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Normality in Uncountable Groups *

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Abstract

The main purpose of this paper is to describe the structure of uncountable groups of cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. Moreover, uncountable groups are studied in which every uncountable non-abelian subgroup is normal.

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1 Introduction

A group G is said to have *finite rank* if there is a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has *infinite rank*. In a long series of papers, it has been shown that the structure of a (generalized) soluble group of infinite rank is strongly influenced by that of its proper subgroups of infinite rank (see for instance [1], where a full reference list on this subject can be found). In particular, M.J. Evans and Y. Kim [4] proved that if G is a locally soluble group of infinite rank in which all subgroups of infinite rank are normal, then G is a Dedekind group, i.e. all subgroups of G are

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normal. Results of this type suggest that the behaviour of *small* subgroups in a *large* group is neglectable, at least for a right choice of the definition of largeness and within a suitable universe. This point of view was adopted also in the recent papers [5] and [6], by considering uncountable groups whose proper uncountable subgroups belong to certain notable group classes.

The aim of this paper is to give a further contribution to this topic by investigating uncountable groups of cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. The main obstacle in the study of groups of large cardinality is the existence of uncountable groups, of cardinality $\aleph$, say, in which all proper subgroups have cardinality strictly smaller than $\aleph$ (the so-called *Jónsson groups*). Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by S. Shelah [14] and V.N. Obraztsov [9].

Actually, Evans and Kim considered the more general situation of groups of infinite rank in which all subgroups of infinite rank are subnormal of bounded defect, and the first part of our paper is dedicated to the corresponding case of uncountable groups whose uncountable subgroups are subnormal of bounded defect. A positive result is obtained when the group contains abelian subgroups of high cardinality. However, there exist uncountable nilpotent groups in which all abelian subgroups are countable, and in particular an example of A. Ehrenfeucht and V. Faber [3] shows that a result similar to the above quoted theorem of Evans and Kim does not hold for groups in which all uncountable subgroups are normal. In Section 3 the structure of non-Dedekind groups whose uncountable subgroups are normal is studied.

The last section of the paper focuses on uncountable metahamiltonian groups. Recall here that a group is *metahamiltonian* if all its non-abelian subgroups are normal; metahamiltonian groups have been introduced and investigated by G.M. Romalis and N.F. Sesekin ([11],[12],[13]), who proved in particular that the commutator subgroup of any locally soluble metahamiltonian group is finite of prime-power order. Locally soluble groups of infinite rank in which all non-abelian subgroups of infinite rank are normal have been proved to be metahamiltonian (see [2]). Here it is shown that if all uncountable non-abelian subgroups of an uncountable group G are normal, then either G is metahamiltonian or all its uncountable subgroups are normal, provided that G has no simple sections of high cardinality.

Most of our notation is standard and can be found in [10].

2 Subnormal subgroups

A celebrated theorem of J.E. Roseblade states that if k is a positive integer and G is a group in which all subgroups are subnormal of defect at most k, then G is nilpotent; moreover, there exists a function f such that the nilpotency class of G is bounded by $f(k)$ (see [10] Part 2, p.71). As expected, Roseblade's function plays a role in our considerations.

Our first two lemmas are designed to produce suitable subgroups of high cardinality in certain large groups.

Lemma 2.1 *Let G be an infinite group of regular cardinality $\aleph$, and let I be a set of cardinality strictly smaller than $\aleph$. If G is the direct product of a collection $(G_i)_{i\in I}$ of subgroups, then there exists $i \in I$ such that G_i has cardinality $\aleph$.*

Proof — Let $\leqslant$ be a well-ordering in the set I, and for each $i \in I$ put

$$G_i^* = \operatorname*{Dr}_{j\leqslant i} G_j.$$

Then $(G_i^*)_{i\in I}$ is a chain of subgroups of G, and

$$G = \bigcup_{i\in I} G_i^*.$$

As the cardinal $\aleph$ is regular, at least one of the G_i^*'s has cardinality $\aleph$, and we may consider the smallest element k of I such that G_k^* has cardinality $\aleph$. Then

$$\bigcup_{h<k} G_h^*$$

has cardinality strictly smaller than $\aleph$, and hence k is the successor of an element k' of I. It follows that $G_k^* = G_{k'}^* \times G_k$, and so G_k has cardinality $\aleph$. □

