Advances in Group Theory and Applications
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3 (2017), pp. 31–53
ISSN: 2499-1287
DOI: 10.4399/97888255036923

Characterizations of Fitting p-Groups whose Proper Subgroups are Solvable *

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(*Received Jun. 6, 2016; Accepted Feb. 21, 2017 — Communicated by F. de Giovanni*)

Abstract

This work continues the study of infinitely generated groups whose proper subgroups are solvable and in whose homomorphic images normal closures of finitely generated subgroups are residually nilpotent. In [4], it has been shown that such a group, if not solvable, is a perfect Fitting p-group for a prime p with additional restrictions. Therefore this work is a study of Fitting p-groups whose proper subgroups are solvable. Here a condition is given for the imperfectness of a Fitting p-group satisfying the normalizer condition, where $p \neq 2$. Hence it follows that if every proper subgroup of the group in question is solvable, then the group itself is solvable. Furthermore some conditions are given for a perfect Fitting p-group whose proper subgroups are solvable in order for the subgroup generated by normal subgroups of a given derived length to be proper, where $p \neq 2$.

Mathematics Subject Classification (2010): 20F19, 20F50, 20E25
Keywords: Fitting group; minimal non-solvable group

1 Introduction

In recent years infinitely generated minimal non-solvable groups (MNS-group for short) have been the subject of several studies (for example see [1],[2],[3],[4],[5]). But very little is known yet about these groups. Let G be such a periodic group. In [4], it has been shown that

* The author is very grateful to Professor F. de Giovanni and the referee for their patience and understanding during the preparation of this work

if G is perfect and in every homomorphic image of G normal closures of finitely generated subgroups are residually nilpotent, then G is a Fitting p-group and has a homomorphic image in whose homomorphic images the $(*)$-condition (see below) cannot be satisfied ([4], Theorem 1.4 (b)). However if the $(*)$-condition is satisfied, then G is solvable ([4], Theorem 1.1). Thus in [5], a perfect Fitting p-group G satisfying the normalizer condition is considered and it has been shown that under an additional condition denoted by $(**)$, G cannot be generated by a subset of finite exponent ([5], Theorem 1.1 and Corollary 1.2) and if in addition G is an MNS-group, then $S_t(G) \neq G$ for every $t \geqslant 1$ ([5], Theorem 1.3).

The studies mentioned above are attempts towards understanding perfect locally finite p-groups about whose structure nothing is known yet. These groups may be divided into two basic types as follows. Those that can be generated by a subset of finite exponent and those that cannot. McLain's group ([11], 12.1.9 (a)) belongs to the first type. Also the group of [2], which is a minimal non-(finite exponent) group belongs to the first type but the author has no knowledge about the existence of such a group. However there exist perfect locally nilpotent groups of exponent p by [12], Theorem 4. On the other hand a perfect totally imprimitive p-group satisfying the cyclic-block-property cannot be generated by a subset of finite exponent (see reference 3 in [5]). In the present work the Fitting MNS-p groups of Theorem 1.3 cannot be generated by normal abelian subgroups but it is not known whether they belong to the first type.

At this point it will be convenient to introduce some definitions and notations.

Let G be a group, $w \in G \setminus Z(G)$ and V be a finitely generated subgroup of G with $w \notin V$. Then the ordered pair (w, V) is called a *Λ-pair* for G. A subgroup E of G which is maximal with respect to the condition that $w \notin E$ but $V \leqslant E$ is called a *(w, V)-maximal subgroup* of G. Let

$$E^*(w, V) = \{E : E \text{ is an } (w, V) - \text{ maximal subgroup of } G\}$$

and

$$W^*(w, V) = \{\mathrm{Core}_G(E) : E \in E^*(w, V)\}$$

Again let (w, V) be a Λ-pair for G. If there exists a proper subgroup L (which need not be unique) of G such that

$$w \notin V \text{ but } w \in \langle V, y \rangle \text{ for every } y \in G \setminus L$$

then (w, V, L) is called a $(*)$- *triple* for G. Note that the statement

$$\text{“}(w, V, L) \text{ is a } (*)\text{-triple”}$$

implies that

$$\bigcap_{y \in G \setminus L} \langle V, y \rangle \neq V$$

and conversely if

$$\bigcap_{y \in G \setminus L} \langle V, y \rangle \neq V,$$

then for each

$$u \in \bigcap_{y \in G \setminus L} \langle V, y \rangle \setminus V$$

the triple (u, V, L) is a $(*)$-triple.

Next suppose that every proper subgroup of G is solvable and let (w, V) be a Λ-pair for G. Then (w, V) is called a *distinguished pair* for G, if there exists no $(*)$-triples (w, U, L) with $V \leqslant U$ and if

$$d(\langle V, y \rangle) > d(V) \text{ implies that } w \in \langle V, y \rangle \text{ for every } y \in G$$

where $d(V)$ denotes the derived length of V. Let (w, V) be a distinguished pair for G and let $E \in E^*(w, V)$. Then $d(\langle V, y \rangle) = d(V)$ for $y \in E$, because if $d(\langle V, y \rangle) > d(V)$, then $w \in \langle V, y \rangle$ by the definition of a distinguished pair, but $w \notin E$ by the definition of E. We note also that if G is an MNS-group, then G has a homomorphic image H whose homomorphic images cannot have $(*)$-triples by [4], Theorem 1.4, but every homomorphic image of H has distinguished pairs. A distinguished pair (w, V) for the group G is called a *dominant pair* if it satisfies the stronger condition that $d(E) = d(V)$ for every $E \in E^*(w, V)$ (for the existence of distinguished pairs and dominant pairs, see [4], Lemma 3.1 and Lemma 4.1). Note that in an MNS-group G a distinguished (dominant) pair (w, V) is obtained from a Λ-pair and the significance of a distinguished (dominant) pair (w, V) is that $E^*(w, V)$ is a smaller class than the class of maximal subgroups of the Λ-pair from which it is obtained. Another point is that if (w, V, L) is a $(*)$-triple ((w, V) is a distinguished pair), then also (wv, V, L) is a $(*)$-triple ((wv, V) a distinguished pair) and $E^*(w, V) = E^*(wv, V)$ for every $v \in V$.

Again let G be a group. A subgroup E of G is said to satisfy the $(**)$-*property* if

$$N_G(E) = N_G(E').$$

A Λ-pair (w, V) is said to satisfy the $(**)$-property if every element of $E^*(w, V)$ satisfies it.

This work continues the work of [5] and contains new characterizations of perfect Fitting p-groups (whose proper subgroups solvable). More precisely let G be a Fitting p-group satisfying the normalizer condition. If the homomorphic images of G satisfy the additional condition $(**)$ in certain subgroups and if $p \neq 2$, then G cannot be perfect (Theorem 1.1). If the group G is an MNS-p-group and satisfies the additional condition $(**)$ in certain subgroups, then G is solvable, where $p \neq 2$. In Theorem 1.3 a perfect Fitting MNS-p-group G is considered such that in every homomorphic image of G dominant pairs satisfy $(**)$ and it is shown that $\langle S_t(G)\rangle \neq 1$ for every $t \geqslant 1$, which generalizes [5], Theorem 1.3.

The property that $\langle S_t(G)\rangle \neq G$ for every $t \geqslant 1$ almost always comes up in the study of perfect MNS-p-groups. For example an important step in the proof of [4], Theorem 1.1, is the showing of

$$\langle S_t(G)\rangle \neq G \text{ for every } t \geqslant 1$$

under the existence of the $(*)$-condition. Also it is the content of Theorem 1.3 in the present work. Therefore the following appears to be a basic question in this area.

