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Corrigendum to: "Characterizations of Fitting p-Groups whose Proper Subgroups are Solvable"

A.O. Asar

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Abstract

The paper entitled *Characterizations of Fitting* p*-Groups whose Proper Subgroups are Solvable* (Adv. Group Theory Appl. 3 (2017), 31-53) contains a serious error. The proof of Lemma 2.8 relating to $p = 3$ is false. This forces a slight change in the implications of Theorem 1.1 and Corollary 1.2 for $p = 3$. The new statements of Theorem 1.1 and Corollary 1.2 are stated below (Theorem 1.3 is correct).

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1 Introduction

For $p \neq 2$, a p-group G is called **powerful** by Lubotzky and Mann if $G' \leqslant \mho_1(G)$.

Theorem 1.1 *Let* G *be a Fitting* p*-group satisfying the normalizer condition, where* $p \neq 2$. *Suppose that in every homomorphic image* H *of* G *every* Λ*-pair* (w_H, V_H) *has a* (w_H, V_H)*-maximal subgroup satisfying the* $(**)$*-property. Then either* G *is not perfect or every homomorphic image* H *of* G *contains a normal nilpotent metabelian* 3*-subgroup* B_H *of class* 3 *and exponent* 9 *which is not powerful* (*that is,* $B_H' \nleqslant \mho_1(B_H)$).

Thus G cannot be perfect for $p > 3$.

Corollary 1.2 *Let* G *be a Fitting* p*-group satisfying the normalizer condition in which every proper subgroup is solvable, where* $p \neq 2$. *Suppose that in every homomorphic image* H *of* G *every dominant pair* (w_H, V_H) *has a* (w_H, V_H)*-maximal subgroup satisfying the* $(**)$*-property. Then either* G *is solvable or every homomorphic image* H *of* G *contains a normal nilpotent metabelian 3-subgroup* B_H *of class 3 and exponent 9 which is not powerful.*

2 Proofs

Lemma 2.1 (2.8′) *Let* G *be a Fitting 3-group. Suppose that the hypothesis of Lemma 2.7 is satisfied in* G. *Thus* B *is a normal nilpotent metabelian subgroup of* G *with* $c(B) \leqslant 3$ *and* B *contains a normal abelian subgroup* A *of* G *with* B/A *is elementary abelian. Then the following hold.*

(a) $\overline{D} \neq 1$.

(b) $\exp(\overline{D}) = 3$.

(c) $|\overline{D}| = 3$.

Proof — The notations are the same as in Lemma 2.5. By the hypothesis

$$B \not\leqslant N = N_G(E)$$

since B is metabelian, $t \in B \setminus N$, $N^t = N$, $t^3 \in N$, $\overline{H} = H/D^*$ and $\overline{T} = \langle \overline{t} \rangle \overline{R}$. If $c(\overline{T}) < 3$, then $\overline{T}$ is abelian by the proof of Lemma 2.7, which is a contradiction since $C_{\overline{T}}(\overline{R}) = \overline{R}$ by Lemma 2.5 (a). Therefore

$$c(\overline{T}) = c = 3.$$

Furthermore $\exp([\overline{R}, \overline{t}]) = 3$ by Lemma 2.6. Finally assume if possible that $\overline{R}$ is infinite. Then $\overline{R} = \overline{R}^o \times \overline{D}$ by Lemma 2.5 (c). Also $\overline{R}^o \lhd \overline{N}$ since $\overline{R} \lhd \overline{N}$. This implies that $\overline{R}^o \leqslant Z(\overline{N})$ since G is Fitting.

(a) Assume if possible that $\overline{D} = 1$. First assume that $\overline{R}$ is infinite. Then

$$\overline{R} = \overline{R}^o \times \overline{D} = \overline{R}^o$$

by Lemma 2.5 (b) and $\overline{R}^o \lhd \overline{T}$. But then $\overline{R}^o \leqslant Z(\overline{T})$ since $\overline{T}$ is nilpotent, which is a contradiction since $\overline{T}$ is not abelian. Next suppose that $\overline{R} = \langle \overline{r} \rangle$ for an $r \in R$ with $|\overline{r}| = 3^n$ and $n > 1$. Then $\overline{T} = \langle \overline{t} \rangle \langle \overline{r} \rangle$. Also $\overline{r}^{\overline{t}} \neq \overline{r}$. Then $\overline{r}^{\overline{t}} = \overline{r}^{1+3^{n-1}}$ and hence $[\overline{r}, \overline{t}] = \overline{r}^{3^{n-1}}$ by [6], Corollary 4.2 (ii). This shows that $[\overline{r}, \overline{t}]^3 = 1$ and so $[\overline{r}, \overline{t}] \in Z(\overline{T})$ since $\langle \overline{r} \rangle \lhd \overline{T}$. But then $c \leqslant 2$, which is another contradiction. Therefore $\overline{D} \neq 1$.