Lemma 2.2 *Let A be an uncountable abelian group of regular cardinality $\aleph$. Then A contains a subgroup of the form*

$$B = \operatorname*{Dr}_{i\in I} B_i,$$

where the set I *has cardinality* $\aleph$ *and either all* B_i *are infinite cyclic or they all have the same prime order.*

Proof — Suppose first that the subgroup T of G consisting of all elements of finite order has cardinality $\aleph$. In this case, it follows from Lemma 2.1 that there exists a prime number p such that also the p-component G_p of G has cardinality $\aleph$. Then the subgroup $G_p[p^k]$ has cardinality $\aleph$ for some positive integer k, and so the socle of G_p is the direct product of a collection of cardinality $\aleph$ consisting of subgroups of order p.

Assume now that T has cardinality strictly smaller than $\aleph$, so that A/T has cardinality $\aleph$. Let U/T be a free abelian subgroup of A/T such that A/U is periodic. Then it is easy to prove that U/T has cardinality $\aleph$, so that it is the direct product of a collection of cardinality $\aleph$ consisting of infinite cyclic subgroups. On the other hand, there exists a subgroup B of U such that $U = B \times T$, so that $B \simeq U/T$ and the statement is proved. □

For our purposes, we also need the following result concerning modules of high cardinality over a principal ideal domain of small cardinality.

Lemma 2.3 *Let* $\aleph$ *be a regular uncountable cardinal number, and let* R *be a principal ideal domain of cardinality strictly smaller than* $\aleph$*. If* M *is an* R*-module of cardinality* $\aleph$*, then* M *contains a submodule which is the direct sum of a collection of cardinality* $\aleph$ *of non-trivial submodules.*

Proof — The injective hull $E_R(M)$ of M can be decomposed into a direct sum

$$E_R(M) = \bigoplus_{i \in I} E_i,$$

where each E_i is an injective indecomposable R-module. In particular, each E_i has cardinality strictly smaller than $\aleph$, since it is a homomorphic image of the fraction field of R, and hence the set I has cardinality $\aleph$. On the other hand, M is an essential submodule of $E_R(M)$, so that $M \cap E_i \neq \{0\}$ for all $i \in I$. Therefore the direct sum

$$\bigoplus_{i \in I} (M \cap E_i)$$

is a submodule of M satisfying the condition of the statement. □

Lemma 2.4 *Let* k *be a positive integer, and let* G *be an uncountable group of regular cardinality* $\aleph$ *in which all subgroups of cardinality* $\aleph$ *are subnormal of defect at most* k*. If* G *is abelian-by-cyclic, then it is nilpotent.*

Proof — Assume for a contradiction that G is not nilpotent, and write $G = A\langle x\rangle$, where A is an abelian normal subgroup and x is an element of G. For each positive integer n, the subgroup

$$A_n = [A, \underbrace{x, \dots, x}_{\longleftarrow n \longrightarrow}]$$

is normal in G, and the factor group G/A_n is nilpotent. Put $c = f(k)$, and let B be any G-invariant subgroup of A of cardinality $\aleph$. Then all subgroups of G/B are subnormal of defect at most k, so that G/B is nilpotent of class at most c and hence A_c is contained in B. It follows that A_c has no proper G-invariant subgroups of cardinality $\aleph$. On the other hand, A_c can be regarded as a module over the ring $R = \mathbb{Z}\langle x\rangle$, and the R-submodules of A_c are precisely its G-invariant subgroups. Thus an application of Lemma 2.3 yields that A_c has cardinality strictly smaller than $\aleph$.

Let m be the smallest non-negative integer such that A_{m+1} has cardinality strictly smaller than $\aleph$, so that A_m has cardinality $\aleph$. As the map

$$\theta : A_m \longrightarrow A_{m+1},$$

defined by putting $a^\theta = [a, x]$ for all $a \in A_m$, is a G-epimorphism, its kernel $\ker\theta$ has cardinality $\aleph$, and hence it contains A_c. It follows that $[A_c, x] = \{1\}$, so that A_c lies in $Z(G)$ and hence G is nilpotent. This contradiction proves the lemma. □

We are now in a position to prove the main result of this section. Recall that a group G is called a *Baer group* if it is generated by abelian subnormal subgroups, or equivalently if all finitely generated subgroups of G are subnormal. Of course, Baer groups are rich in subnormal subgroups.

