14.5.1). The inclusion $K \leqslant L$ implies that $\gamma_{k+1}(K) \leqslant \gamma_{k+1}(L)$, so that $\gamma_{k+1}(K)$ is finite. Then F is finite, which implies that $\gamma_{k+1}(G)$ is locally finite. □

Corollary 2.2 *Let* G *be a group such that* $G/\zeta_k(G)$ *is locally finite. If* $\mathrm{Tor}(G) = \langle 1 \rangle$*, then* G *is nilpotent and torsion-free.*

We can obtain the following more exact result.

Lemma 2.3 *Let* G *be a group,* $Z = \zeta_k(G)$ *and* $T/Z = \mathrm{Tor}(G/Z)$*. If* T/Z *is locally finite, then* $T/\mathrm{Tor}(T)$ *is torsion-free and* $T/\mathrm{Tor}(T) \leqslant \zeta_k(G/\mathrm{Tor}(T))$.

Proof — Let $T_1 = \mathrm{Tor}(T)$. By Lemma 2.1 $\gamma_{k+1}(T)$ is locally finite. Since $T/\gamma_{k+1}(T)$ is nilpotent,

$$(T/\gamma_{k+1}(T))/\mathrm{Tor}(T/\gamma_{k+1}(T)) = (T/\gamma_{k+1}(T))/(T_1/\gamma_{k+1}(T)) \simeq T/T_1$$

is torsion-free. Let

$$\langle 1 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots \leqslant Z_{k-1} \leqslant Z_k = Z$$

be a segment of the upper central series of G. Let t be a positive integer such that $Z_t \leqslant T_1$, but T_1 does not include Z_{t+1}. Put

$$D_1/T_1 = Z_{t+1}T_1/T_1 \text{ and } C_1/D_1 = \mathrm{Tor}(T/D_1).$$

The choice of Z_{t+1} yields that

$$D_1/T_1 \leqslant \zeta_1(G/T_1).$$

If $x \in C_1$, then $x^m \in D_1$ for some positive integer m. Let g be an arbitrary element of G. Then

$$((gT_1)^{-1}(xT_1)(gT_1))^m = (gT_1)^{-1}(xT_1)^m(gT_1) = (xT_1)^m.$$

Since T/T_1 is torsion-free and nilpotent, last equality implies that

$$(gT_1)^{-1}(xT_1)(gT_1) = xT_1.$$

This means that $C_1/T_1 \leqslant \zeta_1(G/T_1)$. Using the similar arguments and ordinary induction, we obtain that $T/T_1 \leqslant \zeta_n(G/T_1)$ where $n \leqslant k$. □

If α is a real number, then by $i(\alpha)$ we denote the smallest integer not less than α.

Lemma 2.4 *Let* G *be a finitely generated group such that* $G/\zeta_k(G)$ *is finite. Let* p *be a prime and suppose that* $sr_p(G/\zeta_k(G)) = r$. *Then thenilpotent residual* L *of* G *is finite and*

$$sr_p(L) \leqslant r + \frac{1}{2}r(3r+1) + r^2 i(\log_2 r) = f_1(r).$$

Proof — Let $Z = \zeta_k(G)$ and $T = Tor(G)$. As we noted above $\gamma_{k+1}(G)$ is finite, so that $\gamma_{k+1}(G) \leqslant T$. It follows that G/T is a nilpotent group of nilpotency class at most k. The finiteness of G/Z implies that Z is finitely generated (see, for example, [10], Corollary 7.2.1). Then $T \cap Z$ is finite. It follows that there exists a normal subgroup H of Z of finite index such that $\langle 1 \rangle = (T \cap Z) \cap H$ (see, for example, [20], 1C, Theorem 1). Let $t = |Z : H|$, then $K = G^t \leqslant H$, so that $(T \cap Z) \cap K = \langle 1 \rangle$ and K normal in G. Being bounded, finitely generated and nilpotent, Z/K is finite, so that G/K is finite. We have

$$T \cap K = T \cap (K \cap Z) = (T \cap Z) \cap K = \langle 1 \rangle.$$

By Remak's theorem we obtain the embedding

$$G \hookrightarrow L = G/T \times G/K.$$

The factor-group $(G/K)/\zeta_k(G/K)$ is an epimorphic image of $G/\zeta_k(G)$, thus it has section p-rank at most r. Using Theorem B of paper [16] we obtain that

