Advances in Group Theory and Applications
© 2017 AGTA - www.advgrouptheory.com/journal
3 (2017), pp. 67–73
ISSN: 2499-1287
DOI: 10.4399/97888255036925

Linear Groups with Many Profinitely Closed Subgroups

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(*Received Nov. 16, 2016; Accepted Jan. 17, 2017 — Communicated by F. de Giovanni*)

Abstract

If G is a linear group with every subgroup of G of infinite Prüfer rank closed in the profinite topology on G, we prove that either every subgroup of G is closed in this topology or G itself has finite Prüfer rank.

Mathematics Subject Classification (2010): 20E26, 20H20, 13E05

Keywords: profinite topology; linear groups; Noetherian and Artinian modules over commutative rings

1 Introduction

Let G be a group in which every subgroup of G of infinite rank is closed in the profinite topology on G. (Rank in this paper always means Prüfer rank.) In [2] de Falco, de Giovanni and Musella prove that if G is also either nilpotent-by-finite or an FC-group, then either G has finite rank or every subgroup of G is profinitely closed in G. Groups with latter property have been called extended residually finite (or ERF for short) in a number of works, e.g. [3] and [11], and we continue that usage here.

In the introduction to [2] the authors point out that their arguments in [2] cannot be used to prove a similar result if the group G is just linear. However a comparable result does exist.

Theorem *Let* G *be a linear group with each of its subgroups of infinite rank profinitely closed in* G*. Then either* G *has finite rank or* G *is ERF. Further if* G *has characteristic 0, then* G *is always soluble-by-finite of finite rank and if* G *has positive characteristic, then* G *is abelian-by-finite and either has finite rank or is ERF.*

Clearly an infinite elementary abelian p-group (p a prime) is isomorphic to a linear group of characteristic p, is ERF but is of infinite rank. Abelian ERF groups are completely characterised in Proposition 3.1 of [3]. They are the abelian groups with no Prüfer sections or equivalently those whose primary components have finite exponent and whose torsion-free quotient has finite rank and no Prüfer sections.

Corollary 1 *Let* G *be a group of automorphisms of the Noetherian module* M *over a commutative ring* R*. Suppose every subgroup of* G *of infinite rank is profinitely closed in* G*. Then* G *is soluble-by-finite and either* G *has finite rank or* G *is ERF.*

Unlike the linear case we will see below that if in Corollary 1 the group G is of infinite rank, then G need not by abelian-by-finite. The same remark applies the Corollary 2 below. Notice that in Corollary 1, the ring $R/\mathrm{Ann}_R M$ is always Noetherian. The following is an immediate consequence of Corollary 1 and [9] Theorem 3.2.

Corollary 2 *Let* G *be a group of automorphisms of the Artinian module* M *over a commutative ring* R*. Assume* $R/\mathrm{Ann}_R M$ *is Noetherian and suppose that every subgroup of* G *of infinite rank is profinitely closed in* G*. Then* G *is soluble-by-finite and either* G *has finite rank or* G *is ERF.*

2 Proof of the Theorem

For brevity let $\mathfrak{X}$ denote the class of all groups G with every subgroup of G of infinite rank profinitely closed in G. Then $\mathfrak{X}$ is subgroup and homomorphic image closed. If G is free abelian of infinite rank there exists $N \leqslant G$ with G/N a Prüfer group. Then N cannot be profinitely closed in G. Also G/N is locally cyclic so $\mathrm{rank}(G)$ is at most $1 + \mathrm{rank}(N)$. Consequently N has infinite rank and $G \notin \mathfrak{X}$. If G is a non-abelian free group, the above applied to G'/G'' yields that $G \notin \mathfrak{X}$.

Suppose G is an $\mathfrak{X}$-subgroup of $GL(n, F)$, n a positive integer and F a field, which we may assume is algebraically closed. By a simple corollary (see [6], 10.17) of Tit's Theorem ([5]) the group G is soluble-by-(locally finite). Also every torsion-free abelian section of G lies in $\mathfrak{X}$ and hence, see above, has finite rank. Thus G is poly (infinite cyclic or locally finite). That is, G has finite Hirsch number and therefore G has a locally finite normal subgroup T with G/T a finite extension of a torsion-free soluble group of finite rank (e.g. Lemma 4 of [10]).