Question 1 *Let* G *be a locally finite perfect* p*-group whose proper subgroups are solvable. Is it true that* $\langle S_t(G)\rangle \neq G$ *for every* $t \geqslant 1$*? What can be said if every proper subgroup of* G *is nilpotent-by-abelian?*

As usual in a group G the derived length (class) of a solvable (nilpotent) subgroup K is denoted by $d(K)$ ($c(K)$). G is called *metabelian* if $d(G) = 2$. Put

$$S_t(G) = \{K \lhd G : d(K) \leqslant t\}$$

and if G is a p-group, put

$$S_t(G)^e = \{K \in S_t(G) : \exp(K) \leqslant p^e\}.$$

Theorem 1.1 *Let* G *be a Fitting* p*-group satisfying the normalizer condition, where* $p \neq 2$*. Suppose that in every homomorphic image* H *of* G *every Λ-pair (w_H, V_H) has a (w_H, V_H)-maximal subgroup satisfying the $(**)$-property. Then G cannot be perfect.*

Corollary 1.2 *Let G be a Fitting p-group satisfying the normalizer condition in which every proper subgroup is solvable, where $p \neq 2$. Suppose that in every homomorphic image H of G every dominant pair (w_H, V_H) has a (w_H, V_H)-maximal subgroup satisfying the $(**)$-property. Then G is solvable.*

We note that if G is a Fitting p-group whose proper subgroups are solvable and $\exp(G) = p$, then G is solvable without the hypotheses of Corollary 1.2. This may be shown as follows. Assume that G is perfect. We may suppose that G has a distinguished pair (w, V) by [4], Theorem 1.4 (b) and Lemma 3.1. Also every proper subgroup of G has exponent p. Then application of [10], Theorem 7.18, shows that every proper subgroup of G is nilpotent. But this contradicts [3], Theorem 1.3, and [4], Lemma 4.6 (b). Therefore the assumption is false and so G is solvable.

Without the normalizer condition the following holds.

Theorem 1.3 *1.3 Let G be a perfect Fitting p-group in which every proper subgroup is solvable, where $p \neq 2$. Suppose that in every homomorphic image H of G every dominant pair (w_H, V_H) satisfies the $(**)$-property. Then $\langle S_t(G)\rangle \neq G$ for every $t \geqslant 1$.*

The above results are special cases of Problems 16.5 and 16.6 in [9].

Notations and definitions are standard and may be found in [6], [7],[10] and [11].

2 Proofs of Theorem 1.1 and Corollary 1.2

We begin with listing some of the properties of Λ-pairs/distinguished pairs (see also [4] and [5] for some properties of distinguished/dominant pairs).

Lemma 2.1 *Let G be a locally finite p-group and let (w, V) be a Λ-pair for G. Then the following hold.*

(a) *Let $v \in V$. Then $E^*(w, V) = E^*(wv, V)$.*

(b) *$W^*(w, V)$ contains maximal elements.*

(c) *Suppose that* G *is perfect. Let* M *be a maximal element of* $W^*(w,V)$. *There exists a finite subgroup* U *of* G *containing* V *such that* w *is not in* $U \nleqslant M$ *and if* $wM \in Z(G/M)$, *then* $wuM \notin Z(G/M)$ *for every* $u \in U \setminus M$. *Thus* $(wuM, UM/M)$ *is a* Λ*-pair for* G/M. *Furthermore*

$$E^*(wuM, UM/M) = E^*(wM, UM/M)$$

and if $R/M \in E^*(wuM, UM/M)$, *then* $R \in E^*(w,V)$. *Thus*

$$W^*(wuM, UM/M) = 1 \text{ and } Z(G/M) \neq 1.$$

(d) *Suppose that* G *is an* MNS*-group and let* (w,V) *be a distinguished/dominant pair for* G. *Let* $L \lhd G$ *such that* $w \notin VL$. *Then there exists* $E \in E^*(w,V)$ *such that* $L \leqslant E$.

Proof — (a) This is obvious since $E \leqslant G$ is (w,V)-maximal if and only if it is (wv,V)-maximal.

(b) This follows from the proof of [4], Lemma 3.4.

(c) Suppose that G is perfect. There exists $E \in E^*(w,V)$ such that $M < E$ by [4], Lemma 4.3. Hence there exists a finite subgroup U of E satisfying $V \leqslant U \nleqslant M$. Also $w \notin U$ since $w \notin E$. Thus (w,U) is a Λ-pair for G and $E^*(w,U) \subseteq E^*(w,V)$ by the proof of [4], Lemma 3.2. It is easy to see that if wM belongs to $Z(G/M)$ and if $u \in U \setminus M$, then $wuM \notin Z(G/M)$. For, in the contrary case, uM lies in $Z(G/M)$ and then $\langle u, M\rangle \lhd G$ but since $\langle u, M\rangle \leqslant E$ this contradicts the maximality of M. Therefore $(wuM, UM/M)$ is a Λ-pair for

$$G/M \text{ and } E^*(wM, UM/M) = E^*(wuM, UM/M)$$

by (a). Furthermore if $R/M \in E^*(wM, UM/M)$, then $R \in E^*(w,V)$ by the proof of [4], Lemma 4.2. Clearly then $W^*(wM, UM/M) = 1$ since M is a maximal element of $W^*(w,V)$. Finally $Z(G/M) \neq 1$ by [4], Lemma 3.5, since G is perfect and contains proper normal subgroups $\neq 1$ by [10], 12.4.1.

(d) Let $L \lhd G$ and $w \notin VL$ and suppose that (w,V) is distinguished. Then there exists a (w,V)-maximal subgroup R such that $VL \leqslant R$. Also $d(\langle V, y\rangle) = d(V)$ for every $y \in R$ since $d(\langle V, y\rangle) > d(V)$ implies that $w \in \langle y, V\rangle$ for a distinguished pair. Therefore $R \in E^*(w,V)$. If (w,V) is a dominant pair, then since each (w,V)-maximal subgroup E satisfies $d(V) = d(E)$ it follows that $R \in E^*(w,V)$. This completes the proof of the lemma. □

Lemma 2.2 *Let G be a locally finite p-group, (w, V) be a Λ-pair for G and A a normal abelian subgroup of G. Then the following hold.*

(a) *There exists an $E \in E^*(w, v)$ with $A \cap E$ is maximal.*

(b) *If G is an MNS-group and (w, V) is a distinguished pair/dominant pair for G, then the conclusion* (a) *holds in $E^*(w, V)$.*

Proof — (a) Let

$$L = \{A \cap E : E \in E^*(w, V)\}$$

be partially ordered by set inclusion. It suffices to show that L contains a maximal element. Let

$$A \cap E_1 \leqslant A \cap E_2 \leqslant \dots$$

be an ascending chain of elements of L and put

$$A^* = \bigcup_{i=1}^{\infty} A \cap E_i.$$

Obviously A^* is normalized by w and V since $\langle w, V\rangle \leqslant N_G(E_i)$ for every $i \geqslant 1$. If $w \in VA^*$, then $w = va_i$ for an $a_i \in A \cap E_i$ and $i \geqslant 1$. But then $w \in E_i$, which is a contradiction. Therefore $w \notin VA^*$ and then there exists $E \in E^*(w, V)$ with $VA^* \leqslant E$. Thus $A \cap E$ is an upper bound for the given chain. Therefore L contains a maximal element by Zorn's Lemma.