(b) Let $d \in D$ and put $\overline{y} = \overline{d d^{t} d^{t^2}}$. Then $\overline{y} \in Z(\overline{T})$. Also it is easy to see that

$$\overline{y} = \overline{d}^3 [\overline{d}, \overline{t}]^3 [\overline{d}, \overline{t}, \overline{t}] = \overline{d}^3 [\overline{d}, \overline{t}, \overline{t}]$$

since $exp([\overline{R}, \overline{t}]) = 3$. Moreover $[\overline{d}, \overline{t}, \overline{t}] \in Z(\overline{T})$ since $c = 3$. Therefore $\overline{d}^3 \in Z(\overline{T})$. But since $Z(\overline{T}) \cap \overline{D} = 1$ it follows that $\overline{d}^3 = 1$. Since d is any element of D it follows that $exp(\overline{D}) = 3$.

(c) We have $exp(\overline{D}) = 3$. Now $\overline{D}^t \leqslant \overline{R}$ and since $\overline{R}/\overline{D}$ is (locally) cyclic it follows that

$$\overline{D}^t \overline{D}/\overline{D} \cong \overline{D}^t / (\overline{D}^t \cap \overline{D})$$

is cyclic and so has order $\leqslant 3$. The same thing holds for $\overline{D}^{t^2} / (\overline{D}^{t^2} \cap \overline{D})$. These imply that each of

$$\overline{D}/(\overline{D}^t \cap \overline{D}) \text{ and } \overline{D}/(\overline{D}^{t^2} \cap \overline{D})$$

have orders $\leqslant 3$. Therefore

$$|\overline{D}/(\overline{D} \cap \overline{D}^t \cap \overline{D}^{t^2})| \leqslant 3^2.$$

But since $\mathrm{Core}_{\overline{H}}(\overline{D}) = 1$, it follows that $|\overline{D}| \leqslant 3^2$.

Now assume if possible that $|\overline{D}| = 3^2$. Then

$$\overline{D} = \langle \overline{u} \rangle \times \langle \overline{v} \rangle \text{ and } |\overline{u}| = |\overline{v}| = 3$$

by (b). Let $Z(\overline{G}) = \langle \overline{z} \rangle$. Then $\overline{z} \in Z(\overline{T})$. Hence

$$[\overline{u}, \overline{t}, \overline{t}], [\overline{v}, \overline{t}, \overline{t}] \in \langle \overline{z_1} \rangle$$

since $c = 3$, where $z_1 \in \langle z \rangle$ and $|\overline{z_1}| = 3$.

First suppose that $[\overline{u}, \overline{t}], [\overline{v}, \overline{t}] \in Z(\overline{T})$. Then since $Z(\overline{T})$ is (locally) cyclic, by replacing $\overline{v}$ by a suitable power of itself we may suppose that $[\overline{v}, \overline{t}] = [\overline{u}, \overline{t}]$ and then

$$[\overline{v}, \overline{t}]([\overline{u}, \overline{t}])^{-1} = [\overline{vu^{-1}}, \overline{t}] = 1.$$

This gives $\overline{vu}^{-1} \in Z(\overline{T})$, which is a contradiction since $\overline{T} \cap \overline{D} = 1$. Hence without loss of generality we may suppose that $[\overline{u}, \overline{t}] \notin Z(\overline{T})$. Now if $[\overline{v}, \overline{t}] \in Z(\overline{T})$, then $[\overline{uv}, \overline{t}] \notin Z(\overline{T})$. Therefore we may also suppose that $[\overline{v}, \overline{t}] \notin Z(\overline{T})$.