Theorem 2.5 *Let* k *be a positive integer, and let* G *be an uncountable group of regular cardinality* $\aleph$ *in which all subgroups of cardinality* $\aleph$ *are subnormal of defect at most* k*.*

(a) *If* G *contains an abelian normal subgroup of cardinality* $\aleph$*, then all subgroups of* G *are subnormal of defect at most* k*, and so* G *is nilpotent of class at most* $f(k)$*.*

(b) *If* G *contains an abelian subgroup of cardinality* $\aleph$, *then* G *is nilpotent and the subgroup* $\gamma_{f(k)+1}(G)$ *has cardinality strictly smaller than* $\aleph$.

Proof — (a) Let A be an abelian normal subgroup of G of cardinality $\aleph$. If x is any element of G, the subnormal subgroup $\langle A, x\rangle$ is nilpotent by Lemma 2.4, and hence G is a Baer group. Let $E = \langle x_1, \dots, x_t\rangle$ be any finitely generated subgroup of G. Then E is subnormal and nilpotent, so that also AE is nilpotent, and so there exists a non-negative integer m such that

$$B = [A, \underbrace{E, \dots, E}_{\leftarrow m \rightarrow}]$$

has cardinality $\aleph$ and $C = [B, E]$ has cardinality strictly smaller than $\aleph$. For each positive integer $i \leqslant t$, let

$$\theta_i : B \longrightarrow C$$

be the homomorphism defined by putting $b^{\theta_i} = [b, x_i]$ for all $b \in B$. Then every $B/\ker\theta_i$ has cardinality strictly smaller than $\aleph$, and so the subgroup

$$U = \bigcap_{i=1}^{t} \ker\theta_i$$

has cardinality $\aleph$. Clearly, U is contained in the centre of AE, and by Lemma 2.2 it contains two subgroups U_1 and U_2 of cardinality $\aleph$ such that

$$U_1 \cap U_2 = E \cap \langle U_1, U_2\rangle = \{1\}.$$

It follows that the subnormal subgroup

$$E = EU_1 \cap EU_2$$

has defect at most k in G. Therefore all finitely generated subgroups of G are subnormal of defect at most k, and so even every subgroup of G is subnormal of defect at most k (see [10] Part 2, Lemma 7.41).

(b) Suppose now that A is an abelian subgroup of G of cardinality $\aleph$, but that A is not necessarily normal. However, by assumption A is at least subnormal in G, and we will first prove that G is nilpotent, by using induction on the defect m of A in G. This is certainly true if $m = 1$ by part (a). If $m > 1$, the subgroup A has defect $m-1$ in its normal closure A^G, and so it can be assumed that A^G is nilpotent. It follows that the abelian group $A^G/(A^G)'$ has cardinality $\aleph$, and so the factor group $G/(A^G)'$ is nilpotent by (a). Therefore G itself is nilpotent.

As G is nilpotent, there exists a positive integer s such that $\gamma_s(G)$ has cardinality $\aleph$ and $\gamma_{s+1}(G)$ has cardinality strictly smaller than $\aleph$. Then the factor group $G/\gamma_{s+1}(G)$ contains the abelian normal subgroup $\gamma_s(G)/\gamma_{s+1}(G)$, which has cardinality $\aleph$, and so it follows from part (a) that $G/\gamma_{s+1}(G)$ is nilpotent of class at most $f(k)$. Therefore $\gamma_{f(k)+1}(G)$ is contained in $\gamma_{s+1}(G)$, and hence it has cardinality strictly smaller than $\aleph$. □

Corollary 2.6 *Let G be an uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are subnormal of defect at most k. Then $\gamma_{f(k)+1}(G)/\gamma_{f(k)+1}(G)'$ has cardinality strictly smaller than $\aleph$.*