$$sr_p(\gamma_{k+1}(G/K)) \leqslant f_1(r).$$

Since G/T is a nilpotent group of nilpotency class at most k, it follows that $\gamma_{k+1}(L) = \gamma_{k+1}(G/K)$ and hence that

$$sr_p(\gamma_{k+1}(G)) \leqslant sr_p(\gamma_{k+1}(L)) \leqslant f_1(r).$$

The result is proved. □

Lemma 2.5 *Let G be a locally finite group and p be a prime. Suppose that G has a local family $\mathfrak{L}$ of finite subgroups such that $sr_p(H) \leqslant r$ for every subgroup $H \in \mathfrak{L}$. Then the section p-rank of G is finite. Moreover, $sr_p(G) \leqslant r$.*

Proof — Let U, V be the subgroups of G such that U is normal in V and V/U is a finite elementary abelian p-group. Then there is a finitely generated (and hence finite) subgroup F such that $V = FU$. Choose in $\mathfrak{L}$ a finite subgroup K such that $F \leqslant K$. Since $sr_p(K) \leqslant r$, $|F/(F \cap U)| \leqslant p^r$. Hence $|V/U| = |FU/U| \leqslant p^r$. It follows that every elementary abelian section of G is finite and has order at most p^r. This means that $sr_p(G) \leqslant r$. □

Proposition 2.6 *Let G be a group such that $G/\zeta_k(G)$ is locally finite. Let p be a prime and suppose that $sr_p(G/\zeta_k(G)) = r$. Then locally nilpotent residual L of G is locally finite, $sr_p(L) \leqslant f_1(r)$ and G/L is locally nilpotent.*

Proof — Let $Z = \zeta_k(G)$. Denote by $\mathfrak{L}$ a family of all finitely generated subgroups of G. If H is a finitely generated subgroup of G, then clearly $H/(H \cap Z) \simeq HZ/Z$ is finite. Using Lemma 2.4 we obtain that the nilpotent residual $H^{\mathfrak{N}}$ of H is finite and $sr_p(H^{\mathfrak{N}}) \leqslant f_1(r)$. If K is a finitely generated subgroup of G such that $H \leqslant K$, then

$$H/(H \cap K^{\mathfrak{N}}) \simeq HK^{\mathfrak{N}}/K^{\mathfrak{N}}$$

is nilpotent. It follows that $H^{\mathfrak{N}} \leqslant H \cap K^{\mathfrak{N}} \leqslant K^{\mathfrak{N}}$. This shows that the family $\{H^{\mathfrak{N}} | H \in \mathfrak{L}\}$ is local, so that $R = \bigcup_{H \in \mathfrak{L}} H^{\mathfrak{N}}$ is a subgroup. Moreover, R is locally finite, because every subgroup $H^{\mathfrak{N}}$ is finite. Lemma 2.4 shows that R has finite section p-rank, moreover, $sr_p(R) \leqslant f_1(r)$.

Let F/R be a finitely generated subgroup of G/R. Then $F = SR/R$ for some finitely generated subgroup S. From the construction of R we obtain that $S^{\mathfrak{N}} \leqslant R$, so that $S^{\mathfrak{N}} \leqslant R \cap S$ and $F/R = SR/R \simeq S/(S \cap R)$ is nilpotent. It follows that G/R is locally nilpotent, so that $L \leqslant R$. On the other hand, for each finitely generated subgroup H a section $H/(H \cap L) \simeq HL/L$ is nilpotent, which implies that $H^{\mathfrak{N}} \leqslant H \cap L$.

Since it is true for every finitely generated subgroup H,

$$R = \bigcup_{H \in \mathfrak{L}} H^{\mathfrak{N}} \leqslant L,$$

so that $R = L$. In particular, L is locally finite, G/L is locally nilpotent and $sr_p(L) \leqslant f_1(r)$. □

Let H and K be normal subgroups of G such that $H \leqslant K$. Then the factor K/H is called G-*central* if $C_G(K/H) = G$. If $C_G(K/H) \neq G$, then we say that the factor K/H is G-*eccentric*.