Let $e \in E$. Then $[\overline{e}, \overline{t}] = \overline{r}^i \overline{d}^j$ for an $r \in R$ with $\langle \overline{r} \rangle \cap \overline{D} = 1$ and $i, j \geqslant 1$ (here $\overline{R}$ may be infinite). Also

$$1 = [\overline{r}, \overline{t}]^3 = [\overline{r}^3, \overline{t}]$$

since $\overline{R} \leqslant Z(\overline{N})$ by Lemma 2.5 (a) and $exp([[\overline{R}, \overline{t}]) = 3$, which implies that $\overline{r}^3 \in Z(\overline{T})$. Thus

$$[\overline{e}^3, \overline{t}] = [\overline{e}, \overline{t}]^3 = (\overline{r}^i \overline{d}^j)^3 = \overline{r}^{3i} \in Z(\overline{T})$$

and hence

$$[\overline{e}^3, \overline{t}] \in Z(\overline{T}) \tag{1}$$

Next

$$1 = [\overline{e}, \overline{t}^3] = [\overline{e}, \overline{t}^2][\overline{e}, \overline{t}]^{\overline{t}^2} = [\overline{e}, \overline{t}]^3 [\overline{e}, \overline{t}, \overline{t}][\overline{e}, \overline{t}, \overline{t}^2]$$

since $[\overline{e}, \overline{t}] \in \overline{R} \leqslant Z(\overline{N})$. Hence

$$1 = ([\overline{e}, \overline{t}]^3 [\overline{e}, \overline{t}, \overline{t}][\overline{e}, \overline{t}, \overline{t}^2])^3 = [\overline{e}, \overline{t}]^9$$

since $exp([\overline{R}, \overline{t}]) = 3$ by Lemma 2.6 of [4]. Since e is any element of E, it follows that

$$[\overline{E}^9, \overline{t}] = [\overline{E}, \overline{t}]^9 = 1 \tag{2}$$

We have $\overline{RE}/\overline{E}$ is (locally) cyclic. First suppose that $\overline{EE}^{\overline{t}}/\overline{E}$ has order 3. Then $\overline{E}^{\overline{t}} \leqslant \langle \overline{z} \rangle \overline{E}$ and in this case $\overline{t}$ normalizes $\langle \overline{z} \rangle \overline{E}$ since then

$$\overline{E}^{\overline{t}^2} \leqslant \langle \overline{z} \rangle \overline{E}^{\overline{t}} \leqslant \langle \overline{z} \rangle \overline{E},$$

which is impossible by $(**)$. Therefore $\overline{EE}^{\overline{t}}/\overline{E}$ and $\overline{EE}^{\overline{t}^2}/\overline{E}$ are (locally) cyclic and

$$|\overline{EE}^{\overline{t}}/\overline{E}| = |\overline{EE}^{\overline{t}^2}/\overline{E}| \geqslant 9. \tag{3}$$

Also (3) holds if $\overline{E}$ is exchanged with $\overline{E}^{\overline{t}}$ or replaced $\overline{E}^{\overline{t}^2}$ since

$$\overline{RE} = \overline{RE}^{\overline{t}} = \overline{RE}^{\overline{t}^2}.$$

Put

$$\overline{K} = \overline{E} \cap \overline{E}^{\overline{t}} \cap \overline{E}^{\overline{t}^2}.$$

Then $\overline{K} \lhd \overline{H}$ and $\overline{E}/\overline{K}$ is abelian. In particular $[\overline{K}, \overline{t}] = 1$ since $\overline{D} \cap \overline{K} = 1$. Furthermore $\overline{E}^9 \leqslant \overline{K}$ by (2) and so $exp(\overline{E}/\overline{K}) \leqslant 9$. Clearly then

$$exp(\overline{E}/\overline{K}) = 9 \tag{4}$$

by (3). Now (3) together (4) gives

$$\overline{E}/(\overline{E} \cap \overline{E}^{\overline{t}}) \text{ and } \overline{E}/(\overline{E} \cap \overline{E}^{\overline{t}^2}) \text{ are cyclic of order } 9 \tag{5}$$

Furthermore

$$\overline{E}/\overline{K} \precsim \overline{E}/(\overline{E} \cap \overline{E}^{\overline{t}}) \times \overline{E}/(\overline{E} \cap \overline{E}^{\overline{t^2}}) \tag{6}$$

Thus it follows from (5) and (6) that

$$\Omega_1(\overline{E}/\overline{K}) = \overline{DK}/\overline{K} \tag{7}$$

since $\overline{K} \cap \overline{D} = 1$ and $\overline{D} = \langle \overline{u} \rangle \times \langle \overline{v} \rangle$.