(b) Suppose that G is an MNS-group and (w, V) is a distinguished pair for G. By (a) there exists a (w, V)-maximal subgroup R of G such that $A \cap R$ is maximal. Since (w, V) is a distinguished pair, $d(V) < d(\langle V, y\rangle)$ implies that $w \in \langle V, y\rangle$ for every $y \in G$. Clearly this implies that $d(V) = d(\langle V, y\rangle)$ for every $y \in R$ and so $R \in E^*(w, V)$. Since $A \cap R$ is maximal (b) follows in this case. If (w, V) is a dominant pair, then the same property holds. □

Lemma 2.3 *Let G be a locally finite p-group, and let (w, V) be a Λ-pair for G such that $W^*(w, V) = 1$. If there exists $E \in E^*(w, V)$ satisfying $(**)$, and $Z(G)$ is infinite, then $N_G(E)$ is self-normalizing.*

Proof — Put $N = N_G(E)$. Assume if possible that there exists g in $N_G(N) \setminus N$. In this case N/E, being infinite, is locally cyclic by Lemma 2.2 of [5], and so $N/E = Z(G)E/E$ and hence $N = EZ(G)$. Thus g

normalizes $EZ(G)$. But then since g normalizes $(EZ(G))' = E'$, it follows that g normalizes E and so $g \in N$, which is a contradiction. Therefore N is self-normalizing. $\square$

Lemma 2.4 *Let G be a perfect locally finite p-group, (w, V) a Λ-pair for G with $W^*(w, V) = 1$ and $E \in E^*(w, V)$ satisfying $(**)$. Put $N = N_G(E)$, and let A be a normal abelian subgroup of G. Then $A \cap N_G(N) \leqslant N$. In particular if G satisfies the normalizer condition and N/E is abelian, then $A \leqslant N_G(E)$ and so G contains a unique maximal normal abelian subgroup.*

Proof — Assume that there exists $a \in A \setminus N$ with $N^a = N$. Put

$$R = N \cap A, \quad D = R \cap E \quad \text{and} \quad H = N\langle a \rangle.$$

Then $D, R \lhd H$. Also R/D is (locally) cyclic since every normal abelian subgroup of N/E is (locally) cyclic by [5], Lemma 2.2. Next $Z(G)$ is non-trivial by Lemma 2.1 (c) since $W^*(w, V) = 1$ and cyclic by Lemma 2.3. Thus $Z(G) = \langle z \rangle$ for some $z \in Z(G)$. Replacing A with $A\langle z \rangle$ we may suppose that $z \in A$. Then $z \in R$. Put $\overline{H} = H/D$. Then $\overline{R}$ is (locally) cyclic and $\langle \overline{z} \rangle$ contains the subgroup of order p of $\overline{R}$ since $\langle z \rangle \cap E = 1$ by the hypothesis. Moreover

$$[\overline{R}, \overline{RE}] = [\overline{R}, \overline{E}] \leqslant \overline{R} \cap \overline{E} = 1$$

since $\overline{R}$ is aelian and in addition H normalizes R and RE which implies that $[R, E] \leqslant D$. Also we may suppose that $a^p \in N$.

Now

$$1 = [\overline{a}^p, \overline{E}] = [\overline{a}, \overline{E}]^p$$

by [6], Lemma 2.2.2 (i), since $\overline{a}^p \in \overline{R}$ and $[\overline{R}, \overline{E}] = 1$. Thus $[\overline{a}, \overline{E}]$ has order p and so contained in $\langle \overline{z} \rangle \overline{E}$. In particular then $[a, E] \leqslant \langle z \rangle E$ and so a normalizes $\langle z \rangle E$. Then since a normalizes $(\langle \overline{z} \rangle E)'$ and $(\langle \overline{z} \rangle E)' = E'$ it follows that a normalizes E'. But then $a \in N$ by $(**)$, which is a contradiction.

Finally suppose that G satisfies the normalizer condition and N/E is abelian. If $A \not\leqslant N$, then there exists $a \in A \setminus N$ with $N^a = N$. But then $a \in N$ by the first part of the proof which is a contradiction. Therefore the assumption is false and so $A \leqslant N$. Let C be another normal abelian subgroup of G. Then also $C \leqslant N$ since G satisfies the normalizer condition. Since N/E is (locally) cyclic it follows that $(AC)' \leqslant E$. But since $\mathrm{Core}_G(E) = 1$ it follows that $(AC)' = 1$ and so AC is abelian. This shows that any two normal abelian subgroups of G are contained in a normal abelian subgroup of G, which means that G contains a unique maximal normal abelian subgroup. □

Lemma 2.5 *Let* G *be a perfect locally finite* p*-group, where* $p \neq 2$, (w, V) *a* Λ*-pair for* G *with* $W^*(w, V) = 1$, *and let* E *be an element of* $E^*(w, V)$ *such that* $N_G(E) = N_G(E')$. *Moreover, let* B *be a normal metabelian subgroup of* G *and* A *be a normal abelian subgroup of* G *contained in* B *such that* B/A *is elementary abelian,* $A \cap Z(G) \neq 1$ *and* $A \leqslant N_G(E)$. *Put*

$$N = N_G(E), \quad R = N \cap B, \quad D = R \cap E,$$

and assume that there exists $t \in B \setminus N$ *with* $N^t = N$ *and* $t^p \in N$. *If* $T = \langle t \rangle R$, $H = TN$ *and* $D^* = \mathrm{Core}_H(D)$, *the following statements hold.*

(a) $Z(T/D^*)$ *is (locally) cyclic and* $Z(T/D^*) \cap E/D^* = 1$. *Also*

$$R/D^* \leqslant Z(N/D^*) \text{ and } C_{T/D^*}(R/D^*) = R/D^*$$

Furthermore $(N/D^*)' \leqslant C_{N/D^*}(\overline{T})$ *and, in particular,* $\overline{D} \cap \overline{N}' = 1$.

(b) *Suppose that* A/D *has finite exponent. Then*

$$A = \langle a \rangle D, \langle a \rangle \cap Z(G) \neq 1, \langle a \rangle \cap D = 1, |a| = \exp(A),$$

$$R =< b > D, \quad < b > \cap D = 1, \quad |b| \leqslant p|a|.$$

Also $|b| > |Z(G)|$ *and if* $\exp([\overline{R}, \overline{t}]D^*/D^*) < |bD^*|$, *then*

$$\langle bD^* \rangle \cap Z(G)D^*/D^*) \neq 1.$$

(c) *Suppose* R/D^* *is infinite and* G *is a Fitting group. Then* R/D^* *is Chernikov and* $R/D^* = (R/D^*)^o \times D/D^*$, *where* $(R/D^*)^o$ *is locally cyclic and* D/D^* *is finite.*