Proof — Assume that $\gamma_{f(k)+1}(G)/\gamma_{f(k)+1}(G)'$ has cardinality $\aleph$. Then it follows from Theorem 2.5 that the group $G/\gamma_{f(k)+1}(G)'$ is nilpotent of class at most $f(k)$, and so $\gamma_{f(k)+1}(G) = \gamma_{f(k)+1}(G)'$. This contradiction proves the statement. □

It turns out that Jónsson groups represent the main obstacle in the study of uncountable groups whose large subgroups satisfy an embedding property. We list here a number of conditions which are enough in order to avoid Jónsson groups. In fact, it is known that if $\aleph$ is any uncountable regular cardinal, then a periodic group of cardinality $\aleph$ cannot be a Jónsson group, provided that it is either locally finite or a 2-group (see [7], Theorem 2.6), and a similar result was proved by S.P. Strunkov [16] for groups in which every 2-generator subgroup is finite. Actually, groups satisfying some weak solubility condition cannot be Jónsson groups, since it was remarked by A. Macintyre that every Jónsson group is simple over the centre; we present here the easy proof of this interesting result.

Lemma 2.7 *Let G be a Jónsson group of cardinality $\aleph$. Then G is perfect, and $G/Z(G)$ is a simple group of cardinality $\aleph$.*

Proof — Let x be any non-central element of G. Then the centralizer $C_G(x)$ is a proper subgroup of G, and so its cardinality is strictly smaller than $\aleph$. It follows that $|G : C_G(x)| = \aleph$, and hence the conjugacy class of x in G has cardinality $\aleph$. Therefore also the normal closure of $\langle x\rangle$ in G has cardinality $\aleph$, and so $\langle x\rangle^G = G$. In particular the factor group $G/Z(G)$ is simple, and of course it has cardinality $\aleph$. Obviously, G' is not contained in $Z(G)$, and if x is an element of $G' \setminus Z(G)$, it follows that $G = \langle x\rangle^G = G'$ is perfect. □

Proposition 2.8 *Let G be an uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are subnormal of defect at most k. If the subgroup $\gamma_{f(k)+1}(G)$ has cardinality $\aleph$, then it is a Jónsson group. In particular, if G has no Jónsson normal subgroups of cardinality $\aleph$, then $\gamma_{f(k)+1}(G)$ has cardinality strictly smaller than $\aleph$.*

Proof — Assume for a contradiction that $L = \gamma_{f(k)+1}(G)$ is not a Jónsson group, so that it contains a proper subgroup H of cardinality $\aleph$. As all subgroups of G containing H are subnormal of defect at most k, the normal closure H^L is properly contained in L and the factor group L/H^L is nilpotent by Roseblade's theorem. In particular L' is a proper subgroup of L. On the other hand, it is clear that L is contained in any normal subgroup of G of cardinality $\aleph$, and so L' has cardinality strictly smaller than $\aleph$. Thus L/L' has cardinality $\aleph$, which is impossible by Corollary 2.6. Therefore L is a Jónsson group. □

Note that the main results of this section depend on the existence of abelian subgroups of high cardinality. This assumption is much stronger than that of being a non-Jónsson group. In fact, it was proved by B.H. Neumann [8] that there exists an uncountable periodic nilpotent group of class 2 all of whose abelian subgroups are countable. Neumann's example shows in particular that there is no chance of improvement of the celebrated theorem of Hall-Kulatilaka and Kargapolov on the existence of infinite abelian subgroups in locally finite groups.

3 Normal subgroups

If G is an uncountable group whose uncountable subgroups are normal, it follows from Theorem 2.5 that G is a Dedekind group, provided that it contains an uncountable abelian subgroup. However, the existence of such a subgroup is not assured even in the case of an uncountable nilpotent group. In fact, A. Ehrenfeucht and V. Faber [3] constructed an extraspecial p-group G of cardinality $\aleph_1$ in which all abelian subgroups are countable; of course, in this case every uncountable subgroup of G contains G', and so it is normal in G. In their construction the Continuum Hypothesis was used, but in a later paper S. Shelah and J. Steprans [15] showed that this is unnecessary. The aim of this section is to describe the structure of uncountable non-Dedekind groups in which every uncountable subgroup is normal.