Furthermore

$$(\overline{E} \cap \overline{E}^{\overline{t}})(\overline{E} \cap \overline{E}^{\overline{t}^2})/(\overline{E} \cap \overline{E}^{\overline{t}^2})$$

$$\cong (\overline{E} \cap \overline{E}^{\overline{t}})/(\overline{E} \cap \overline{E}^{\overline{t}} \cap \overline{E} \cap \overline{E}^{\overline{t}}) = (\overline{E} \cap \overline{E}^{\overline{t}})/\overline{K}$$

and so

$$(\overline{E} \cap \overline{E}^{\overline{t}})/\overline{K} \text{ is cyclic of order } \leqslant 9 \tag{8}$$

by (5).

Next let $\overline{y} \in \overline{E} \cap \overline{E}^{\overline{t}}$. Then $\overline{y} = \overline{f}^{\overline{t}}$ for an $f \in E$. Substituting this in (1) gives

$$[\overline{f}^3, \overline{t}]^{\overline{t}} = [(\overline{f}^3)^{\overline{t}}, \overline{t}] \in Z(\overline{T})$$

and then

$$[\overline{f}^3, \overline{t}] = [\overline{f}, \overline{t}]^3 \in Z(\overline{T})$$

But also $\bar{f}^{-1}\bar{y} = [\bar{f}, \bar{t}]$. Therefore we get $(\bar{f}^{-1}\bar{y})^3 \in Z(\bar{T}) \cap \bar{E} = 1$. This implies that

$$(\bar{f}^{-1}\overline{yK})^3 = 1 \text{ and hence } (\overline{yK})^3 = (\overline{fK})^3$$

since $\bar{E}/\bar{K}$ is abelian by (5). Now since $\bar{y} = \bar{f}^{\bar{t}}$, it follows that

$$(\overline{fK})^3 = (\overline{yK})^3 = (\bar{f}^{\bar{t}}\bar{K})^3 = ((\overline{fK})^3)^{\bar{t}}$$

which implies that $[\bar{y}^3\bar{K}, \overline{tK}] = 1$. Then also $\bar{y}^3 \in \bar{E}^{\bar{t}} \cap \bar{E}^{\bar{t}^2}$ and so it follows that $\bar{y}^3 \in \bar{K}$. Hence it follows that $\bar{E} \cap \bar{E}^{\bar{t}}/\bar{K}$ is cyclic of order $\leqslant 3$ by (7). But since $\overline{DE}^{\bar{t}}/\bar{E}^{\bar{t}}$ is cyclic and $\bar{D} = \langle\bar{u}\rangle \times \langle\bar{v}\rangle$, we have $\bar{D} \cap \bar{E}^{\bar{t}} \neq 1$ but $\bar{D} \cap \bar{K} = 1$ due to the fact that $\bar{K}$ is normalized by $\bar{t}$. Therefore

$$\bar{E} \cap \bar{E}^{\bar{t}}/\bar{K} \text{ is cyclic of order } 3 \qquad (9)$$

Next without loss of generality we may suppose that $\bar{u} \notin \bar{E}^{\bar{t}}$ and $\bar{v} \in \bar{E} \cap \bar{E}^{\bar{t}}$ since $\bar{D} = \langle\bar{u}\rangle \times \langle\bar{v}\rangle$. Then $\bar{v} \notin \bar{E} \cap \bar{E}^{\bar{t}^2}$ since $\bar{D} \cap \bar{K} = 1$. Thus

$$|\bar{u}(\bar{E} \cap \bar{E}^{\bar{t}})| = 3 = |\bar{v}(\bar{E} \cap \bar{E}^{\bar{t}^2})|.$$

Next there are $e_u, e_v \in \bar{E}$ so that

$$\bar{E}/(\bar{E} \cap \bar{E}^{\bar{t}}) = \langle\overline{e_u}(\bar{E} \cap \bar{E}^{\bar{t}})\rangle$$

$$\bar{E}/(\bar{E} \cap \bar{E}^{\bar{t}^2}) = \langle\overline{e_v}(\bar{E} \cap \bar{E}^{\bar{t}^2})\rangle$$

and

$$|\overline{e_u}(\bar{E} \cap \bar{E}^{\bar{t}})| = |\overline{e_v}(\bar{E} \cap \bar{E}^{\bar{t}})| = 9$$

