Advances in Group Theory and Applications
3 (2017), pp. 55–66 ISSN: 2499-1287
DOI: 10.4399/97888255036924

Some New Local Properties Defining Soluble PST-Groups

J.C. Beidleman

(*Received Aug. 14, 2016; Accepted Nov. 24, 2016 — Communicated by F. de Giovanni*)

Abstract

Let G be a group and p a prime number. G is said to be a Y_p-group if whenever K is a p-subgroup of G every subgroup of K is an S-permutable subgroup in $N_G(K)$. The group G is a soluble PST-group if and only if G is a Y_p-group for all primes p. It is our purpose here to define a number of local properties related to Y_p which lead to several new characterizations of soluble PST-groups.

Mathematics Subject Classification (2010): 20D10, 20D20, 20D35

Keywords: S-permutable subgroup; semipermutable subgroup; seminormal subgroup; PST-group

1 Introduction and Statement of Results

All groups considered in this article are finite.

A subgroup H of a group G is said to permute with a subgroup K of G if HK is a subgroup of G. H is said to be permutable (resp. S-permutable) if it permutes with all the subgroups (resp. Sylow subgroups) of G. Examples of permutable subgroups include the normal subgroups of G. However, if G is a modular, non-Dedekind p-group, p a prime, we see permutability is quite different from normality. For instance, letting C_n denote the cyclic group of order n, we see that C_2 is permutable but not normal in the group $C_8 \rtimes C_2$ where the generator for C_2 maps a generator of C_8 to its fifth power. Kegel [12] proved that an S-permutable subgroup is always subnormal. In particular, a permutable subgroup of a group is subnormal. A group G is called a PST-group (resp. PT-group) if S-permutability (resp. permutability) is a transitive relation. By Kegel's result, a group G is a PST-group (resp. PT-group) if every subnormal subgroup of G is S-permutable (resp. permutable) in G.

A number of research papers have been written on these groups. See for example [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[16].

Another class of groups is the so called T-groups. A group G is a T-group if normality in G is transitive, that is, if $H \trianglelefteq K \trianglelefteq G$ then $H \trianglelefteq G$. There are several nice characterizations of T-groups in [15].

Soluble PST, PT and T-groups were characterized by Agrawal [1], Zacher [18] and Gaschütz [11], respectively.

Theorem 1.1

1. *A soluble group* G *is a* PST*-group if and only if the nilpotent residual* L *of* G *is an abelian Hall subgroup of* G *on which* G *acts by conjugation as power automorphisms.*

2. *A soluble* PST*-group* G *is a* PT*-group (respectively* T*-group) if and only if* G/L *is a modular (respectively Dedekind) group.*

Note that if G is a soluble T, PT or PST-group then every subgroup and every quotient of G inherits the same properties.

We mention that in [6, Chapter 2] many of the beautiful results on these classes of groups are presented.

Subgroup embedding properties closely related to permutability and S-permutability are semipermutability and S-semipermutability. A subgroup X of a group G is said to be *semipermutable* (respectively, *S-semipermutable*) in G provided that it permutes with every subgroup (respectively, Sylow subgroup) K of G such that $gcd(|X|, |K|) = 1$. A semipermutable subgroup of a group need not be subnormal. For example a 2-Sylow subgroup of the non-abelian group of order 6 is semipermutable but not subnormal.

Note that a subnormal semipermutable (respectively, S-semipermutable) subgroup X of a group G must be normalised by every subgroup (respectively, Sylow subgroup) P of G such that $gcd(|X|, |P|) = 1$.

This observation was the basis for Beidleman and Ragland [10] to introduce the following subgroup embedding properties.

A subgroup X of a group G is said to be *seminormal* (respectively, S-*seminormal*)[1] in G if it is normalised by every subgroup (respectively, Sylow subgroup) K of G such that $gcd(|X|, |K|) = 1$.

By [10, Theorem 1.2], a subgroup of a group is seminormal if and only if it is S-seminormal. Furthermore, seminormal subgroups are not necessarily subnormal: it is enough to consider a non-subnormal subgroup H of a group G such that $\pi(H) = \pi(G)$. To see some of the properties of these subgroups see Examples 1, 2 and 3 in Section 3.