Lemma 2.7 *Let G be a group and A be a normal periodic abelian subgroup of G. Suppose also that A is divisible. If A includes a bounded G-invariant subgroup B such that A/B is G-central, then $G = C_G(A)$.*

Proof — Suppose the contrary, let $C_G(A) \neq G$. Choose an element $g \notin C_G(A)$. Then $C_A(g) \neq A$. Consider now the mapping $\xi_g : A \to A$ defined by the rule $\xi_g(a) = [a, g]$. Clearly this mapping is an endomorphism of A. By the choice of g we have $\mathrm{Ker}(\xi_g) = C_A(g) \neq A$. Since the factor A/B is G-central, we have $\mathrm{Im}(\xi_g) = [A, g] \leqslant B$, in particular a subgroup $\mathrm{Im}(\xi_g)$ is bounded. Then the isomorphism

$$\mathrm{Im}(\xi_g) \simeq A/\mathrm{Ker}(\xi_g)$$

implies that the factor-group $A/\mathrm{Ker}(\xi_g)$ must be bounded. On the other hand, being divisible, A has no non-trivial bounded factor-groups. This contradiction proves the result. □

Let G be a group and A be a normal abelian subgroup of G. A subgroup A is called G-*quasifinite* if every proper G-invariant subgroup of A is finite.

Lemma 2.8 *Let G be a group and A be a normal periodic subgroup of G. Suppose that $G/C_G(A)$ is hypercentral and A satisfies the following conditions:*

(i) $A/(A \cap \zeta(G))$ *is an abelian Chernikov group;*

(ii) $A/(A \cap \zeta(G))$ *is G-quasifinite;*

(iii) $C_G(A/(A \cap \zeta(G)) \neq G$.

Then A includes a G-invariant abelian subgroup D which is Chernikov and G-quasifinite, $A = (A \cap \zeta(G))D$ and the intersection $(A \cap \zeta(G)) \cap D$ is finite.

Proof — Put $C = A \cap \zeta(G)$. Being Chernikov and G-quasifinite, A/C is divisible. Suppose that A is non-abelian. Then $Z = \zeta(A) \neq A$. Since $C \leqslant Z$, A/Z is divisible. Then A/Z includes a Prüfer p-subgroup P/Z. Since P/Z is locally cyclic, P is abelian. It follows that $P \leqslant C_A(P)$, so that $A/C_A(P)$ is divisible. It follows that $P \leqslant \zeta(A)$ (see [15], Lemma 3.10). But it contradicts the choice of P. This contradiction proves that A is abelian. Let

$$C_G(A) \neq gC_G(A) \in \zeta(G/C_G(A)).$$

Then the mapping $\xi_g : A \to A$, defined by the rule $\xi_g(a) = [a, g]$, is an endomorphism of A. Since $g \notin C_G(A)$,

$$\mathrm{Ker}(\xi_g) = C_A(g) \neq A.$$

The following obvious inclusion $C \leqslant C_A(g)$ implies that

$$[A, g] = \mathrm{Im}(\xi_g) \simeq A/\mathrm{Ker}(\xi_g).$$

The fact that $gC_G(A) \in \zeta(G/C_G(A))$ implies that $C_A(g)$ and $[A, g]$ are G-invariant subgroups. If $\mathrm{Ker}(\xi_g) \neq C$, then $\mathrm{Ker}(\xi_g)/C$ is a G-invariant subgroup of A/C. Since $\mathrm{Ker}(\xi g) \neq A$, we obtain that $\mathrm{Ker}(\xi g)/C$ is finite. It follows that $A/\mathrm{Ker}(\xi_g) \simeq A/C$, so that

$$D = [A, g] \simeq A/C.$$

Suppose that $g \in C_G(A/C)$, then $[A, g] \leqslant C$. Let x be an arbitrary element of G. The choice of g implies that $g^x = gz$ for some element $z \in C_G(A)$. If $a \in A$, then

$$\xi_g(a) = [a, g] = [a, g]^x = [a^x, g^x] = [a^x, gz] = [a^x, g] = \xi_g(a^x).$$

It follows that $a^x a^{-1} \in \mathrm{Ker}(\xi_g)$. This shows that the factor $A/\mathrm{Ker}(\xi_g)$ is G-central. As we noted above, $\mathrm{Ker}(\xi_g)/C$ is finite. Using Lemma 2.7 we obtain that factor A/C is G-central. However, it contradicts condition (iii). This contradiction shows that $g \notin C_G(A/C)$. Let B be a proper G-invariant subgroup of D and let V be a preimage of B. Let $x \in G$, $v \in V$, then

$$\xi_g(v^x) = [v^x, g] = [v^x, gz] = [v^x, g^x] = [v, g]^x \in B^x = B.$$

It follows that $v^x \in V$, so that V is G-invariant. Since $B \neq \mathrm{Im}(\xi_g)$,