Observe first that the above example is not residually countable, and in fact the first result of this section shows that the situation can be controlled in the case of uncountable groups of cardinality $\aleph$ which are residually of cardinality strictly smaller than $\aleph$. Recall here that if $\mathfrak{X}$ is a class of groups, a group G is called *residually* $\mathfrak{X}$ if it has a collection of normal subgroups $(N_i)_{i \in I}$ such that G/N_i belongs to $\mathfrak{X}$ for each i and

$$\bigcap_{i \in I} N_i = \{1\}.$$

In particular, a group is residually finite if the intersection of all its (normal) subgroups of finite index is trivial.

Theorem 3.1 *Let* G *be an uncountable group of regular cardinality* $\aleph$ *in which all subgroups of cardinality* $\aleph$ *are normal. If* G *is residually a group of cardinality strictly smaller than* $\aleph$*, then all subgroups of* G *are normal.*

Proof — Let $\mathfrak{X}$ be a collection of normal subgroups of G such that each G/X has cardinality strictly smaller than $\aleph$ for each element X of $\mathfrak{X}$ and

$$\bigcap_{X \in \mathfrak{X}} X = \{1\}.$$

In particular, every element X of $\mathfrak{X}$ has cardinality $\aleph$, and so G/X is a Dedekind group. Clearly, it can be assumed that G/N is not abelian for some $N \in \mathfrak{X}$, so that the factor group $G/N \cap X$ is not abelian for all elements X of $\mathfrak{X}$. On the other hand, $G/N \cap X$ has cardinality strictly smaller than $\aleph$, and so it is a Dedekind group. Thus

$$|G' : G' \cap N| = |G' : G' \cap N \cap X| = 2,$$

and hence $G' \cap N = G' \cap N \cap X$. It follows that

$$G' \cap N = \bigcap_{X \in \mathfrak{X}} G' \cap N \cap X = \{1\},$$

and in particular N is an abelian group of cardinality $\aleph$. Therefore G is a Dedekind group by Theorem 2.5. □

Let H be a Jónsson group of cardinality $\aleph$, and let K be a Dedekind group whose cardinality is strictly smaller than $\aleph$. The direct product $G = H \times K$ has cardinality $\aleph$, and if X is any subgroup of G of cardinality $\aleph$, then X contains H and so it is normal in G. On the other hand, H is perfect by Lemma 2.7, and hence $G'' = H$ is a Jónsson group. Observe that in this example the commutator subgroup G' is not a Jónsson group if K is not abelian, for instance when K is a quaternion group of order 8.

As $f(1) = 2$, it follows directly from Proposition 2.8 that if G is an uncountable group of regular cardinality $\aleph$ whose subgroups of cardinality $\aleph$ are normal, then the subgroup $\gamma_3(G) = [G', G]$ has cardinality strictly smaller than $\aleph$, provided that G has no Jónsson normal subgroup of cardinality $\aleph$. However, this result can be improved in the following way.

Lemma 3.2 *Let G be an uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. If the commutator subgroup G' of G has cardinality $\aleph$, then G'' is a Jónsson group of cardinality $\aleph$. In particular, if G has no Jónsson normal subgroups of cardinality $\aleph$, then G' has cardinality strictly smaller than $\aleph$.*

Proof — Assume that the abelian group G'/G'' has cardinality $\aleph$, so that there exists a subgroup X of G such that $G'' < X < G'$ and both groups X/G'' and G'/X have cardinality $\aleph$. Then X is normal in G and G/X is a Dedekind group, which is impossible as its commutator subgroup is infinite. It follows that G'/G'' has cardinality strictly smaller than $\aleph$, and hence G'' has cardinality $\aleph$. On the other hand, G'' is obviously contained in every subgroup of G of cardinality $\aleph$, so that all its proper subgroups have cardinality strictly smaller than $\aleph$. Therefore G'' is a Jónsson group. □

Recall that a group G is said to be 2-*Engel* if the identity $[x, y, y] = 1$ holds in G. We refer to [10] (Part 2, Chapter 7) for the main properties of 2-Engel groups.