However, a p-subgroup of a group G, p a prime, which is also seminormal is subnormal [10, Theorem 1.3].

Semipermutable, S-semipermutable and seminormal subgroups have been investigated in [10],[17],[19],[20].

The following result is an interesting characterisation of soluble PST-groups.

Theorem 1.2 ([10, 1.5]) *Let G be a soluble group. Then the following statements are pairwise equivalent:*

1. G *is a* PST*-group.*
2. *All the subnormal subgroups of* G *are seminormal in* G.
3. *All the subnormal subgroups of* G *are semipermutable in* G.
4. *All the subnormal subgroups of* G *are S-semipermutable in* G.

The following beautiful result is due to H. Wielandt.

Theorem 1.3 ([13, 7.3.3]) *Let H be a subgroup of a group G. Then the following statements are equivalent:*

1. H *is subnormal in* G.
2. H *is subnormal in* $\langle H, H^g\rangle$ *for all* $g \in G$.
3. H *is subnormal in* $\langle H, g\rangle$ *for all* $g \in G$.

[1] Note that the term *seminormal* has several different meanings in the literature

Wielandt's result seems to have inspired the authors of [5] to introduce the concept of weakly S-permutable subgroups of a subgroup H of a group G. This concept led to several new characterizations of soluble PST-groups which are presented in the following theorem from [5].

Theorem 1.4 *Let G be a group. The following statements are pairwise equivalent:*

1. *G is a soluble PST-group.*

2. *Every subgroup of G is weakly S-permutable in G.*

3. *For every prime number p, every p-subgroup of G is weakly S-permutable in G.*

Theorems 1.3 and 1.4 motivate the following definition.

Definition 1.5 *Let H be a subgroup of a group G.*

1. *H is said to be weakly S-permutable in G if whenever $g \in G$ and H is S-permutable in $\langle H, H^g \rangle$, then H is S-permutable in $\langle H, g \rangle$.*

2. *H is said to be weakly semipermutable in G if whenever $g \in G$ and H is semipermutable in $\langle H, H^g \rangle$, then H is semipermutable in $\langle H, g \rangle$.*

3. *H is said to be weakly S-semipermutable in G if whenever $g \in G$ and H is S-semipermutable in $\langle H, H^g \rangle$, then H is S-semipermutable in $\langle H, g \rangle$.*

4. *H is said to be weakly seminormal in G if whenever $g \in G$ and H is seminormal in $\langle H, H^g \rangle$, then H is seminormal in $\langle H, g \rangle$.*

The next theorem relates the concept of S-permutable subgroups of a group G with weakly S-permutable subgroups of G.

Theorem 1.6 ([5]) *A subgroup H of a group G is S-permutable in G if and only if H is S-permutable in $\langle H, g \rangle$ for every $g \in G$.*

Theorem 1.6 and its proof are used to establish the following result.

Theorem A *Let H be a subnormal subgroup of a group. Then:*

1. *H is S-semipermutable in G if and only if H is S-semipermutable in $\langle H, g \rangle$ for every $g \in G$.*

2. H *is seminormal in* G *if and only if* H *is seminormal in* $\langle H, g\rangle$ *for every* $g \in G$.

A class of groups G is a PST-group if and only if Sylow permutability is a transitive relation in G.

Definition 1.7 *Let* G *be a group and* p *a prime. Then*

1. G *is a* Y_p*-group if for every* p*-subgroup* K *of* G *every subgroup of* K *is S-permutable in* $N_G(K)$.

2. G *is a* $\widehat{Y}_p$*-group if for every* p*-subgroup* K *of* G *every subgroup of* K *is semipermutable in* $N_G(K)$.

3. G *is a* $\widetilde{Y}_p$*-group if for every* p*-subgroup* K *of* G *every subgroup of* K *is S-semipermutable in* $N_G(K)$.

4. G *is a* $\widetilde{\widetilde{Y}}_p$*-group if for every* p*-subgroup* K *of* G *every subgroup of* K *is seminormal in* $N_G(K)$.

5. G *is a* $\underline{Y}_p$*-group if for every* p*-subgroup* K *of* G *every subgroup of* K *is weakly S-permutable in* $N_G(K)$.