$V/\mathrm{Ker}(\xi_g)$ is a proper G-invariant subgroup of $A/\mathrm{Ker}(\xi_g)$. The fact that $A/\mathrm{Ker}(\xi_g)$ is G-quasifinite implies that $V/\mathrm{Ker}(\xi_g)$ is finite. Then $B \simeq V/\mathrm{Ker}(\xi_g)$ is finite. It follows that D is G-quasifinite. Since the intersection $D \cap C$ is a G-invariant subgroup of G, then we have two possibility: either $D \cap C = D$ or $D \cap C$ is finite. If we suppose that $D \cap C = D$, then

$$D = [A, g] \leqslant C.$$

But in this case $g \in C_G(A/C)$, what is impossible. Hence $D \cap C$ is finite. In this case $D/(D \cap C)$ is infinite. In other words, DC/C is an infinite G-invariant subgroup of A/C. The fact that A/C is G-quasifinite implies that $A/C = DC/C$ and $A = DC$. □

Corollary 2.9 *Let* G *be a group and* A *be a normal periodic subgroup of* G*. Suppose that* $G/C_G(A)$ *is hypercentral and* A *satisfies the following conditions:*

(i) $A/(A \cap \zeta_k(G))$ *is an abelian Chernikov group for some positive integer* k*;*

(ii) $A/(A \cap \zeta_k(G))$ *is* G*-quasifinite;*

(iii) $C_G(A/(A \cap \zeta_k(G)) \neq G$.

Then A *includes a* G*-invariant abelian subgroup* D *such that* D *is Chernikov and* G*-quasifinite,* $A = (A \cap \zeta_k(G))D$ *and the intersection*

$$(A \cap \zeta_k(G)) \cap D$$

is finite.

Corollary 2.10 *Let* G *be a group and* A *be a normal periodic subgroup of* G*. Suppose that* $G/C_G(A)$ *is hypercentral and* A *satisfies the following conditions:*

(i) $A/(A \cap \zeta_k(G))$ *is an abelian Chernikov group for some positive integer* k*;*

(ii) A *has a series of* G*-invariant subgroups*

$$A \cap \zeta_k(G) = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_m = A$$

in which the factors A_j/A_{j-1} *are* G*-quasifinite,* $1 \leqslant j \leqslant m$*;*

(iii) $C_G(A_j/A_{j-1}) \neq G$.

Then A *includes a* G*-invariant abelian subgroup* D *such that* D *is Chernikov,* $A = (A \cap \zeta_k(G))D$ *and the intersection* $(A \cap \zeta_k(G)) \cap D$ *is finite.*

Corollary 2.11 *Let* G *be a group and* A *be a normal divisible Chernikov subgroup of* G*. Suppose that* $G/C_G(A)$ *is hypercentral. Then* $A = C \times D$ *where* $C \leqslant \zeta_k(G)$ *for some positive integer* k *and* D *has a series of* G*-invariant subgroups*

$$\langle 1 \rangle = D_0 \leqslant D_1 \leqslant \ldots \leqslant D_m = D$$

in which the factors D_j/D_{j-1} *are* G*-quasifinite and* $C_G(D_j/D_{j-1}) \neq G$*, for* $1 \leqslant j \leqslant m$*.*

Let G be a group and A be a normal abelian torsion-free subgroup of G. A subgroup A is called G-*rationally irreducible* if for every non-trivial G-invariant subgroup B of A the factor A/B is periodic. A group G is said to have finite 0-rank $r_0(G) = r$ if G has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly r.

Lemma 2.12 *Let* G *be a group and* A *be a normal abelian subgroup of* G*. Suppose that* $G/C_G(A)$ *is a finite nilpotent group and* A *satisfies the following conditions:*

(i) $A \cap \zeta(G)$ *is a* p*-subgroup for some prime* p*;*

(ii) $A/(A \cap \zeta(G))$ *is torsion-free of finite* 0*-rank;*

(iii) $A/(A \cap \zeta(G))$ *is* G*-rationally irreducible;*

(iv) $C_G(A/(A \cap \zeta(G)) \neq G$.