Proof — (a) Put $\overline{H} = H/D^*$. Clearly $Z(G)$ is finite by Lemma 2.3 since $T \nleqslant N$ and $Z(G) \neq 1$ by Lemma 2.1 (c) since $W^*(w, V) = 1$. Also $Z(G) \cap E = 1$ since $\mathrm{Core}_G(E) = 1$. Next $Z(\overline{T}) \cap \overline{E} = 1$ since $\mathrm{Core}_H(D) = D^*$. Clearly then $Z(\overline{T}) \cap \overline{N}$ is (locally) cyclic since N/E is (locally) cyclic. Let $z \in Z(G)$ with $|z| = p$. Then $1 \neq \overline{z} \in \overline{R} \setminus \overline{D}$. Now since $\overline{R} \lhd \overline{H}$ it follows that $[\overline{R}, \overline{N}] \lhd \overline{H}$. Also $[\overline{R}, \overline{N}] \leqslant \overline{R} \cap \overline{E}$ since N/E is abelian. This implies that $[\overline{R}, \overline{N}] = 1$ and so $\overline{R} \leqslant Z(\overline{N})$. Next assume if possible that $[\overline{t}, \overline{R}] = 1$. Then

$$1 = [\overline{t}^p, \overline{N}] = [\overline{t}, \overline{N}]^p$$

and so $[\bar{t}, \overline{N}]$ is a subgroup of order $\leqslant p$ of $\overline{R}$. Clearly then $[\bar{t}, \overline{N}]\overline{E}/\overline{E}$ is contained in $\langle\bar{z}\rangle\overline{E}/\overline{E}$ since $\overline{RE}/\overline{R}$ is (locally) cyclic. This implies that $[t, E] \leqslant \langle z\rangle E$ and then t normalizes $\langle z\rangle E$. Then since t normalizes $(\langle z\rangle E)' = E'$ it follows that $t \in N$ by the hypothesis, which is a contradiction. Therefore $C_{\overline{T}}(\overline{R}) = \overline{R}$. Now since $Z(\overline{T}) \leqslant \overline{N}$ it follows that $Z(\overline{T})$ is (locally) cyclic.

Since $\overline{N}/\overline{E}$ is abelian, it follows that $\overline{N}' \leqslant \overline{E}$ (in fact inclusion is proper since $\overline{E} \ntriangleleft \overline{H}$). Hence $[\overline{N}', \overline{T}] \leqslant \overline{E} \cap \overline{T} \leqslant \overline{D}$ and then $[\overline{N}', \overline{T}] = 1$ since $[\overline{N}', \overline{T}] \lhd \overline{H}$. In particular $\overline{D} \cap \overline{N}' = 1$ by definition of D^*.

(b) Now suppose that $\exp(A/D)$ is finite. Also $Z(G)$ is finite by the hypothesis and by Lemma 2.3. So we may suppose that $Z(G) \leqslant A$. Clearly $A \nleqslant E$ since $\mathrm{Core}_G(E) = 1$. In particular $Z(G) \cap E = 1$. Also AE/E is cyclic by [5], Lemma 2.2. Let $z \in Z(G)$ with $|z| = p$. Then z has finite height, say h, in A since $\langle z\rangle \cap E = 1$ and $\exp(A/D)$ is finite. Therefore there exists $a \in A$ such that

$$a^{p^h} = z \quad \text{and} \quad A = \langle a\rangle \times A_1$$

for a subgroup A_1 of A by [8], Lemma, p.180, or [11], 4.3.3. Then we have also $\langle a\rangle \cap D = 1$ since $\langle z\rangle \cap E = 1$. If $|a| < \exp(A)$, then

$$A^* = \langle y^{|a|} : y \in A\rangle$$

is a non-trivial normal subgroup of G with $Z(G) \cap A^* = 1$. Then A^* contains a normal subgroup $L \neq 1$ of G such that $w \notin VL$ by [4], Lemma 3.5. But then $L \leqslant E_1$ for an $E_1 \in E^*(w, V)$, which contradicts the hypothesis $W^*(w, V) = 1$ (also holds if (w, V) is a distinguished pair by Lemma 2.1 (d)). Therefore $|a| = \exp(A)$ and then $A = \langle a\rangle \times D$ since A/D is cyclic and so $|A/D| = \exp(A)$. Also $\exp(B) \leqslant p \cdot \exp(A)$. Hence $R = \langle b\rangle D$ for a $b \in R$ since RE/E is cyclic and $|b| \leqslant p|a|$. Next we show that $\langle b\rangle \cap D) = 1$. If $|bD| = |aD|$, then we may let $b = a$. Then

$$R = \langle a\rangle D \quad \text{and} \quad \langle a\rangle \cap D = 1.$$

So suppose that $|bD| > |aD|$. Then $|bD| > |a|$ since $|aD| = |a|$ which implies that $|bD| = b$ since $\exp(B) \leqslant p|a|$. Clearly then $\langle b\rangle \cap D = 1$.

Assume if possible that $|\bar{b}| = |\overline{Z(G)}|$. Then $\overline{R} = \overline{Z(G)}\,\overline{D}$ and hence $\overline{R}\,\overline{E} = \overline{Z(G)}\,\overline{E}$, which implies that $\bar{t}$ normalizes $\overline{Z(G)}\,\overline{E}$. Clearly then t normalizes both $Z(G)E$ and $(Z(G)E)' = E'$. But then $t \in N$ by $(**)$, which is a contradiction. Therefore it follows that $|b| > |Z(G)|$. Next assume that $\exp([\overline{R},\overline{t}]) = |\overline{b}^{p^s}|$ for some $s \geqslant 1$. Then

$$[\overline{b}^{p^s},\overline{t}] = [\overline{b},\overline{t}]^{p^s} = 1$$

since $\overline{R}$ is abelian and so $\overline{b}^{p^s} \in Z(\overline{T})$ and then $\overline{z} \in \langle\overline{b}\rangle$ since $Z(\overline{T})$ is cyclic and contains $Z(G)$.

(c) Suppose that $\overline{R}/\overline{D}$ is infinite and G is a Fitting group. Then $\overline{R}/\overline{D}$ is locally cyclic by [5], Lemma 2.2. For each element $\overline{x} \in \langle\overline{t}\rangle$ we have $\overline{R}/\overline{D}^{\overline{x}} \simeq \overline{R}/\overline{D}$. Hence if

$$\overline{L} = \bigcap_{\overline{x}\in\langle\overline{t}\rangle} \overline{D}^{\overline{x}},$$

then $\overline{R}/\overline{L}$ is isomorphic to a subgroup of

$$\overline{R}/\overline{D} \times \cdots \times \overline{R}/\overline{D}^{\overline{t}^{p-1}}$$

and so $\overline{R}/\overline{L}$ is Chernikov. Also since $\mathrm{Core}_H(D) = D^*$ it follows that $\overline{L} = 1$ and so $\overline{R}$ is Chernikov. In this case $\overline{R} = (\overline{R})^o \times \overline{F}$ by [11], 4.1.4, where $(\overline{R})^o$ is the divisible part of $\overline{R}$ and $\overline{F}$ is finite. Put $\overline{S} = [(\overline{R})^o,\overline{t}]$. Then since $\langle\overline{t}\rangle\overline{S}$ is nilpotent due to the fact that G is Fitting, $\overline{t}$ commutes with the divisible group $\overline{S}$. Let $|\overline{t}| = m$. Then

$$1 = [(\overline{R})^o,\overline{t}^m] = [(\overline{R})^o,\overline{t}]^m = [((\overline{R})^o)^m,\overline{t}] = [(\overline{R})^o,\overline{t}]$$

and so $[(\overline{R})^o,\overline{t}] = 1$. Now since $(\overline{R})^o \leqslant Z(\overline{H})$ it follows that $\overline{R}^o \cap \overline{E} = 1$ and so $\overline{R}^o$ is locally cyclic. Also $\overline{N} = \overline{R}^o\overline{E}$. Hence

$$\overline{R} = \overline{R}^o(\overline{R}\cap\overline{E}) = \overline{R}^o\overline{D} = \overline{R}^o \times \overline{D}$$

since $\overline{z} \in \overline{R}^o$. In particular $\overline{D}$ is finite. □

Lemma 2.6 *Let* B *be a nilpotent metabelian* p*-group with class* $c \leqslant p$ *and let* A *be a normal abelian subgroup of* B *such that* B/A *is elementary abelian. Then* $\exp([A,B]) \leqslant p$ *and* $\exp(B') \leqslant p^2$.