6. G *is a* $\widetilde{\underline{Y}}_p$*-group if for every* p*-subgroup* K *of* G *every subgroup of* K *is weakly S-semipermutable in* $N_G(K)$.

7. G *is a* $\widetilde{\widetilde{\underline{Y}}}_p$*-group if for every* p*-subgroup* K *of* G *every subgroup of* K *is weakly seminormal in* $N_G(K)$.

The following result is a very nice local characterization of soluble PST-groups.

Theorem 1.8 ([6, 2.2.9] and [4, Theorem 4]) *A group* G *is a soluble* PST*-group if and only if it satisfies* Y_p *for all primes* p.

Our next result shows how some of the classes in Definition 1.7 are related to the class Y_p.

Theorem 1.9 ([6, 1.8]) *Let* p *be a prime and* G *a group. Then*

$$Y_p = \widehat{Y}_p = \widetilde{Y}_p = \widetilde{\widetilde{Y}}_p.$$

Using Theorems 1.8 and 1.9 we note that the next result shows that all of the classes $\underline{Y}_p$, $\widetilde{\underline{Y}}_p$ and $\widetilde{\widetilde{\underline{Y}}}_p$ are just Y_p.

Theorem B *Let* p *be a prime and* G *a group. Then*

1. $G \in Y_p$ *if and only if* $G \in \underline{Y}_p$.
2. $G \in \widetilde{Y}_p$ *if and only if* $G \in \widetilde{\underline{Y}}_p$.
3. $G \in \widetilde{\widetilde{Y}}_p$ *if and only if* $G \in \widetilde{\widetilde{\underline{Y}}}_p$.

From Theorem 1.9 and Theorem B we obtain several results that yield new local characterizations of soluble PST-groups.

Corollary 1.10 *Let* p *be a prime. Then*

$$Y_p = \underline{Y}_p = \widehat{Y}_p = \widetilde{Y}_p = \widetilde{\underline{Y}}_p = \widetilde{\widetilde{Y}}_p = \widetilde{\widetilde{\underline{Y}}}_p.$$

Using Theorem B and Corollary 1.10 we obtain the main result of this paper.

Theorem C *Let* G *be a group. Then the following statements are pairwise equivalent:*

1. G *is a soluble* PST-*group.*
2. G *is a* Y_p*-group for all primes* p.
3. G *is a* $\underline{Y}_p$*-group for all primes* p.
4. G *is a* $\widehat{Y}_p$*-group for all primes* p.
5. G *is a* $\widetilde{Y}_p$*-group for all primes* p.
6. G *is a* $\widetilde{\underline{Y}}_p$*-group for all primes* p.
7. G *is a* $\widetilde{\widetilde{Y}}_p$*-group for all primes* p.
8. G *is a* $\widetilde{\widetilde{\underline{Y}}}_p$*-group for all primes* p.

2 Preliminaries

The lemmas which follow are used in the proof of Theorems A – C.

The first lemma follows from the definitions of the terms given in parts 1 – 4 of Lemma 2.1.

Lemma 2.1 *Let* H *and* K *be subgroups of a group* G*. Then*

1. *If* $H \leqslant K$ *and* H *is* S*-permutable in* G*, then* H *is* S*-permutable in* K.
2. *If* $H \leqslant K$ *and* H *is semipermutable in* G*, then* H *is semipermutable in* K.
3. *If* $H \leqslant K$ *and* H *is* S*-semipermutable in* G*, then* H *is* S*-semipermutable in* K.
4. *If* $H \leqslant K$ *and* H *is seminormal in* G*, then* H *is seminormal in* K.

The next lemma is a result of H. Wielandt.

Lemma 2.2 ([14, 13.3.7]) *Let* N *be a minimal normal subgroup of a group* G*. Then* N *normalizes every subnormal subgroup of* G.

3 Examples

Example 3.1 *Let* S_4*,* A_4 *and* K_4 *denote, respectively, the symmetric group of order* 4*, the alternating group of order* 4*, and the Klein* 4*-group. Let* $G = S_4$ *and let* $H = \langle (123) \rangle$*. Then* H *is* S*-semipermutable in* G *but it is not semipermutable in* G *since it does not permute with an element of order* 2 *in* K_4*, the Sylow* 2*-subgroup of* A_4.