since $\bar{E}/(\bar{E} \cap \bar{E}^{\bar{t}})$ and $\bar{E}/(\bar{E} \cap \bar{E}^{\bar{t}^2})$ are cyclic of order 9 by (5). Then

$$\bar{u}(\bar{E} \cap \bar{E}^{\bar{t}}) = \overline{e_u}^3(\bar{E} \cap \bar{E}^{\bar{t}})$$

and $\bar{v}(\bar{E} \cap \bar{E}^{\bar{t}^2}) = \overline{e_v}^3(\bar{E} \cap \bar{E}^{\bar{t}^2})$. Thus $\bar{E}/\bar{K} = \langle\overline{e_u}\bar{K}, \overline{vK}\rangle$ since $\bar{v} \in \bar{E} \cap \bar{E}^{\bar{t}}$. Hence it follows that $\langle\overline{uK}, \overline{vK}\rangle = \langle\overline{e_u}^3\bar{K}, \overline{vK}\rangle$ since $\Omega_1(\bar{E}/\bar{K}) = \overline{DK}/\bar{K}$

by (7). Similarly if

$$\overline{d} \in \overline{D} \cap (\overline{E} \cap \overline{E}^{\overline{t}^2}),$$

then $\overline{E}/\overline{K} = \langle \overline{e_v}\overline{K}, \overline{dK} \rangle$ and hence $\langle \overline{u}\overline{K}, \overline{v}\overline{K} \rangle = \langle \overline{e_v}^3\overline{K}, \overline{dK} \rangle$.

Finally $\overline{t}$ normalizes $\Omega_1(\overline{R}) = \langle \overline{z_1}, \overline{u}, \overline{v} \rangle$ and $[\overline{u}, \overline{t}], [\overline{v}, \overline{t}] \notin \langle \overline{z_1} \rangle$ by our assumption. But $[\overline{u}, \overline{t}, \overline{t}], [\overline{v}, \overline{t}, \overline{t}] \in \langle \overline{z_1} \rangle$ since $c(B) = 3$. It follows from this that either $[\overline{uv}, \overline{t}] \in \langle \overline{z_1} \rangle$ or $[\overline{uv^2}, \overline{t}] \in \langle \overline{z_1} \rangle$ but not both of them. Without loss of generality let $[\overline{uv}, \overline{t}] \in \langle \overline{z_1} \rangle$. But also

$$[\overline{e_u}^3, \overline{t}], [\overline{e_v}^3, \overline{t}] \in \langle \overline{z_1} \rangle$$

by (1). Therefore we must have $\overline{e_u}^3\overline{K}, \overline{e_v}^3\overline{K} \in \langle \overline{uv}\overline{K} \rangle$. This implies that

$$\overline{e_v}^3\overline{K} = (\overline{e_u}^3\overline{K})^i$$

for an $i = 1, 2$. But since

$$\overline{e_v}^3(\overline{E} \cap \overline{E}^{\overline{t}^2}) = \overline{v}(\overline{E} \cap \overline{E}^{\overline{t}^2}) \neq \overline{u}(\overline{E} \cap \overline{E}^{\overline{t}}) = \overline{e_u}^3(\overline{E} \cap \overline{E}^{\overline{t}})$$

by (6) and $\overline{K} = (\overline{E} \cap \overline{E}^{\overline{t}}) \cap (\overline{E} \cap \overline{E}^{\overline{t}^2})$, this cannot happen. Therefore the assumption is false and so $|\overline{D}| = 3$ must be the case. This completes the proof of the lemma. □

Lemma 2.2 (2.9′) *Let* G *be a Fitting* 3*-group. Suppose that the hypothesis of Lemma 2.5 is satisfied in* G*. Thus* B *is a normal nilpotent metabelian subgroup of* G *and contains a normal abelian subgroup* A *of* G *with* B/A *is elementary abelian and* $A \leqslant N$*. Assume that* $c(B) = 3$*. Then*

$$\exp(B) = \exp(B')^2 = \exp(Z(B))^2 = \exp(B/Z(B))^2 = 9.$$

Proof — Clearly B is not abelian since $c(B) = 3$. Then

$$B \not\leqslant N, \; Z(G) \leqslant A \leqslant N, \; t \in B \setminus N, \; N^t = N, \; t^3 \in N,$$

$$\overline{H} = H/D^*, \; \overline{R} \leqslant Z(\overline{N}) \text{ and } \overline{T} = \langle \overline{t} \rangle \overline{R}.$$

Also $\overline{T}$ is not abelian since $C_{\overline{T}}(\overline{R}) = \overline{R}$ and then $c(\overline{T}) = 3$ by the proof of Lemma 2.7. Put $c = c(\overline{T})$. Then $c = c(B) = 3$. Furthermore $\exp([\overline{R}, \overline{t}]) \leqslant 3$ by Lemma 2.6 and $\overline{D} = \langle \overline{u} \rangle$ by Lemma 2.8′ (c), where $|\overline{u}| = 3$.