Then A *includes a* G*-invariant abelian torsion-free subgroup* D *such that* D *is* G*-rationally irreducible,* $G \neq C_G(D)$ *and* $A/(A \cap \zeta(G))D$ *is a Chernikov* p*-subgroup.*

Proof — Put $C = A \cap \zeta(G)$. Choose in A/C a finitely generated subgroup B/C such that $(A/C)/(B/C)$ is periodic. Let X be a finitely generated subgroup of B such that $B = XC$. Put $Y = \langle X \rangle^G$. Since $G/C_G(A)$ is finite, a subgroup Y is finitely generated. Then $Y = (Y \cap C) \times W$ for some torsion-free subgroup W. Let $m = |Y \cap C|$, then $Y^m = W^m$. The subgroup $Z = Y^m$ is G-invariant. The inclusion $Z \leqslant W$ shows that Z is torsion-free. By the choice of Z we have $r_0(Z) = r_0(A/C)$. It follows that A/Z is periodic. Let D/Z be a Sylow p'-subgroup of A/Z. Then D is G-invariant and $D \cap C = \langle 1 \rangle$. It follows that D is torsion-free. By this choice $(A/C)/(D/C)$ is a p-group. Since A/C is a torsion-free group of finite 0-rank, its p-images are Chernikov. Finally, if we suppose that $C_G(D) = D$, then $DC/C \leqslant C_{A/C}(G)$. It follows that $C_{A/C}(G)$ is a non-trivial G-invariant subgroup of A/C. The fact that A/C is G-rationally irreducible implies that $(A/C)/C_{A/C}(G)$ is periodic. Since A/C is torsion-free, $A/C = C_{A/C}(G)$, and we obtain a contradiction. This contradiction proves that $C_G(D) \neq D$. It is not hard to prove that D is G-rationally irreducible. □

Lemma 2.13 *Let G be a group and A be a normal abelian torsion-free subgroup of G such that $G/C_G(A)$ is hypercentral. Suppose that A has a series of G-invariant pure subgroups*

$$\langle 1 \rangle = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = A$$

whose factors A_j/A_{j-1} are G-rationally irreducible. If

$$G = C_G(A_m/A_{m-1})$$

for some positive integer m, then $C_A(G) \neq \langle 1 \rangle$.

Proof — Put $H = G/C_G(A)$. We can consider A as a $\mathbb{Z}H$-module. Put

$$B = A \otimes_{\mathbb{Z}} \mathbb{Q}, \quad B_j = A_j \otimes_{\mathbb{Z}} \mathbb{Q},$$

for $1 \leqslant j \leqslant m$. Then B naturally is an $\mathbb{Q}H$-module and every factor B_j/B_{j-1} is a simple $\mathbb{Q}H$-module, $1 \leqslant j \leqslant m$. The equality

$$H = C_H(A_m/A_{m-1})$$

implies that $H = C_H(B_m/B_{m-1})$. Using Corollary 2.4 of paper [14] we obtain that $C_B(H)$ is non-trivial. It implies that $C_A(G)$ is also non-trivial. □

Lemma 2.14 *Let G be a group and A be a normal abelian subgroup of G. Suppose that $G/C_G(A)$ is abelian and A satisfies the following conditions:*

(i) $A \cap \zeta(G)$ *is periodic and* $A/(A \cap \zeta(G))$ *is torsion-free of finite* 0*-rank;*

(ii) $A/(A \cap \zeta(G))$ *is* G*-rationally irreducible;*

(iii) $C_G(A/(A \cap \zeta(G))) \neq G$.

Then A *includes a* G*-invariant abelian torsion-free subgroup* D *such that* D *is* G*-rationally irreducible and* $(A \cap \zeta(G)D$ *has finite index in* A*.*