An S-permutable subgroup of a group is subnormal. That this is not the case with S-semipermutable subgroups can be seen in the subgroup H in S_4. Notice that H is not seminormal in S_4.

Example 3.2 *Let*

$$D_{10} = \langle x, y \mid x^5 = y^2 = 1, x^y = x^{-1} \rangle,$$

the dihedral group of order 10, and

$$C_{15} = \langle t, s \mid t^5 = s^3 = 1, ts = st \rangle,$$

the cyclic group of order 15*. Let* $G = D_{10} \times C_{15}$ *and let* $K = \langle t \rangle \times \langle y \rangle$*. Since* $\langle s \rangle$ *centralizes* K *it follows that* K *is seminormal in* G*. Note that* K *is not subnormal in* G.

Example 3.3 *Let* $H = \langle x \rangle \rtimes \langle y \rangle$ *be a semidirect product of a cyclic group,* $\langle x \rangle$*, of order* 11 *by a cyclic group,* $\langle y \rangle$*, of order* 5*. Let* $G = H \times S_4$*. Set* $K = \langle x \rangle \times S_3$ *where* S_3 *is a copy of the symmetric group on three elements in* S_4*. Then* K *is a seminormal subgroup of* G *which is not subnormal.*

4 Proof of the Theorems

PROOF OF THEOREM A — Let H be a subnormal subgroup of G.

1. Assume H is S-semipermutable in G and let $g \in G$. Let p be a prime divisor of $|\langle H, g\rangle|$ such that $gcd(p, |H|) = 1$. We are to show that H is S-semipermutable in $\langle H, g\rangle$. Let $P \in Syl_p(\langle H, g\rangle)$ and let $Q \in Syl_p(G)$ be such that $P \leqslant Q$. Note that $HQ = QH$. Consider $HQ \cap P = HP$ so that H is S-semipermutable in $\langle H, g\rangle$.
   Now assume that G is a group of minimal order such that H is not S-semipermutable in G. Note that $H \triangleleft\triangleleft G$. Let M be a maximal normal subgroup of G such that $H \leqslant M$.
   There is a prime p such that $gcd(p, |H|) = 1$ and a Sylow p-subgroup P of G such that HP is not a subgroup of G. Let M_1 be a maximal subgroup of G such that $H \leqslant M_1$ and $M \nleqslant M_1$. Then $G = MM_1$ and H is S-semipermutable in both M_1 and M. So there exists a Sylow p-subgroup Q_1 of M_1 and a Sylow p-subgroup Q of M such that QQ_1 is a Sylow p-subgroup of G and H permutes with QQ_1. Let $QQ_1 = P_0 \in Syl_p(G)$, so that there is an element $x \in G$ such that $P_0 = P^x$.
   Let N be a minimal normal subgroup of G with $N \leqslant M$. Hence $HN \leqslant M$ and by a result due to Wielandt (Lemma 2.2) N normalizes H. Notice by the minimality of G, HN/N permutes with PN/N and hence P permutes with HN in G. Assume that $P(HN) = X$ is a proper subgroup of G. Then $H \triangleleft\triangleleft X$ and H is S-semipermutable in $\langle H, t\rangle$ for all $t \in X$. By choice of G, we have $HP = PH$, a contradiction and so $P(HN) = X = G$.
   Let $x = x_1 x_2$ where $x_1 \in P$ and $x_2 \in HN$. It follows that

   $$P_0 = P^x = P^{x_2} \quad \text{and} \quad H^{x_2^{-1}} = H.$$

   This means that $HP = PH$ or H permutes with P. This is a contradiction, so that $H \leqslant M \leqslant M_1$.
   Now $HN \leqslant M$ and $|G : M|$ is a power of p. Hence all the maximal subgroups of G/M are normal. This means that M is a maximal subgroup of G containing H. Therefore, if $t \in G \backslash M$, it follows that $G = \langle M, t\rangle$. From the hypothesis, H is S-semipermutable in $\langle M, t\rangle = G$, a final contradiction.