Proof — Since $[A,B]$ is abelian it suffices to show that $\exp([A,t]) \leqslant p$ for every $t \in B\setminus A$. So let $t \in B\setminus A$ and $a \in A$. We claim that $[a,t]^p = 1$. Since $t^p \in A$ we have

$$1 = [a,t^p] = \prod_{k=1}^{p} [a,_k t]^{\binom{p}{k}}.$$

For $p = 2$, this gives $1 = [a,t]^2$ since $c = 2$. So we may suppose that $p \geqslant 3$. Clearly

$$p \mid \binom{p}{k} \quad \text{but} \quad p^2 \nmid \binom{p}{k} \quad \text{for} \quad 1 \leqslant k < p$$

and $[a,_p t] = 1$ by the hypothesis. Put

$$\binom{p}{k} = pu_k$$

for every $1 \leqslant k < p$. Then $p \nmid u_k$. Now since $[a,_{p-1} t] \in Z(B)$ it follows that

$$[a,_{p-1} t]^p = [a,_{p-2} t, t]^p = [a,_{p-2} t, t^p] = 1.$$

Substituting these values above we get

$$1 = ([a,t]^{u_1} \cdots [a,_{p-2} t]^{u_{p-2}}])^p$$

Here if $p = 3$, then $[a,t]^{3u_1} = 1$ and so we are done in this case. Therefore we may suppose that $p > 3$ and use induction on c. Put $\overline{B} = B/\gamma_c(B)$. Then $[\overline{a}, \overline{t}]^p = 1$ by the induction hypothesis which implies that $[a,t]^p \in Z(B)$. Now since

$$[a,_k t]^p = [[a,t]^p,_{k-1} t] = 1$$

for every $1 < k \leqslant p-2$ it follows that $[a,t]^p = 1$. Since a is any element of A it follows that $\exp([A,t]) \leqslant p$, which was to be shown.

Next let $s, t \in B$. Then

$$[t, s^p] = \prod_{i=1}^{p} [t,_i s]^{\binom{p}{i}} = [t,s]^p$$

since $[t,a]^p = 1$ by the first pat of the proof and $c(B) \leqslant p$. Hence it follows that

$$[t,s]^{p^2} = [t,s^p]^p = 1$$

by the first part of the proof. Therefore $\exp(B') \leqslant p^2$. □

Lemma 2.7 *Let G be a perfect locally finite p-group, where $p \neq 2$, and let (w,V) be a Λ-pair for G such that $W^*(w,V) = 1$ and there exists E*

in $E^*(w, V)$ *with* $N_G(E) = N_G(E')$. *Moreover, let* B *be a normal nilpotent subgroup of* G *with* $c(B) = c < p$ *and* A *a normal abelian subgroup of* G *contained in* $B \cap N_G(E)$ *such that* B/A *is elementary abelian . Suppose furthermore that* $B \cap N_G(N_G(E)) \setminus N_G(E) \neq 1$ *whenever* $B \nleqslant N_G(E)$. *Then* B *is abelian.*

Proof — Assume that B is not abelian. First we show that B is not contained in $N_G(E)$. Assume if possible that $B \leqslant N_G(E)$. Then $B' \leqslant E$ since $N_G(E)/E$ is locally cyclic by [5], Lemma 2.2, due to the fact that $p \neq 2$, which is impossible since $\mathrm{Core}_G(E) = 1$ by the hypothesis. Therefore $B \nleqslant N_G(E)$.

Choose $t \in B \setminus N_G(E)$ with $N_G(E)^t = N_G(E)$ and $t^p \in N_G(E)$. Put

$$N = N_G(E), \quad R = N \cap B, \quad D = R \cap E, \quad T = \langle t \rangle R$$

and $H = TN$. Then $A \leqslant N$ by the hypothesis and $T \nleqslant N$ but $t^p \in A$. Also $Z(G) \neq 1$ and is cyclic by Lemma 2.3. Let $Z(G) = \langle z \rangle$. Then $z \neq 1$ by Lemma 2.1 (c) since $W^*(w, V) = 1$. Without loss of generality we may suppose that $z \in A$. Next let $D^* = \mathrm{Core}_H(D)$ and put $\overline{H} = H/D^*$. Then

$$\overline{R} \leqslant Z(\overline{N}) \quad \text{and} \quad C_{\overline{T}}(\overline{R}) = \overline{R}$$

by Lemma 2.5 (a). Let $y \in N$. Then

$$1 = [\overline{y}, \overline{t}^p] = \prod_{k=1}^{p} [\overline{y},_k \overline{t}]^{\binom{p}{k}}$$

since $\overline{t}^p \in \overline{R}$ and $\overline{R} \leqslant Z(\overline{N})$. Also $\langle \overline{t} \rangle \overline{R}/\overline{R}$ is elementary abelian and hence $\exp([\overline{R}, \overline{t}]) \leqslant p$ by Lemma 2.6 since $c < p$. Using this in the above equality we get

$$1 = [\overline{y}, \overline{t}]^p [\overline{y},_p \overline{t}].$$

Moreover $[\overline{y},_p \overline{t}] = 1$ since $c < p$. Using this above we get finally

$$1 = [\overline{y}, \overline{t}]^p.$$

Here since y is any element of N it follows that $\exp([\overline{N}, \overline{t}]) \leqslant p$ and so

$$[\overline{N}, \overline{t}]\overline{E}/\overline{E} \leqslant \langle \overline{z} \rangle \overline{E}/\overline{E}.$$

This follows because N/E is locally cyclic and $Z(G) \cap E = 1$ owing to the fact that $\mathrm{Core}_G(E) = 1$. Clearly this implies that $[\overline{E}, \overline{t}] \leqslant \langle \overline{z} \rangle \overline{E}$ and hence it follows that t normalizes $\langle z\rangle E$. But then t normalizes E' and hence $t \in N$ by the hypothesis, which is a contradiction and so the proof of the lemma is complete. □

Lemma 2.8 *Let* G *be a perfect Fitting* p*-group satisfying the normalizer condition, where* $p \neq 2$*, and suppose furthermore that in every homomorphic image of* G *the* $(**)$ *condition is satisfied exactly as in Theorem 1.1. Then in every homomorphic image* H *of* G *the following holds: if* H *has a* Λ*-pair* (w_H, V_H) *with* $W^*(w_H, V_H) = 1$ *and if* B_H *is a nilpotent normal metabelian subgroup of* H *containing a normal abelian subgroup* A_H *of* H *such that* B_H/A_H *is elementary abelian, then* $[B'_H, H] \nleqslant \gamma_{c(B_H)}(B_H)$*, where* $c(B_H)$ *is the class of* B_H.

Proof — Assume that G has a Λ-pair (w, V) with $W^*(w, V) = 1$ and a nilpotent normal abelian-by-elementary abelian subgroup B such that $[B', G] \leqslant \gamma_{c(B)}(B)$. Furthermore suppose that for every homomorphic image H of G the following holds. If H satisfies the hypothesis of the lemma but has a nilpotent normal abelian-by-elementary abelian subgroup B_H not satisfying the conclusion of the lemma, then $c(B_H) \geqslant c(B)$.

Let A be a largest normal abelian subgroup of G such that B/A is elementary abelian. We may also suppose that $Z(G) \leqslant B$. Note that $Z(G)$ is finite and cyclic by Lemma 2.3 and non-trivial by Lemma 2.1 (c) since $W^*(w, V) = 1$. Put $c = c(B)$. If $c < 3$, then B is abelian by Lemma 2.7, which is a contradiction. Therefore $c \geqslant 3$. Now $[B', G] \leqslant \gamma_c(B)$ by the assumption. This means that $[B, B, B]$ is contained in $\gamma_c(B)$ and then $c(B) = 3$ since $c(B) \geqslant 3$. If $p > 3$, then this gives a contradiction by Lemma 2.7. Therefore $p = 3$.