Assume if possible that $\overline{R}$ is infinite. Then $\overline{R} = \overline{R}^o \times \langle \overline{u} \rangle$ by Lemma 2.5 (b) and then $\overline{R}^o \leqslant Z(\overline{T})$ since $\overline{T}$ is nilpotent. But then $c(\overline{T}) \leqslant 2$ which is impossible since $c = 3$. Therefore $\overline{R}$ is finite and so

$$\overline{R} = \langle \overline{b} \rangle \times \langle \overline{u} \rangle$$

by Lemma 2.8′ (c).

Clearly if $\overline{b} \in Z(\overline{T})$, then $c(\overline{T}) \leqslant 2$ since then $\overline{R}\langle \overline{b} \rangle = \langle \overline{u} \rangle / \langle \overline{b} \rangle$ and so $[\overline{u}, \overline{t}] \in \langle \overline{b} \rangle$, which is impossible since $c = 3$. Therefore $\overline{b} \notin Z(\overline{T})$. In particular $\overline{b}^3 \neq 1$ since in the contrary case

$$|\overline{b}| = |\overline{a}| = |\overline{z}|$$

in Lemma 2.5 (a) and then $\overline{RE} = \langle \overline{z} \rangle \overline{E}$. But now, as

$$[\langle \overline{z} \rangle \overline{E}, \overline{t}] \leqslant \overline{B} \cap \overline{N} = \overline{R} \leqslant \langle \overline{z} \rangle \overline{E},$$

it follows that $\overline{t}$ normalizes $\langle \overline{z} \rangle \overline{E}$, which gives a contradiction by $(**)$. Consequently it follows that

$$|\overline{b}| \geqslant 3^2 \text{ and } \overline{b} \notin Z(\overline{T}) \tag{1}$$

Since $exp([\overline{R}, \overline{t}]) = 3$ and $\overline{R} \leqslant Z(\overline{N})$, it follows that $[\overline{b}^3, \overline{t}] = [\overline{b}, \overline{t}]^3 = 1$ and so $\overline{b}^3 \in Z(\overline{T})$ and then $Z(\overline{T}) = \langle \overline{b}^3 \rangle$. This follows since

$$Z(\overline{T}) \cap \overline{E} = 1 \text{ and } Z(\overline{T})\overline{E}/\overline{E} < \langle \overline{b} \rangle \overline{E}/\overline{E}.$$

Notice that if

$$Z(\overline{T})\overline{E}/\overline{E} = \langle \overline{b} \rangle \overline{E}/\overline{E},$$

then $\overline{b}\overline{u}^k \in Z(\overline{T})$ for a $k \geqslant 1$. But then we may replace $\overline{b}$ by $\overline{b}\overline{u}^k$ which is impossible. In particular $\overline{t}^3 \in \langle \overline{b}^3 \rangle$.

We have seen above that

$$[\overline{u}, \overline{t}] \in \langle \overline{z} \rangle \tag{2}$$

and $[\overline{u}, \overline{t}] \neq 1$ since $Z(\overline{T}) \cap \overline{E} = 1$, where $z \in Z(G)$ with $|z| = 3$.

Next let $e \in E$. Then

$$1 = [\overline{e}, \overline{t}^3] = [\overline{e}, \overline{t}]^3 [\overline{e}, \overline{t}, \overline{t}]^3 [\overline{e}, \overline{t}, \overline{t}, \overline{t}]$$

$$= [\overline{e}, \overline{t}]^3 [\overline{e}, \overline{t}, \overline{t}, \overline{t}]$$

and hence

$$[\overline{e},\overline{t}]^3 = [\overline{e},\overline{t},\overline{t},\overline{t}]^{-1} \in Z(\overline{T}) \tag{3}$$