Proof — Put $C = A \cap \zeta(G)$. Since the factor A/C is G-eccentric, there exists an element $g \notin C_G(A/C)$. The mapping

$$\xi_g : A \to A,$$

defined by the rule $\xi_g(a) = [a, g]$, is an endomorphism of A. Since $g \notin C_G(A)$,

$$\mathrm{Ker}(\xi_g) = C_A(g) \neq A.$$

The fact that $G/C_G(A)$ is abelian implies that $C_A(g)$ and $[A, g]$ are G-invariant subgroups. By our conditions we have the inclusion $C \leqslant \mathrm{Ker}(\xi_g)$. If we suppose that $\mathrm{Ker}(\xi_g) \neq C$, then $\mathrm{Ker}(\xi_g)/C$ is a non-trivial G-invariant subgroup of A/C. Since A/C is G-rationally irreducible, $(A/C)/(\mathrm{Ker}(\xi_g)/C)$ is periodic. Then the fact that A/C is torsion-free implies that $g \in C_G(A/C)$, and we obtain a contradiction. This contradiction shows that $\mathrm{Ker}(\xi_g) = C$. The isomorphism

$$[A, g] = \mathrm{Im}(\xi_g) \simeq A/\mathrm{Ker}(\xi_g)$$

shows that $D = [A, g]$ is a torsion-free abelian subgroup of A. Let B be a non-trivial G-invariant subgroup of D and let V be a preimage of B. Let $x \in G$, $v \in V$. Then

$$\xi_g(v^x) = [v^x, g] = [v^x, gz] = [v^x, g^x] = [v, g]^x \in B^x = B.$$

It follows that $v^x \in V$, so that V is G-invariant. Since B is non-trivial, $V/\mathrm{Ker}(\xi_g)$ is a non-trivial G-invariant subgroup of $A/\mathrm{Ker}(\xi_g)$. The fact that $A/\mathrm{Ker}(\xi_g)$ is G-rationally irreducible implies that A/V is periodic. Then for every element $a \in A$ there is a positive integer t such that $a^t \in V$. We have now $[a, g]^t = [a^t, g] \in B$. It follows that D/B is periodic. In other words, D is G-rationally irreducible. The isomorphism $D \simeq A/C$ and equality $D \cap C = \langle 1 \rangle$ imply that

$$DC/C \simeq A/C.$$

It follows that DC/C has finite index in A/C (see [3], Theorem 2). Thus the index $|A : DC|$ is finite. □

Corollary 2.15 *Let* G *be a group and* A *be a normal abelian subgroup of* G*. Suppose that* $G/C_G(A)$ *is abelian-by-finite and* A *satisfies the follow-*

ing conditions:

(i) $A \cap \zeta(G)$ *is periodic and* $A/(A \cap \zeta(G))$ *is torsion-free of finite* 0*-rank;*

(ii) $A/(A \cap \zeta(G))$ *is* G*-rationally irreducible;*

(iii) $G/C_G(A/(A \cap \zeta(G))$ *is infinite.*

Then A *includes a* G*-invariant abelian torsion-free subgroup* D *such that* D *is* G*-rationally irreducible and* $(A \cap \zeta(G))D$ *has finite index in* A.

Proof — Let $H/C_G(A)$ be a normal abelian subgroup of $G/C_G(A)$, having finite index. Put $C = A \cap \zeta(G)$. Let

$$\mathfrak{S} = \{X \mid C \leqslant X \leqslant A,\ X/C \text{ is a pure } H\text{-invariant subgroup of } A/C\}.$$

Since $r_0(A)$ is finite, we can choose in $\mathfrak{S}$ a subgroup B_1 such that $r_0(B_1) \leqslant r_0(X)$ for every subgroup $X \in \mathfrak{S}$. By this choice B_1/C is an H-rationally irreducible subgroup. If we suppose that $H = C_H(B_1/C)$, then $C_{A/C}(H) \neq \langle 1 \rangle$. The fact that H is normal in G implies that $C_{A/C}(H)$ is a G-invariant subgroup of A/C. Then the factor $(A/C)/C_{A/C}(H)$ is periodic. On the other hand, A/C is torsion-free, and so it follows that $A/C = C_{A/C}(H)$. But in this case $G/C_G(A/C)$ is finite, and we obtain a contradiction. Using similar arguments, we find in A an H-invariant pure subgroup B_2 such that $B_1 \leqslant B_2$ and the factor B_2/B_1 is H-rationally irreducible. If we suppose that

$$H = C_H(B_2/B_1),$$

then using Lemma 2.13 we obtain that $C_{A/C}(H) \neq \langle 1 \rangle$. Now with the help of the above arguments we obtain a contradiction, which proves that $H \neq C_H(B_2/B_1)$.