2. Assume that G is a group of minimal order such that H is seminormal in $\langle H, g\rangle$ for each $g \in G$ but H is not seminormal

in G. Let M be a maximal normal subgroup of G such that $H \leqslant M$. There is a prime p and a Sylow p-subgroup P of G such that $gcd(p, |H|) = 1$ and P does not normalize H.
Let M_1 be a maximal subgroup of G such that $H \leqslant M_1$ but M is not contained in M_1. Then $G = M_1 M$ and H is S-seminormal in both M and M_1. Hence there exists a Sylow p-subgroup Q of M and a Sylow p-subgroup Q_1 of M_1 such that QQ_1 is a Sylow p-subgroup of G and QQ_1 normalizes H. Let $P_0 = QQ_1$ and note there is an element $x \in G$ such that $P_0 = P^x$.
For the next part of the proof consider the last part of the proof of 1. □

Proof of Theorem B — Let G be a group and K a p-subgroup of G.

1. Assume that $G \in Y_p$ and let H be a subgroup of the p-subgroup K of G and consider $H \leqslant K \leqslant N_G(K)$. Suppose that H is S-permutable in $\langle H, H^g \rangle$ where $g \in N_G(K)$. Since $G \in Y_p$, H is S-permutable in $N_G(K)$. But $\langle H, g \rangle \leqslant N_G(K)$ and hence H is S-permutable in $\langle H, g \rangle$, by part 1 of Lemma 2.1, for all $g \in N_G(K)$. Note that H is subnormal in $N_G(K)$ and H is weakly S-permutable in $N_G(K)$. Therefore $G \in \underline{Y}_p$.
   Conversely, assume that $G \in \underline{Y}_p$ and let H be a subgroup of the p-subgroup K of G and note $H \leqslant K \leqslant N_G(K)$. We are to show that H is S-permutable under the assumption that H is weakly S-permutable in $N_G(K)$. Let $g \in N_G(K)$ and assume that H is S-permutable in $\langle H, H^g \rangle$. Then

   $$\langle H, H^g \rangle \leqslant \langle H, g \rangle \leqslant N_G(K)$$

   and H is S-permutable in $\langle H, g \rangle$ for all $g \in N_G(K)$ and by Theorem 1.6 H is S-permutable in $N_G(K)$. Thus $G \in Y_p$.

2. Assume first that G is a $\widetilde{Y}_p$-group and let H be a subgroup of the p-subgroup K of G such that $H \leqslant K \leqslant N_G(K)$. Since $G \in \widetilde{Y}_p$, H is S-semipermutable in $N_G(K)$. Assume that H is S-semipermutable in $\langle H, H^g \rangle$ where $g \in N_G(K)$. Now $\langle H, g \rangle \leqslant N_G(K)$ so that H is S-semipermutable in $\langle H, g \rangle$, for all $g \in G$, by part 3 of Lemma 2.1. Therefore, H is weakly S-semipermutable in $N_G(K)$ and so $G \in \underline{\widetilde{Y}}_p$.
   Conversely, assume that G is a $\underline{\widetilde{Y}}_p$-group and assume H is a subgroup of the p-subgroup K of G such that $H \leqslant K \leqslant N_G(K)$.

Also let H be weakly S-semipermutable in $N_G(K)$. We are to show that H is S-semipermutable in $N_G(K)$. Let g be an element in $N_G(K)$ such that H is S-semipermutable in $\langle H, H^g \rangle$. Then H is S-semipermutable in $\langle H, g \rangle$ and we note that this is true for all $g \in N_G(K)$. By part 1 of Theorem A G is S-semipermutable in $N_G(K)$. It follows that $G \in \tilde{Y}_p$.

3. Assume that G is in $\tilde{\tilde{Y}}_p$ and let H be a subgroup of the p-subgroup K of G such that $H \leqslant K \leqslant N_G(K)$. Let g be an element of $N_G(K)$ and assume H is seminormal in $\langle H, H^g \rangle$. Note, since G is seminormal in $N_G(K)$, H is seminormal in $\langle H, g \rangle \leqslant N_G(K)$, by part 4 of Lemma 2.1. This is true for all $g \in N_G(K)$ so that H is weakly seminormal in $N_G(K)$ and $G \in \underline{\tilde{\tilde{Y}}}_p$.