Suppose first that $\mathrm{char}(F) = 0$. Then T is abelian-by-finite ([6], 9.4) and therefore T^0, the Zariski connected component of T containing 1, is an abelian normal subgroup of G of finite index in T, see [6] Chapter 5, especially 5.11. But $\mathrm{rank}(T^0) \leqslant n$ by [6], 2.2. It follows that G is a finite extension of a soluble group of finite rank.

Now assume that $\mathrm{char}(F) = p > 0$. Suppose T is infinite and simple. Let P be a maximal p-subgroup of T. Then P is nilpotent of finite exponent. Clearly T has no proper subgroups of finite index, so P cannot be profinitely closed in T. Thus P has finite rank and hence is finite. But then T is abelian-by-finite by a corollary ([6], 9.7) of the Brauer-Feit Theorem [1]. Therefore T cannot be infinite and simple.

Now suppose T has no non-trivial soluble normal subgroups. Then T has a normal subgroup E such that E is a direct product of a finite number of simple groups and $C_T(E) = \langle 1 \rangle$, see [4], 5.1.5; note that the Hirsch-Plotkin radical of T is soluble and hence here is trivial. Then E must be finite by the infinite simple case discussed above. Hence $T/C_T(E)$ is also finite. But $C_T(E) = \langle 1 \rangle$; therefore T is finite in this case.

Now consider the general case. Let S be the product of all the soluble normal subgroups of T. Then S is soluble ([6], 3.8). Since S is Zariski closed in T ([6], 5.11), so T/S is isomorphic to a linear group over F ([6], 6.4). Consequently T/S is finite by the previous case. Clearly S is normal in G and therefore G is soluble-by-finite.

Suppose G has infinite rank. We need to prove that G is ERF. By Lemma 4.2 of [3] a finite extension of an ERF group is also ERF. Thus we may replace G by any one of its subgroups of finite index. Consequently we may assume that G is soluble, (Zariski) connected and a subgroup of the lower triangular group $\mathrm{Tr}(n, F)$ (see [6], 5.8; recall F is algebraically closed).

Linear groups over F are abelian-by-finite if either they have finite rank ([6], 10.9) or they are ERF ([11], Theorem 1.1 (ii)). We induct on n; if $n = 1$ then G is abelian and the result is clear, e.g. by [2] Theorem A, so assume $n \geqslant 2$. By induction the projections of G onto the first $n-1$ of its rows and columns and onto the last $n-1$ of its rows and columns satisfy the Theorem and in particular are abelian-by-finite. Thus we may assume that $A = G' \leqslant 1 + Fe_{n,1}$, where

$$\{e_{i,j} : i, j = 1, 2, \ldots, n\}$$

denotes the standard set of n by n matrix units.

Now A is an elementary abelian p-group. Suppose A is finite. Then the centralizer $C = C_G(A)$ has finite index in G. Also C is nilpotent of infinite rank, so C and G are ERF by Theorem A of [2]. Therefore we may assume that A is infinite and hence of infinite rank. By hypothesis every subgroup of G of infinite rank is profinitely closed in G. Thus consider an arbitrary subgroup H of G of finite rank. Note that $H \cap A$ is finite.

Suppose first that $H \cap A \neq \langle 1 \rangle$. Since A embeds into the additive group of F and G acts on it via a map of G into the multiplicative group of F, so $C_H(A) = C_H(H \cap A)$ and the latter clearly has finite index in H. If $a \in A \backslash H$, there exists $B_a \geqslant H \cap A$ with $A = \langle a \rangle \times B_a$. Then

$$C_a = \bigcap_{h \in H} (B_a)^h \geqslant H \cap A$$

has finite index in A and hence has infinite rank. Therefore HC_a is profinitely closed in G. Consequently so is

$$K = \bigcap_{a \in A \backslash H} HC_a.$$

Clearly $K \leqslant HA$. Also

$$K \cap A = \bigcap_a (HC_a \cap A) = \bigcap_a (H \cap A)C_a = \bigcap_a C_a = H \cap A.$$

Therefore $K = K \cap HA = H(K \cap A) = H(H \cap A) = H$ and hence H is profinitely closed in G.