Using the same arguments, after finitely many steps, we construct a series

$$\langle 1 \rangle = B_0 \leqslant B_1 \leqslant \ldots \leqslant B_n = A$$

of G-invariant pure subgroups, whose factors are H-rationally irreducible and H-eccentric. Using Lemma 2.14, after finitely many steps, we obtain an H-invariant torsion-free subgroup K such that CK has finite index. Let $t = |A/CK|$ and put $E/C = (A/C)^t$. Then E is a G-invariant subgroup and $E = C \times L$ where $L = E \cap K$. There exists a G-invariant subgroup U of E such that $(U \cap C)^s = \langle 1 \rangle$ and $E^s \leqslant UC$, where $s = |G/H|$ (see [15], Theorem 5.9). In particular, the factor A/UC is a bounded group. On the other hand, A/C has finite 0-rank, which follows that A/UC is finite. Finally, $\mathrm{Tor}(U) = U \cap C$ is also bounded subgroup, so that $U = (U \cap C) \times V$ for some torsion-free subgroup V (see, for example, [9], Theorem 27.5). We have now $U^s = V^s$, in particular, $D = U^s$ is a G-invariant torsion-free subgroup. Furthermore, the fact that V has finite 0-rank implies that V/V^s is finite, which follows that $U/(U \cap C)D$ is finite. In turn out, it follows that UC/DC is finite, so that A/DC is also finite. □

Corollary 2.16 *Let G be a group and A be a normal abelian subgroup of G. Suppose that $G/C_G(A)$ is abelian-by-finite and A satisfies the following conditions:*

(i) $A \cap \zeta_k(G)$ *is periodic and* $A/(A \cap \zeta_k(G))$ *is torsion-free of finite* 0*-rank for some positive integer* k*;*

(ii) $A/(A \cap \zeta_k(G))$ *is* G*-rationally irreducible;*

(iii) $G/C_G(A/(A \cap \zeta_k(G))$ *is infinite.*

Then A includes a G-invariant abelian torsion-free subgroup D such that D is G-rationally irreducible and $(A \cap \zeta_k(G))D$ has finite index in A.

Lemma 2.17 *Let G be a group and A be a normal abelian torsion-free subgroup of G. Suppose that $G/C_G(A)$ is locally soluble. If A is G-rationally irreducible and $r_0(A)$ is finite, then $G/C_G(A)$ is abelian-by-finite.*

Proof — Put $H = G/C_G(A)$. We can consider A as a $\mathbb{Z}H$-module. Put $B = A \otimes_{\mathbb{Z}} \mathbb{Q}$, then B naturally is an $\mathbb{Q}H$-module, moreover, B is a simple $\mathbb{Q}H$-module. Since $r_0(A) = n$ is finite, $\dim_{\mathbb{Q}}(B) = n$ is finite. Thus we can consider H as an irreducible subgroup of $GL_n(\mathbb{Q})$. Being locally soluble, H is soluble (see, for example, [21], Corollary 3.8). Being irreducible, H is abelian-by-finite (see, for example, [21], Lemma 3.5). □

By Lemma 2.12, Corollary 2.16 and Lemma 2.17 we obtain

Corollary 2.18 *Let G be a group and A be a normal abelian subgroup of G. Suppose that $G/C_G(A)$ is abelian-by-finite and A satisfies the following conditions:*

(i) *there exists a positive integer k such that $A \cap \zeta_k(G)$ is p-subgroup for some prime p and $A/(A \cap \zeta_k(G))$ is torsion-free group of finite* 0*-rank;*

(ii) A *has a series of* G*-invariant subgroups*

$$A \cap \zeta_k(G) = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_m = A$$

in which the factors A_j/A_{j-1} *are torsion-free and* G*-quasifinite, for* $1 \leqslant j \leqslant m$*;*

(iii) $C_G(A_j/A_{j-1}) \neq G$*, for* $1 \leqslant j \leqslant m$*;*

Then A *includes a* G*-invariant abelian torsion-free subgroup* D *such that* $A/(A \cap \zeta_k(G))D$ *is a Chernikov* p*-subgroup.*

3 Proof of the Theorem

Let

$$\langle 1 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots \leqslant Z_{k-1} \leqslant Z_k = Z$$

be a segment of the upper central series of G. Since G/Z has finite section p-rank, then G includes normal subgroups P, L, D such that $Z \leqslant P \leqslant L \leqslant D$, P/Z is locally finite, L/P is a nilpotent torsion-free subgroup of finite 0-rank, D/L is a finitely generated torsion-free abelian group and G/D is finite (see, for example, [6], Theorem 3.6).