Now B' is not contained in $Z(B)$ since $c = 3$ and so there exists an element $t \in B' \setminus Z(B)$. Let F be a finite subgroup of $Z(B)$ with $Z(G) \leqslant F$ and consider the Λ-pair (t, F). There exists $E \in E^*(t, F)$ such that $Z(B) \leqslant E$ since $t \notin FZ(B)$. Also $A \leqslant N_G(E)$ by Lemma 2.4. Furthermore there exists a maximal element M of $W^*(t, R)$ such that $Z(B) \leqslant M$ by Lemma 2.1 (b).

Put $\overline{G} = G/\gamma_3(B)$. Then $c(\overline{B}) = 2$. Also $\overline{B}/\overline{A}$ is elementary abelian. Hence if $W^*(\overline{t}, \overline{F}) = 1$, then $\overline{B}$ is abelian by Lemma 2.7, which is a contradiction. Therefore $\overline{M} \neq 1$.

Now consider $\overline{G}/\overline{M}$. Then there exists a finite subgroup U of G containing F such that $t \notin U \nleqslant M$ and $\overline{t}\overline{u} \notin Z(\overline{G})$ by Lemma 2.1 (c) for some $u \in U$. Thus $(\overline{t}\overline{u}, \overline{UM}/\overline{M})$ is a Λ-pair for $\overline{G}/\overline{M}$. Also

$$W^*((\overline{t}\overline{u})\overline{M}, \overline{UM}/\overline{M}) = 1$$

since, by Lemma 4.2 of [4], $E^*((\bar{t}\bar{u})\overline{M}, \overline{UM}/\overline{M})$ consists of all $\overline{R}/\overline{M}$ such that $\overline{M} \leqslant \overline{R} \in E^*(\bar{t}, \overline{F})$. Moreover $(\overline{BM}/\overline{M})/(\overline{AM}/\overline{M})$ is elementary abelian. Therefore again $\overline{BM}/\overline{M}$ is abelian by Lemma 2.7. But this means that $\overline{B}' \leqslant \overline{M}$, which is impossible since $\bar{t} \in \overline{B}' \setminus \overline{M}$. Consequently the assumption is false and the proof of the lemma is complete. □

Proof of Theorem 1.1 — Let G be Fitting p-group satisfying the normalizer condition and $p \neq 2$. Suppose that in each homomorphic image of G every Λ-pair has a maximal element satisfying the property $(**)$. Assume that G is perfect. First we show the following. G has a homomorphic image H with the following property. H has a Λ-pair (w_H, V_H) satisfying $(**)$ and the condition $W^*(w_H, V_H) = 1$ such that every normal nilpotent subgroup of H which is abelian-by-elementary abelian is abelian. Assume that there exists no such H. For each homomorphic image X of G satisfying the above properties let $n(X)$ be the minimum of the classes of all the normal nilpotent abelian-by-elementary abelian subgroups of X which are not abelian. Among all the homomorphic images X of G having a Λ-pair (w_X, V_X), satisfying $(**)$ and the condition $W^*(w_X, V_X) = 1$ there is a homomorphic image H such that $n(H) \leqslant n(X)$ for all such X. Without loss of generality we may suppose that $H = G$. Thus G admits a Λ-pair (w, V) such that $(**)$ and the condition $W^*(w, V) = 1$ are satisfied. Also $n(G)$ is minimal in the above sense and $n(G) > 1$ by the assumption. Let B be a normal nilpotent abelian-by-elementary abelian subgroup of G so that $c(B) = n(G)$. Let A be the largest normal abelian subgroup of G contained in B such that $exp(B/A) = p$ and $B' \leqslant A$. By the hypothesis there exists $E \in E^*(w, V)$ satisfying $(**)$. Put $N = N_G(E)$. Then N/E is (locally) cyclic by [5], Lemma 2.2, since $p \neq 2$. Also $A \leqslant N$ by Lemma 2.4. Furthermore $B \nleqslant N$ as in Lemma 2.7 since B is not abelian and thus there exists $t \in B \setminus N$ such that $N^t = N$ and $t^p \in N$ since G satisfies the normalizer condition.

If $c(B) < 3$, then B is abelian by Lemma 2.7 since $p \geqslant 3$. Therefore $c(B) \geqslant 3$. Let $c(B) = c$ and put $\overline{G} = G/\gamma_c(B)$, so that $c(\overline{B}) = c - 1$. Assume first if possible that $\overline{B}' \leqslant Z(\overline{G})$. Then $[\overline{B}', \overline{G}] = 1$ and so $[B', G]$ is contained in $\gamma_c(B)$. Also (w, V) is a Λ-pair for G with $W^*(w, V) = 1$. But then $\overline{B}' \nleqslant Z(\overline{G})$ by Lemma 2.8, which is a contradiction. Therefore there exits $\bar{s} \in \overline{B}' \setminus Z(\overline{G})$.

Let T be a finite subgroup of G such that $\bar{s} \notin \overline{T}$ (for example $\overline{T} = 1$).

Then $(\overline{s}, \overline{T})$ is a Λ-pair for $\overline{G}$. Let $\overline{M}$ be a maximal element of $W^*(\overline{s}, \overline{T})$. If $\overline{M} = 1$, then $\overline{B}$ is abelian by the induction hypothesis since $c(\overline{B}) < c$. But then $c(B) = 2$, which is impossible. Therefore $\overline{M} \neq 1$. Now consider $\overline{G}/\overline{M}$. By Lemma 2.1 (c) there exists a finite subgroup U of G such that

$$\overline{s} \notin \overline{U}, \quad \overline{T} \leqslant \overline{U} \nleq \overline{M}$$

and there exists $u \in U \setminus M$ such that $(\overline{su}\overline{M}, \overline{UM}/\overline{M})$ is a Λ-pair for $\overline{G}/\overline{M}$. Also $(\overline{su}\overline{M}, \overline{UM}/\overline{M})$ satisfies the hypothesis and

$$W^*(\overline{su}\overline{M}, \overline{UM}/\overline{M}) = 1.$$

In this case the group $\overline{BM}/\overline{M}$ is abelian since $c(\overline{B}) < c$ and this implies that $\overline{B}' \leqslant \overline{M}$. However since $\overline{s} \in \overline{B}'$ but $\overline{s} \notin \overline{M}$ this gives another contradiction. Therefore the assumption is false and so it follows that B is abelian.

Thus we have shown that every normal nilpotent abelian-by-elementary abelian subgroup of G is in fact abelian. Now let A be a maximal normal abelian subgroup of G. Let $g \in G \setminus A$ and put $H = \langle g^G \rangle A$. Then H is nilpotent since G is a Fitting group. Then

$$B/A = \Omega_1(Z(H/A))$$

is elementary abelian and $B \neq A$ since H is nilpotent. But since B must be abelian by the first part of the proof this contradicts the maximality of A. Therefore the assumption is false and so G is not perfect. This completes the proof of the theorem. □

Proof of Corollary 1.2 — Let G be a Fitting p-group satisfying the hypothesis of the corollary but G is not solvable, where $p \neq 2$. Thus every proper homomorphic image of G is an MNS-group and, in particular, is perfect. By [4], Theorem 1.4 (b), we may suppose that G has no homomorphic images having $(*)$-triples for non-central elements. Then in every homomorphic image of G there exist distinguished pairs and dominant pairs by [4], Lemmas 3.1 and 4.1 (b).