Corollary 3.3 *Let G be an uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. If G has no Jónsson normal subgroups of cardinality $\aleph$, then it is a 2-Engel group. In particular, G is metabelian and nilpotent of class at most 3.*

Proof — The commutator subgroup G' has cardinality strictly smaller than $\aleph$ by Lemma 3.2. If x is any element of G, the conjugacy class of x is contained in the subgroup $\langle x, G'\rangle$, and so it has cardinality strictly smaller than $\aleph$. Then the centralizer $C_G(x)$ has cardinality $\aleph$, so that it is normal in G and hence x commutes with all its conjugates. Therefore the normal closure $\langle x\rangle^G$ is abelian, and G is a 2-Engel group. Finally, it is known that any 2-Engel group is nilpotent of class at most 3, and so also metabelian (see for instance [10] Part 2, p.47). □

The above corollary holds in particular in the case of locally soluble groups. Moreover, it has also the following consequence.

Corollary 3.4 *Let G be an uncountable locally finite group of regular cardinality $\aleph$ whose subgroups of cardinality $\aleph$ are normal. Then the commutator subgroup G' has cardinality strictly smaller than $\aleph$, and G is a 2-Engel group.*

Proof — The statement follows from Corollary 3.3, as it is known that Jónsson groups cannot be locally finite (see [7], Theorem 2.6). □

We can now prove our main result on the structure of uncountable locally finite groups whose uncountable subgroups are normal.

Theorem 3.5 *Let G be an uncountable locally finite group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. Then either G is a Dedekind group or there is a prime number p such that $G = H \times K$, where H is a nilpotent p-group of cardinality $\aleph$ and K is a Dedekind p'-group of cardinality strictly smaller than $\aleph$. Moreover:*

(a) *If $p > 2$, then the subgroup $\Omega_1(H)$ has exponent p and cardinality $\aleph$, while the groups H^p and $H/\Omega_1(H)$ have cardinality strictly smaller than $\aleph$.*

(b) *If $p = 2$, then either G has a homomorphic image isomorphic to Q_8 and $H^2 = H'$ or G has no homomorphic images isomorphic to Q_8 and $(G')^2 = \{1\}$.*

Proof — Assume that G is not a Dedekind group, so that by Theorem 2.5 it has no abelian subgroups of cardinality $\aleph$. Moreover, G is nilpotent by Corollary 3.4, so that it can be decomposed into the direct product of its Sylow subgroups and it follows from Lemma 2.1 that there is a prime number p such that the unique Sylow p-subgroup H of G has cardinality $\aleph$. Let K be the subgroup of G consisting of all elements of order prime to p, so that $G = H \times K$ and $K \simeq G/H$ is a Dedekind group which has cardinality strictly smaller than $\aleph$, because $H \simeq G/K$ is not a Dedekind group.

Suppose first that $p > 2$. As G is a 2-Engel group, all its 2-generator subgroups have nilpotency class at most 2 (see for instance [10] Part 2, Theorem 7.15), and so for each non-negative integer n the group

$$\Omega_{n+1}(H)/\Omega_n(H)$$

has exponent p. In particular, $\Omega_1(H)$ has exponent p. Since

$$H = \bigcup_{n\in\mathbb{N}} \Omega_n(H),$$

by the regularity of $\aleph$ we may consider the smallest positive integer m such that the subgroup $\Omega_m(H)$ has cardinality $\aleph$. Assume for a contradiction that $m > 1$. Let x be any element of the set

$$X = \{h^p \mid h \in \Omega_m(H)\}.$$

If y and z are elements of $\Omega_m(H)$ such that $y^p = z^p = x$, we have

$$(y^{-1}z)^p = y^{-p}z^p[z, y^{-1}]^{p(p-1)/2} = [z^p, y^{-1}]^{(p-1)/2} = 1,$$

and hence $y^{-1}z$ belongs to $\Omega_1(H)$. It follows that the set

$$W_x = \{y \in \Omega_m(H) \mid y^p = x\}$$

is contained in a suitable coset of $\Omega_1(H)$, and so it has cardinality strictly smaller than $\aleph$. On the other hand, X lies in $\Omega_{m-1}(H)$, so that is has likewise cardinality strictly smaller than $\aleph$, and hence