since $\exp([\overline{R},\overline{t}]) = 3$, $[\overline{e},\overline{t}] \in \overline{R}$ and $c = 3$. Clearly then $([\overline{e},\overline{t}]^3)^3 = 1$. If $[\overline{e},\overline{t}]^3 = 1$ for every $e \in E$, then since

$$[\overline{E},\overline{t}]\overline{E}/\overline{E} \leqslant \langle\overline{z}\rangle\overline{E}/\overline{E},$$

it follows that $[\overline{E},\overline{t}] \leqslant \langle\overline{z}\rangle\overline{E}$, which gives a contradiction as before. Therefore $[\overline{e},\overline{t}]^3 \neq 1$ for an $e \in E$ and hence

$$|[\overline{e},\overline{t}]| = 9. \tag{4}$$

On the other hand $[\overline{e},\overline{t}] = \overline{b}^i\overline{u}^j$ for some $i, j \geqslant 1$ since $\overline{R} = \langle\overline{b}\rangle \times \langle\overline{u}\rangle$. Hence $\overline{b}^{9i} = 1$ since $\overline{u}^3 = 1$ and $[\overline{t},\overline{e}]^9 = 1$. Assume if possible that

$$[\overline{b},\overline{t}] \in Z(\overline{T}).$$

Then

$$[\overline{e},\overline{t},\overline{t}] = [\overline{b}^i,\overline{t}][\overline{u}^j,\overline{t}] \in Z(\overline{T})$$

by (2) and then $[\overline{e},\overline{t}]^3 = 1$ by (3) which is impossible by (4). This means that $[\overline{e}^3,\overline{t}] = [\overline{e},\overline{t}]^3 = 1$. But also $(\overline{e})^3(\overline{u})^{-1} \in C_{\overline{E}}(\overline{t})$ by (4). Therefore $\overline{u} \in Z(\overline{T})$, which is impossible since $Z(\overline{T}) \cap \overline{D} = 1$. Consequently it follows that $[\overline{b},\overline{t}] \notin Z(\overline{T})$.

First suppose that $3|i$. Then $[\overline{e},\overline{t}] = \overline{b}^{3k}\overline{u}^j$, where $i = 3k$. Hence

$$[\overline{e},\overline{t},\overline{t}] = [(\overline{b})^{3k}(\overline{u})^j,\overline{t}] = [(\overline{u})^j,\overline{t}]$$

since $\overline{b}^3 \in Z(\overline{T})$. But also $[\overline{u},\overline{t}] \in Z(\overline{T})$ by (2). Therefore

$$[\overline{e},\overline{t},\overline{t}] \in Z(\overline{T})$$

and then $[\overline{e},\overline{t}]^3 = 1$ by (3), which gives a contradiction as above. Thus $3 \nmid i$ and so $|\overline{b}| = 9$. In this case $\langle\overline{b}\rangle \cap \langle\overline{D}\rangle = 1$ by Lemma 2.5 (b). Therefore $|\overline{b}| = 9 = |b|$.

Now since $Z(\overline{T}) = \overline{b}^3$, as was shown above, and since $\overline{z} \in \langle\overline{b}\rangle$, it follows that $Z(\overline{T}) = \langle\overline{z}\rangle$. In this case $\exp(Z(B)) = 3$. To see this let

$$K = \Omega_n(Z(B))$$

for an $n \geqslant 2$ and let h be the height of z in K. Then there exists a $y \in K$ so that $y^{3^h} = z$ and $K = \langle y \rangle \times B_1$. If $|y| < exp(K)$, then $1 \neq (K)^{|y|} \lhd G$ and $z \notin K^{|y|}$, which is a contradiction as in the proof of Lemma 2.5. Therefore

$$exp(K) = |y|.$$

Since $\langle y \rangle \cap E = 1$ and $y \in Z(B)$, it follows that $|y| = 3$ and so $exp(K) = 3$. Since $n \geqslant 2$ is arbitrary, it follows that $exp(\gamma_3(B)) = 3$. Now since $[B', B] \leqslant \gamma_3(B)$ it follows that

$$[(B')^3, B] = [B', B]^3 = 1$$

and so $(B')^3 \leqslant Z(B)$. If $exp(B') = 9$, then again we may choose $b \in B'$. But this is impossible since $[b, t] \notin Z(B)$ as was shown above. Therefore $exp(B') = 3$. Then it follows from the last part of the proof of Lemma 2.6 that $exp(B/Z(B)) = 3$. Consequently