Put $C = C_G(Z)$. Then G/C is nilpotent (see [13]). The center of C includes $C \cap Z$, so that $C/(C \cap Z)$ has finite section p-rank. Then $D = [C, C]$ has finite section p-rank, moreover, there exists a function f_2 such that $sr_p(D) \leqslant f_2(r)$. The factor C/D is abelian, so that

$$C/D \leqslant C_{G/D}(C/D).$$

On the other hand, G/C is nilpotent, so that $(G/D)/(C/D)$ is nilpotent. In particular, $(G/D)/C_{G/D}(C/D)$ is nilpotent. Further we will consider the factor-group G/D, therefore without loss of generality we may assume that $D = \langle 1 \rangle$. In other words, we can suppose that C is abelian and G/C is nilpotent. Put $T = Tor(C)$ and let Q be the Sylow p'-subgroup of T. As above without loss of generality we can suppose that $Q = \langle 1 \rangle$. In other words, we can assume that T is a p-subgroup. The factor-group C/T is abelian and torsion-free. It is not hard to prove that $\zeta_j(G/T) \cap C/T$ is a pure subgroup of C/T for each positive integer j. On the other hand, $ZT/T \leqslant \zeta_k(G/T)$, which shows that $(C/T)/(\zeta_k(G/T) \cap C/T)$ is a torsion-free abelian group having section p-rank at most r. Put

$$Y/T = \zeta_k(G/T) \cap C/T \text{ and } X = T \cap \zeta_k(G).$$

Then T/X is a p-group having finite section p-rank at most r. It follows that T/X is a Chernikov group. Let U/X be the divisible part of T/X. By Corollary 2.11 $U/X = E/X \times W/X$ where E has a series of G-invariant subgroups

$$X = E_0 \leqslant E_1 \leqslant \ldots \leqslant E_s = E,$$

in which the factors E_j/E_{j-1} are G-quasifinite and G-eccentric, for $1 \leqslant j \leqslant s$, and $W/X \leqslant \zeta_r(G/X)$. By Corollary 2.10 E includes a G-invariant subgroup E_1 such that $E = XE_1$ and the intersection $X \cap E_1$ is finite. It implies the isomorphism $E_1 \simeq E/X$, which shows that E_1 has a section p-rank at most r. Since further we will consider the factor-group G/E_1, as above without loss of generality we may suppose that $E_1 = \langle 1 \rangle$. In other words, we can assume that $T/(T \cap \zeta_{k+r}(G))$ is finite. Using now Corollary 2.6 of the paper [14] we obtain that T includes a finite G-invariant subgroup E_2 such that $T/E_2 \leqslant \zeta_{k+r}(G/E_2)$. Again we can assume that $E_2 = \langle 1 \rangle$. In other words, further we will suppose that $T \leqslant \zeta_{k+r}(G)$.

Since $G/C_G(C)$ is nilpotent, Proposition 3.7 of the paper [7] shows that C has a series of G-invariant subgroups

$$\langle 1 \rangle = C_0 = T \leqslant C_1 \leqslant \ldots \leqslant C_s \leqslant C$$

such that the factors C_j/C_{j-1} are torsion-free of finite 0-rank, rationally irreducible and G-eccentric, where $1 \leqslant j \leqslant s \leqslant r$ and also $C/C_s \leqslant \zeta_{k+r}(G/C_s)$. Using Corollary 2.18 we obtain that C_s includes a G-invariant torsion-free subgroup E_3 such that $r_0(E_3) \leqslant r$, C_s/TE_3 is a Chernikov p-group. Above we have already considered the similar case. Using the above arguments we obtain that C_s includes a G-invariant subgroup $E_4 \geqslant E_3$ such that $sr_p(E_4/E_3) \leqslant 2r$, and $C_s/E_4 \leqslant \zeta_j(G/E_4)$ for some positive integer j. Then

$$C/E_4 \leqslant \zeta_{k+r+j}(G/E_4).$$

Since G/C is nilpotent, G/E_4 is likewise nilpotent. Put $L = E_4$. Finally, we note that $sr_p(L) \leqslant f_2(r) + 5r = g(r)$. □

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