$$\Omega_m(H) = \bigcup_{x\in X} W_x$$

has cardinality strictly smaller than $\aleph$. This contradiction shows that $m = 1$, and so $\Omega_1(H)$ has cardinality $\aleph$. It follows that all subgroups of $H/\Omega_1(H)$ are normal, so that $H/\Omega_1(H)$ is an abelian group and hence the commutator subgroup H' lies in $\Omega_1(H)$. Let h and k be arbitrary elements of H. Then

$$[h^p, k] = [h, k]^p = 1,$$

so that H^p is contained in $Z(H)$ and in particular H^p must have cardinality strictly smaller than $\aleph$ by Theorem 2.5. Moreover, the map

$$\varphi : H \longrightarrow H^p,$$

defined by putting $h^\varphi = h^p$ for each element h of H, is an epimorphism with kernel $\Omega_1(H)$, and so also $H/\Omega_1(H) \simeq H^p$ has cardinality strictly smaller than $\aleph$.

Suppose now that $p=2$, so that in particular K is abelian and $G' = H'$. Moreover, it follows from a classical result of Levi that $[G', G]^3 = \{1\}$ (see [10] Part 2, Theorem 7.14), so that $[G', G] = \{1\}$ and G has class 2. Assume that G has a homomorphic image which is isomorphic to the quaternion group Q_8 of order 8, so that there exists a normal subgroup L of H such that $H/L \simeq Q_8$. As H' has cardinality strictly smaller than $\aleph$ by Corollary 3.4, it follows that $L/L \cap H'$ is an abelian normal subgroup of cardinality $\aleph$ of $H/L \cap H'$, and hence $H/L \cap H'$ is a Dedekind non-abelian group by Theorem 2.5. Therefore H^2 is contained in H', and so $H^2 = H'$.

Suppose finally that $p = 2$, but Q_8 cannot be obtained as homomorphic image of G (and so neither of H), so that all Dedekind homomorphic images of G are abelian. Let U be the subgroup of all elements h of H such that h^2 belongs to G'. Since H/G' is an abelian 2-group of cardinality $\aleph$, also the group U/G' has cardinality $\aleph$. It follows that also the subgroup V, consisting of all elements u of U such that u^2 lies in U', has cardinality $\aleph$. Thus G/V is a Dedekind group, so that it is abelian and hence $G' \leqslant V$. If a and b are arbitrary elements of G', the element a^2 belongs to U', and so $[a, b]^2 = [a^2, b]$ lies in $[U', U] = \{1\}$. Therefore $(G')^2 = \{1\}$. □

In the non-periodic case, we have the following information.

Theorem 3.6 *Let G be an uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. If G has no Jónsson normal subgroups of cardinality $\aleph$, then the commutator subgroup G' of G is periodic-by-divisible.*

Proof — The group G is 2-Engel by Corollary 3.3, so that the elements of finite order of G form a subgroup T. Clearly, it can be assumed without loss of generality that G' is not periodic. Then the torsion-free group G/T is not abelian and hence T has cardinality strictly smaller than $\aleph$. It follows that G/T has cardinality $\aleph$, so that we may replace G by G/T, and suppose that G is torsion-free, and so in particular of class 2.

Assume for a contradiction that G' is not divisible, so that it contains a subgroup K such that $|G' : K| = m$, where either m is an odd prime number or $m = 4$. Then $(G/K)^m$ is contained in $Z(G/K)$. The centre $Z(G)$ has cardinality strictly smaller than $\aleph$ by Theorem 2.5, and hence

$$G/Z(G) \simeq \big(G/Z(G)\big)^m$$

is a torsion-free abelian group of cardinality $\aleph$. It follows that also G^m has cardinality $\aleph$, and so $(G/K)^m$ is an abelian normal subgroup of cardinality $\aleph$ of G/K. Another application of Theorem 2.5 yields that G/K is a Dedekind group, which is impossible as $|G'/K| > 2$. This contradiction proves the statement. □

Corollary 3.7 *Let G be a torsion-free uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal. If the commutator subgroup G' of G is residually finite, then G is abelian.*

Proof — As G' is residually finite, the group G has no Jónsson normal subgroups, and so it follows from Theorem 3.6 that G' is divisible. Thus $G' = \{1\}$ and G is abelian. □

4 Metahamiltonian groups

The aim of this section is to compare the class of uncountable groups whose uncountable non-abelian subgroups are normal to the class of metahamiltonian groups. As we mentioned in the introduction, a group G is metahamiltonian if all its non-abelian subgroups are normal.