   Conversely, assume that G is contained in $\underline{\tilde{\tilde{Y}}}_p$ and let H be a subgroup of G such that $H \leqslant K \leqslant N_G(K)$ where K is a p-subgroup of G. Let g be an arbitrary element of $N_G(K)$. Note that H is weakly seminormal in $N_G(K)$ so that H is seminormal in $\langle H, H^g \rangle$ and hence in $\langle H, g \rangle$. But this is true for all $g \in N_G(K)$ so that H is seminormal in $N_G(K)$ by part (2) of Theorem A. Thus $G \in \tilde{\tilde{Y}}_p$.

This completes the proof of Theorem B. □

Proof of Theorem C — The proof of Theorem C follows from Theorems 1.8, 1.9 and B. □

REFERENCES

[1] R.K. Agrawal: "Finite groups whose subnormal subgroups permute with all Sylow subgroups", *Proc. Amer. Math. Soc.* 47 (1975), 77–83.

[2] K.A. Al-Sharo – J.C. Beidleman – H. Heineken – M.F. Ragland: "Some characterizations of finite groups in which semipermutability is a transitive relation", *Forum Math.* 22 (2010), 855–862.

[3] A. Ballester-Bolinches – R. Esteban-Romero: "Sylow permutable subnormal subgroups of finite groups II", *Bull. Austr. Math. Soc.* 64 (2001), 479–486.

[4] A. Ballester-Bolinches – R. Esteban-Romero: "Sylow permutable subnormal subgroups of finite groups", *J. Algebra*, 251 (2002), 727–738.

[5] A. Ballester-Bolinches – R. Esteban-Romero: "On finite soluble groups in which Sylow permutability is a transitive relation", *Acta. Math. Hungar.* 101 (2007), 193–202.

[6] A. Ballester-Bolinches – R. Esteban-Romero – M. Asaad: "Products of Finite Groups", *de Gruyter*, Berlin (2010).

[7] J.C. Beidleman – H. Heineken: "Finite soluble groups whose subnormal subgroups permute with certain classes of subgroups", *J. Group Theory* 6 (2003), 139–158.

[8] J.C. Beidleman – H. Heineken: "Pronormal and subnormal subgroups and permutability", *Boll. Un. Mat. Ital.* 6 B (2003), 605–615.

[9] J.C. Beidleman – H. Heineken – M.F. Ragland: "Solvable PST-groups, strong Sylow bases and mutually permutable products", *J. Algebra* 321 (2009), 2022–2027.

[10] J.C. Beidleman – M.F. Ragland: "Subnormal, permutable, and embedded subgroups in finite groups", *Central Eur. J. Math.* 9 (2011), 915–921.

[11] W. Gaschütz: "Gruppen, in denen das Normalteilersein transitiv ist", *J. Reine Angew. Math.* 198 (1957), 87–92.

[12] O.H. Kegel: "Sylow-Gruppen und Subnormalteiler endlicher Gruppen", *Math. Z.* 78 (1962), 205–221.

[13] J.C. Lennox – S.E. Stonehewer: "Subnormal Subgroups of Groups", *Oxford Mathematical Monographs*, Oxford (1987).

[14] D.J.S. Robinson: "A Course in the Theory of Groups", *Springer*, Berlin (1996).

[15] D.J.S. Robinson: "A note on finite groups in which normality is transitive", *Proc. Amer. Math. Soc.* 19 (1968), 933–937.

[16] P. Schmid: "Subgroups permutable with all Sylow subgroups", *J. Algebra* 207 (1998), 285–293.

[17] L. Wang, Y. Wang: "Finite groups in which S-semipermutability is a transitive relation", *Int. J. Algebra* 2 (2008), 143–152.

[18] G. Zacher: "I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasinormali", *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* 37 (1964), 150–154.

[19] Q. Zhang: "s-semipermutability and abnormality in finite groups", *Comm. Algebra* 27 (1999), 4515–4524.

[20] Q.H. Zhang, L.F. Wang: "The influence of s-semipermutable subgroups on finite groups", *Acta. Math. Sinica (Chin. Ser.)* 48 (2005), 81–88.

J.C. Beidleman
Department of Mathematics
University of Kentucky
715 Patterson Office Tower
Lexington, KY (USA)
e-mail: james.beidleman@uky.edu