$$exp(B) = exp(B')^2 = exp(Z(B))^2 = exp(B/Z(B))^2 = c(B)^2 = 9$$

since $|b| = 9$ and so the proof of the lemma is complete. □

Proof of Theorem 1.1 — Let G be Fitting p-group satisfying the normalizer condition and $p \neq 2$. Suppose that in each homomorphic image of G every Λ-pair has a maximal element satisfying the $(**)$-property and every normal nilpotent metabelian 3-subgroup of class 3 and of exponent 9 is powerful. Assume that G is perfect. First we show the following. G has a homomorphic image H with the following property. H has a Λ-pair (w_H, V_H) satisfying $(**)$ and the condition $W^*(w_H, V_H) = 1$ such that every normal nilpotent subgroup of H which is abelian- by-elementary abelian is abelian. Assume that there exists no such H. For each homomorphic image X of G satisfying the above properties let $n(X)$ be the minimum of the classes of all the normal nilpotent abelian-by-elementary abelian subgroups of X which are not abelian. Among all the homomorphic images X of G having a Λ-pair (w_X, V_X), satisfying $(**)$ and the condition $W^*(w_X, V_X) = 1$ there is a homomorphic image H such that $n(H) \leqslant n(X)$ for all such X. Without loss of generality we may suppose that $H = G$. Thus G admits a Λ-pair (w, V) such that $(**)$ and the condition $W^*(w, V) = 1$ are satisfied. Also $n(G)$ is minimal in the above sense and $n(G) > 1$ by the assumption. Let B be a normal nilpotent abelian-by-elementary abelian subgroup of G so that $c(B) = n(G)$. Let A be the largest normal abelian subgroup of G contained in B

such that $\exp(B/A) = p$ and $B' \leqslant A$. By the hypothesis there exists an $E \in E^*(w, V)$ satisfying $(**)$. Put $N = N_G(E)$. Then N/E is (locally) cyclic by [5], Lemma 2.2 since $p \neq 2$. Also $A \leqslant N$ by Lemma 2.4. Furthermore $B \nleq N$ as in Lemma 2.7 since B is not abelian and thus there exists $t \in B \setminus N$ so that $N^t = N$ and $t^p \in N$ since G satisfies the normalizer condition.

If $c(B) < 3$, then B is abelian by Lemma 2.7 since $p \neq 2$. Therefore $c(B) \geqslant 3$. Let $c(B) = c$ and put $\overline{G} = G/\gamma_c(B)$, so that $c(\overline{B}) = c - 1$.

Assume first if possible that $\overline{B}' \leqslant Z(\overline{G})$. Then $[\overline{B}', \overline{G}] = 1$ and hence $[B', G] \leqslant \gamma_c(B)$ which implies that $[B', G, B] = 1$. This implies that

$$[B, B, B, B] = 1$$

and hence $c \leqslant 3$. It follows that $c = 3$ since $c \geqslant 3$ and a second application of Lemma 2.7 shows that $p = 3$.

By the hypothesis G satisfies the normalizer condition. By the assumption G has a Λ-pair (w, V) such that $W^*(w, V) = 1$ and there exists $E \in E^*(w, V)$ such that $N_G(E) = N_G(E')$. Now applying Lemma 2.9′ we get

$$\exp(B) = 9 \text{ and } \exp(B/Z(B)) = 3$$

But also $B' \leqslant \mho_1(B)$ by the assumption, and it follows that $B' \leqslant Z(B)$. But then $c(B) = 2$, which is a contradiction. Therefore there exists an element $\overline{s} \in \overline{B}' \setminus Z(\overline{G})$. The rest of the proof is the same as the original proof. □

Proof of Corollary 1.2 — Let G be a Fitting p-group satisfying the hypothesis of the corollary, where $p \neq 2$. Assume that in every homomorphic of G every normal nilpotent metabelian 3-subgroup of class 3 and of exponent 9 is powerful but G is not solvable. Thus every proper homomorphic image of G is an MNS-group and, in particular, is perfect. By [4], Theorem 1.4 (b) we may suppose that G has no homomorphic images having $(*)$-triples for non-central elements. Then in every homomorphic image of G there exist distinguished pairs and dominant pairs by [4], Lemma 3.1 and Lemma 4.1 (b). The rest of the proof is the same as the original proof. □

A.O. Asar

Yargic Sokak 11/6, Cebeci
Ankara (Turkey)
e-mail: aliasar@gazi.edu.tr