Now assume that $H \cap A = \langle 1 \rangle$. If $A = \langle a \rangle \times B$, then B has infinite rank. Thus B is profinitely closed in G and hence there exists a normal subgroup N of G of finite index with $N \cap A \leqslant B$. Clearly $N \cap A$ has infinite rank and is normalized by H. Hence $H(N \cap A)$ is profinitely closed in G and consequently so is $K = \bigcap_N H(N \cap A)$. Clearly $\bigcap_N (N \cap A) = \langle 1 \rangle$ and $K \leqslant HA$. Thus

$$K \cap A = \bigcap_N H(N \cap A) \cap A = \bigcap_N (H \cap A)(N \cap A) = \bigcap_N (N \cap A) = \langle 1 \rangle$$

and $K = K \cap HA = H(K \cap A) = H$. Therefore H is profinitely closed in G and the proof of the theorem is complete.

3 Proof of Corollary 1

By 6.1 of [8] (or see [7] for a somewhat less explicit version) there exists a finite number of homomorphisms $\rho_i : G \to GL(m, F_i)$, where m is a positive integer and the F_i are fields with distinct characteristics, such that $\cap_i \ker \rho_i = \langle 1 \rangle$. We can apply the Theorem to each $G\rho_i$, so after replacing G by one of its subgroups of finite index we may assume that each $G\rho_i$ is either soluble of finite rank or is abelian and ERF. Since abelian groups are ERF if and only if they have no Prüfer sections, so direct products of finitely many abelian ERF groups are ERF. It follows easily that G has normal subgroups M and N such that G/M is soluble of finite rank, G/N is abelian and ERF and $M \cap N = \langle 1 \rangle$.

We may assume that G has infinite rank and seek to prove that G is ERF. Let H be any subgroup of G of finite rank. It suffices to prove that H is profinitely closed in G. Now $G' \leqslant N$, so

$$[M, G] \leqslant M \cap N = \langle 1 \rangle.$$

Also G has infinite rank and G/M has finite rank. Hence M has infinite rank, while $H \cap M$ has finite rank. If B is a subgroup of M of finite index, then B has infinite rank, H centralizes B and therefore HB is profinitely closed in G. Consequently so is $K = \bigcap_B HB$.

Clearly $K \leqslant HM$. Also

$$K \cap M = \bigcap_B (HB \cap M) = \bigcap_B (H \cap M)B.$$

Now G/N is ERF, so M is too. Consequently $M/(H \cap M)$ is residually finite. Therefore

$$\bigcap_B (H \cap M)B = H \cap M.$$

But then $K \cap M = H \cap M$ and $K = H(K \cap M) = H$. Thus H is profinitely closed in G and G is ERF, completing the proof of Corollary 1.

Polycyclic groups are always ERF. Arguments similar to those above show the if G is the direct product of a polycyclic group X and an infinite elementary abelian p-group Y for some prime p, then G is ERF. (Here if B has finite index in Y, then G/B is polycyclic and hence HB is profinitely closed in G.) Now X embeds into $GL(n, \mathbf{Q})$ for some integer n and $\mathbf{Q}$ the rationals and Y embeds into $GL(2, F)$ for some field F of characteristic p. Then G embeds into $\mathrm{Aut}_R M$ for $M = \mathbf{Q}^{(n)} \oplus F^{(2)}$ and $R = \mathbf{Q} \oplus F$. Thus in both corollaries if G is of infinite rank and ERF, there is no need for G to be abelian-by-finite. Of course here M and R are both Noetherian and Artinian.

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