Lemma 4.1 *Let G be an uncountable group of regular cardinality $\aleph$ in which all non-abelian subgroups of cardinality $\aleph$ are normal. If A is an abelian normal subgroup of G of cardinality $\aleph$, then there exists a G-invariant subgroup B of A such that both B and A/B have cardinality $\aleph$.*

Proof — It can obviously be assumed that A is not contained in the centre $Z(G)$, so that the centralizer $C = C_G(A)$ is a proper subgroup of G containing A. Let X/C be any non-trivial subgroup of G/C. Then X is a non-abelian subgroup of cardinality $\aleph$, so that it is normal in G and hence G/C is a Dedekind group. Let zC be a non-trivial element of $Z(G/C)$, and let a be an element of A such that $[a, z] \neq 1$. As the subgroup

$$E = \langle a, A \cap \langle z \rangle \rangle^{\langle z \rangle}$$

is countable, the factor group A/E has cardinality $\aleph$, and hence it follows from Lemma 2.3 that A/E contains a proper $\langle z \rangle$-submodule B/E of cardinality $\aleph$ such that also A/B has cardinality $\aleph$. Then $B\langle z \rangle$ is a non-abelian subgroup of cardinality $\aleph$, so that it is normal in G, and hence also $B = B\langle z \rangle \cap A$ is a normal subgroup of G. □

Lemma 4.2 *Let G be an uncountable group of regular cardinality $\aleph$ in which all non-abelian subgroups of cardinality $\aleph$ are normal. If G has no simple sections of cardinality $\aleph$, then its commutator subgroup G' has cardinality striclty smaller than $\aleph$.*

Proof — Assume for a contradiction that G' has cardinality $\aleph$. Since G has no simple sections of cardinality $\aleph$, it is known that either G' is abelian or it contains two proper non-abelian subgroups X and Y of cardinality $\aleph$ such that $X < Y$ (see [5], Corollary 3.3). But in the latter case the subgroup X is normal in G and G/X is a Dedekind group, which is impossible as $|G' : X| > 2$. Therefore G' is abelian, and so It follows from Lemma 4.1 that G' contains a G-invariant subgroup N of cardinality $\aleph$ such that also G'/N has cardinality $\aleph$. Thus G/N is metahamiltonian, and hence it has finite commutator subgroup. This contradiction proves that G' has cardinality strictly smaller than $\aleph$. □

Our last result shows that the study of uncountable groups whose uncountable non-abelian subgroups are normal can be reduced to that of uncountable groups whose uncountable subgroups are normal, at least when the group has no large simple sections.

Theorem 4.3 *Let G be an uncountable group of regular cardinality $\aleph$ in which all non-abelian subgroups of cardinality $\aleph$ are normal. If G has no simple sections of cardinality $\aleph$, then either all subgroups of G of cardinality $\aleph$ are normal or G is metahamiltonian.*

Proof — Assume that G contains a non-normal subgroup A of cardinality $\aleph$, which of course must be abelian. As the commutator subgroup G' of G has cardinality strictly smaller than $\aleph$ by Lemma 4.2, also the conjugacy classes of elements of G have cardinality strictly smaller than $\aleph$, and so $|G : C_G(x)| < \aleph$ for each element x of G. Let E

be any finitely generated non-abelian subgroup of G. Then the centralizer $C_A(E)$ has cardinality $\aleph$, and hence by Lemma 2.2 it contains two subgroups U and V of cardinality $\aleph$ such that

$$U \cap V = E \cap \langle U, V \rangle = \{1\}.$$

The non-abelian subgroups EU and EV are normal in G, and hence also $E = EU \cap EV$ is a normal subgroup of G. Therefore G is a metahamiltonian group. □

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