First we show that in every proper homomorphic of G every Λ-pair has a maximal element satisfying $(**)$. Thus let $H \neq 1$ be a homomorphic image of G and let (w_H, V_H) be a Λ-pair for H. Clearly without loss of generality we may let $H = G$ and let (w, V) be a Λ-pair for G. We must show that there exists $E \in E^*(w, V)$ satisfying $(**)$. Since $w \notin V$, applying [4], Lemma 3.1, to (w, V) we obtain a finite subgroup T of G containing V and excluding t such that (w, T) is a distinguished pair for G. Next applying [4], Lemma 4.1 (a), to (w, T) we obtain a finite subgroup U of G containing T and excluding t such that (w, U) is a dominant pair for G. Also

$$E^*(w, U) \subseteq E^*(w, V)$$

since $V \leqslant U$ by [4], Lemma 3.2. Now by the hypothesis there exists $E \in E^*(w, U)$ satisfying $(**)$. Since $E \in E^*(w, V)$, the assertion is verified. Thus we have shown that in every homomorphic image of G every Λ-pair has a maximal element satisfying $(**)$. But then G cannot be perfect by Theorem 1.1. Therefore the assumption is false and so G must be solvable. □

3 Proof of Theorem 1.3

Lemma 3.1 *Let* G *be a locally finite* p*-group such that* $Z(G) \neq G$*. Then there exists a proper normal subgroup* M *of* G *such that the set*

$$\{\exp(AM/M) : A \in S_1(G)\}$$

is bounded.

Proof — If $A \leqslant Z(G)$ for every $A \in S_1(G)$, then we may let $M = Z(G)$. Therefore we may suppose that there exists $A_1 \in S_1(G)$ with the property that $A_1 \nleqslant Z(G)$. Choose $a_1 \in A_1 \setminus Z(G)$ and let $m = |a_1|$. Then for every $A \in S_1(G)$

$$1 = [a_1^m, A] = [a_1, A]^m = [a_1, A^m]$$

by [6], Lemma 2.2.2, and hence $A^m \leqslant C_G(a_1)$. Hence it follows that

$$\langle A^m : A \in S_1(G)\rangle \leqslant C_G(a_1).$$

Put

$$M = \langle A^m : A \in S_1(G)\rangle.$$

Then obviously $M \lhd G$ and $M \neq G$ since $a_1 \notin Z(G)$. Moreover $\exp(AM/M) \leqslant m$ for every $A \in S_1(G)$. □

Lemma 3.2 *Let* G *be a locally finite* p*-group such that* $Z(G) \neq G$*, and suppose that* $\langle S_1(G)\rangle = G$*. Then there exists a homomorphic image* H *of* G *such that* $\langle S_1(H)^1\rangle = H$*.*

Proof — By Lemma 3.1 there exists a proper subgroup $M \lhd G$ and $m \geqslant 1$ such that

$$|\{\exp(AM/M) : A \in S_1(G)\}| \leqslant p^m.$$

Thus

$$\langle A^{p^m} : A \in S_1(G)\rangle \leqslant M \neq G.$$

In this case there exists a smallest number $1 \leqslant r \leqslant m$ such that

$$M^* := \langle A^{p^r} : A \in S_1(G)\rangle$$

is a proper subgroup of G. Also $M \leqslant M^*$. Clearly

$$\langle AM^*/M^* : A \in S_1(G)\rangle = G/M^*$$

by the hypothesis and $\exp(AM^*/M^*) \leqslant p^r$. If $r = 1$, we are done since then each AM^*/M^* is elementary abelian. So suppose $r > 1$. In this case put

$$R = \langle A^{p^{r-1}} : A \in S_1(G)\rangle.$$

Then $R = G$ by the minimality of r and so $R/M^* = G/M^*$. Also since $A^{p^{r-1}}M^*/M^*$ is elementary abelian for every $A \in S_1(G)$ it follows that

$$\langle S_1(G/M^*)^1\rangle = G/M^*$$

and so the proof is complete. □

Lemma 3.3 *If* $G = \langle A : A \in S_1(G)\rangle$ *and* $g \in G$, *then*

$$[g, G] = \prod_{A \in S_1(G)} [g, A].$$

Proof — Let $y \in [g, G]$. Then there are $g_1, \dots, g_n \in G$ such that

$$y = [g, g_1] \cdots [g, g_n].$$

Also there are $A_1, \dots, A_r \in S_1(G)$ such that

$$\{g_1, \dots, g_r\} \subseteq A_1 \cdots A_r$$

by the hypothesis. Then $y \in [g, A_1 \cdots A_r]$. Also an easy induction shows that

$$[g, A_1 \cdots A_r] = [g, A_1] \cdots [g, A_r]$$

since each A_i is a normal abelian subgroup of G. Therefore

$$y \in \prod_{A \in S_1(G)} [g, A].$$

Since y is any element of $[g, G]$ it follows that

$$[g, G] \leqslant \prod_{A \in S_1(G)} [g, A].$$

But the reverse inclusion is obvious. Hence the equality follows. □

Lemma 3.4 *Let G be a perfect locally finite p-group, where $p \geqslant 3$ and suppose that $\langle S_1(G)\rangle = G$. Moreover, let (w, V) be a Λ-pair for G with $W^*(w, V) = 1$ such that every element of $E^*(w, V)$ satisfies $(**)$, and let A be a normal abelian subgroup of G with $Z(G) \neq A$. If*

$$\Gamma_{(w,V)}(A) = \{E \in E^*(w, V) : A \leqslant N_G(E)\},$$

then $\langle\Gamma_{(w,V)}(A)\rangle = G$. Furthermore if A_1, A_2 are two normal abelian subgroups of G with $(A_1A_2)' \neq 1$, then

$$\Gamma_{(w,V)}(A_1) \cap \Gamma_{(w,V)}(A_2) = \emptyset.$$

Proof — Note that $E \in \Gamma_{(w,V)}(A)$ means that $A \leqslant N_G(E)$ and $A/(A \cap E)$ is (locally) cyclic by [5], Lemma 2.2, since $p \neq 2$. First we show that $\Gamma_{(w,A)}(A) \neq \emptyset$. By Lemma 2.2 (a) there exists $E \in E^*(w, V)$ such that $A \cap E$ is maximal and then $A/(A \cap N)$ is finite by [5], Lemma 2.3, where $N = N_G(E)$. In this case if $A \nleqslant N$, then there exists an element a of $A \cap N_G(N) \setminus N$ with $N^a = N$. However Lemma 2.4 implies that $a \in N$ since E satisfies $(**)$. Therefore $A \leqslant N$ and so $\Gamma_{(w,V)}(A) \neq \emptyset$. Assume if possible that $\langle\Gamma_{(w,V)}(A)\rangle \neq G$. Then without loss of generality we may assume that A is a maximal normal abelian subgroup of G. Now there exists $C \in S_1(G)$ such that $C \nleqslant \langle\Gamma_{(w,V)}(A)\rangle$ by the hypothesis. Also $[C, A] \neq 1$ by the maximality of A. Furthermore there exists $E \in E^*(w, V)$ such that $C \leqslant N_G(E)$ as in the case of A. Hence it follows that $E \nleqslant \langle\Gamma_{(w,V)}(A)\rangle$ since $[C, A] \neq 1$ and so there exists $y \in E \setminus \langle\Gamma_{(w,V)}(A)\rangle$. Put $V_1 = \langle V, y\rangle$. Then (w, V_1)

is a Λ-pair for G since $w \notin V_1$. Also

$$E^*(w, V_1) \subseteq E^*(w, V)$$

by [4], Lemma 3.2. If $R \in E^*(w, V_1)$, then $R \notin \Gamma_{(w,V)}(A)$ since V_1 is not contained in $\langle \Gamma_{(w,V)}(A) \rangle$. However applying Lemma 2.2 (a) to $E^*(w, V_1)$ we can find $T \in E^*(w, V_1)$ such that $T \cap A$ is maximal and then $A/(A \cap N_G(T))$ is finite by [5], Lemma 2.3. But then $A \leqslant N_G(T)$ as above. This is a contradiction since $T \nleqslant \langle \Gamma_{(w,V)}(A) \rangle$. Therefore the assumption is false and so $\langle \Gamma_{(w,V)}(A) \rangle = G$.

Next let A_1, A_2 be two normal abelian subgroups of G such that A_1A_2 is not abelian. Assume if possible that there exists

$$E \in \Gamma_{(w,V)}(A_1) \cap \Gamma_{(w,V)}(A_2).$$

Then $A_1, A_2 \leqslant N_G(E)$. In this case $(A_1A_2)' \leqslant E$ since $N_G(E)/E$ is (locally) cyclic by [5], Lemma 2.2. But then $(A_1A_2)' = 1$ since $\mathrm{Core}_G(E) = 1$, which is a contradiction. □

Proof of Theorem 1.3 — Let G be a perfect locally finite p-group whose proper subgroups are solvable, where $p \neq 2$. Suppose that in every homomorphic image of G every dominant pair satisfies $(**)$. First we show that $\langle S_1(G) \rangle \neq G$. Assume that $\langle S_1(G) \rangle = G$. Then there exists a homomorphic image H of G such that $\langle S_1(H)^1 \rangle = H$ by Lemma 3.2. Without loss of generality we may assume that $G = H$ and thus $\langle S_1(G)^1 \rangle = G$. By Theorem 1.4 (b) of [4] we may suppose that G has no homomorphic images having $(*)$-triples for non-central elements. Then in every homomorphic image of G there exist dominant pairs by [4], Lemmas 3.1 and 4.1 (a). Let (w, V) be a dominant pair for G. Every element of $E^*(w, V)$ satisfies $(**)$ by the hypothesis. Now $W^*(w, V)$ contains a maximal element, say M, by [4], Lemma 3.4. Since

$$\langle AM/M : A \in S_1(G)^1 \rangle = G/M$$

and G/M is not abelian, there are $A_1, A_2 \in S_1(G)^1$ such that the group A_1A_2M/M is not abelian.

First suppose that $M = 1$. Then A_1A_2 is not abelian. Assume that $w \notin [V, G]V$. Then $[V, G]V \leqslant E$ for an $E \in E^*(w, V)$ by Lemma 2.1 (d). But since $\mathrm{Core}_G(E) = 1$ this is impossible, and so $w \in [V, G]V$. Now

$$[V, G] = \prod_{A \in S_1(G)^1} [V, A]$$

since

$$[v, G] = \prod_{A \in S_1(G)^1} [v, A]$$

for every $v \in V$ by Lemma 3.3. Hence there are

$$A_1, \dots, A_r \in S_1(G)^1$$

for some $r \geqslant 2$ such that

$$w \in (\prod_{i=1}^{r} [V, A_i])V.$$

In this case there exists a finite subset Y of

$$\langle [V, A_2] \cdots [V, A_r] \rangle$$

such that $w \in \langle [V, A_1], Y, V \rangle$. Also there are finite subsets $Y_i \neq \emptyset$ of $[V, A_i]$ for $i = 2, \dots, r$ such that $Y \subseteq \langle Y_i : i = 2, \dots, r \rangle$. Now define

$$V_i = \langle V, Y_2, \dots, Y_i \rangle$$

for $i = 2, \dots, r$. Then each (w, V_i) is a Λ-pair for G. To see this choose $E \in \Gamma_{(w,V)}(A_2)$. This is possible by Lemma 3.4. Then $A_2 = \langle z \rangle (A_2 \cap E)$ by Lemma 2.5 (b) since A_2 is elementary abelian, where $z \in Z(G)$ with $|z| = p$. Hence $[V, A_2] \leqslant E$ and so $w \notin \langle V, Y_2 \rangle$ since $Y_2 \subseteq [V, A_2]$ and $V \leqslant E$, which means that (w, V_2) is a Λ-pair. Next consider $\Gamma_{(w,V_2)}(A_3)$. In the same way $w \notin V_3 = \langle V_2, Y_3 \rangle$, where $Y_3 \subseteq [V, A_3]$. Continuing in this way we see that

$$w \notin V_r = \langle V, Y_2, \dots, Y_r \rangle$$

and so it follows that (w, V_r) is a Λ-pair for G. By [4], Lemmas 3.1 and 4.1 (a), there exists a finite subgroup U of G containing V_r such that (w, U) is a dominant pair for G and every element of $E^*(w, U)$ satisfies $(**)$.

Now there exists $E \in \Gamma_{(w,U)}(A_1)$ since $\Gamma_{(w,U)}(A_1) \neq \emptyset$ by Lemma 3.4. Then $[U, A_1] \leqslant E$ as above and hence $w \notin \langle [U, A_1], U \rangle$. But also $V, Y \leqslant V_r \leqslant U$. Therefore $w \notin \langle [V, A_1], Y, V \rangle$, which is a contradiction.

Next suppose that $M \neq 1$ and put $\overline{G} = G/M$. By Lemma 2.1 (c) there exists a finite subgroup U of G containing V with $w \notin U \nleq M$ and $u \in U$ scuh that $(\overline{wu}, \overline{U})$ is a Λ-pair for $\overline{G}$ and $W^*(\overline{wu}, \overline{U}) = 1$. Then there exists a finite subgroup T of G containing U such that $(\overline{wu}, \overline{T})$ is a dominant pair for $\overline{G}$ and $W^*(\overline{wu}, \overline{T}) = 1$. Also $(\overline{wu}, \overline{T})$ satisfies $(**)$. Therefore $\langle S_1(\overline{G})^1\rangle \neq \overline{G}$ by the first part of the proof. Obviously then also

$$\langle \overline{A} : A \in S_1(G)^1\rangle \neq \overline{G}$$

and taking that inverse images it follows that $\langle S_1(G)^1\rangle \neq G$, which is another contradiction. Therefore the assumption that $\langle S_1(G)^1\rangle = G$ is false and so $\langle S_1(G)^1\rangle \neq G$. Clearly then also $\langle S_1(G)\rangle \neq G$.

Finally assume if possible that $\langle S_t(G)\rangle = G$ for a $t \geqslant 1$. Then $t > 1$ by the first part of the proof. We may suppose that t is the least integer with this property. Put $M = \langle S_{t-1}(G)\rangle$. Then $M \neq G$ by the induction assumption and

$$\{AM/M : A \in S_t(G)\} \subseteq S_1(G/M).$$

Also $\langle S_1(G/M)\rangle \neq G/M$ by the first part of the proof since G/M satisfies the hypothesis of the theorem. But then also

$$\langle AM/M : A \in S_t(G)\rangle \neq G/M$$

and taking the inverse images gives $\langle S_t(G)\rangle \neq G$, which contradicts the assumption that $\langle S_t(G)\rangle = G$. Therefore $\langle S_t(G)\rangle \neq G$ and so the proof of the theorem is complete. □

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