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G-Groups and Biuniform Abelian Normal Subgroups [1]

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Abstract

We prove a weak form of the Krull-Schmidt Theorem concerning the behavior of direct-product decompositions of G-groups, biuniform abelian G-groups, G-semidirect products and the G-set Hom(H,A). Here G and A are groups and H is a G-group. Our main result is the following. Let P be any group. Let $H_1, \dots, H_n$, $H'_1, \dots, H'_t$ be $n+t$ biuniform abelian normal subgroups of P. Suppose that the products $H_1 \dots H_n, H'_1 \dots H'_t$ are direct, that is, $H_1 \dots H_n = H_1 \times \dots \times H_n$ and $H'_1 \dots H'_t = H'_1 \times \dots \times H'_t$. Then the normal subgroups $H_1 \times \dots \times H_n$ and $H'_1 \times \dots \times H'_t$ of P are P-isomorphic if and only if $n = t$ and there exist two permutations σ and τ of $\{1, 2, \dots, n\}$ such that $[H_i]_m = [H'_{\sigma(i)}]_m$ and $[H_i]_e = [H'_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.

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1 Introduction

In some previous papers [1, 5, 8], the second author studied a phenomenon concerning direct-sum decompositions in some classes of

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modules, consisting essentially in the validity of a weak form of the Krull-Schmidt Theorem. Here is an example. A right module U over a ring R is said to be a *biuniform* module if it is non-zero, the intersection of any two non-zero submodules of U is non-zero and the sum of any two proper submodules of U is a proper submodule of U. For instance, *uniserial* non-zero R-modules, that is, the modules whose lattice of submodules is linearly ordered under inclusion, are biuniform modules.

Two right R modules U and V are said to belong to

1. *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;

2. *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

The weak form of the Krull-Schmidt Theorem we refer to above is the following.

Theorem A ([5, Theorem 1.9]) *Let* $U_1, \dots, U_n, V_1, \dots, V_t$ *be* $n+t$ *biuniform right modules over a ring* R. *Then the direct sums* $U_1 \oplus \dots \oplus U_n$ *and* $V_1 \oplus \dots \oplus V_t$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \dots, n\}$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i = 1, 2, \dots, n$.

In the previous two papers [9, 2], we looked for a similar result in the setting of groups. In this paper we continue in this investigation. Our main result is the following Theorem B:

Theorem B *Let* P *be any group. Let* $H_1, \dots, H_n, H'_1, \dots, H'_t$ *be* $n+t$ *biuniform abelian normal subgroups of* P. *Suppose that the products*

$$H_1 \dots H_n, \; H'_1 \dots H'_t$$

are direct, that is, $H_1 \dots H_n = H_1 \times \dots \times H_n$ *and* $H'_1 \dots H'_t = H'_1 \times \dots \times H'_t$. *Then the normal subgroups* $H_1 \times \dots \times H_n$ *and* $H'_1 \times \dots \times H'_t$ *of* P *are* P*-isomorphic if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \dots, n\}$ *such that* $[H_i]_m = [H'_{\sigma(i)}]_m$ *and* $[H_i]_e = [H'_{\tau(i)}]_e$ *for every* $i = 1, 2, \dots, n$.

Here, by a biuniform abelian normal subgroup H of P, we mean an abelian group H belonging to the modular lattice $\mathcal{N}(P)$ of all normal subgroups of P for which the interval $[1, H]$ is a modular lattice of Goldie dimension 1 and dual Goldie dimension 1. For the other exact definitions, see Section 5. Notice that the automorphisms of the G-group G in the category of G-groups are exactly the central automorphisms of G. Coherently, the classical Krull-Schmidt-Remak Theorem concerns the existence of a *central* automorphism of the group G of which we study the direct-product decompositions. In Theorem B, in a similar way, we get a P-*isomorphism*, which is an isomorphism in the category of P-groups (see Proposition 5.2).

In this paper, we also present further results related to Theorem A concerning the behaviour of direct-product decompositions of G-groups, biuniform abelian G-groups, G-semidirect products and the G-set $\mathrm{Hom}(H, A)$. Here G and A are groups and H is a G-group.

We denote by $\mathbb{Z}$ the ring of integers and by $J(R)$ the Jacobson radical of a ring R.

2 Basic notions on G-groups

Let G be a group. A G-*group* is a pair (H, φ), where H is a group and $\varphi\colon G \to \mathrm{Aut}(H)$ is a group homomorphism. Equivalently, it is a group H endowed with a mapping

$$\cdot\colon G \times H \to H, \quad (g, h) \mapsto gh$$

such that

(a) $g(hh') = (gh)(gh')$
(b) $(gg')h = g(g'h)$
(c) $1_G h = h$

for every $g, g' \in G$ and every $h, h' \in H$.

A G-*group morphism* $f\colon (H, \varphi) \to (H', \varphi')$ is a group homomorphism $f\colon H \to H'$ such that $f(gh) = gf(h)$ for every $g \in G$, $h \in H$. We will denote by G-**Grp** the category of G-groups.

The symbol $H' \leqslant_G H$ will denote that H' is a G-subgroup of H. We say that H is an *abelian* G-*group* if H is a G-group and H is abelian. For such an H, the set $\mathrm{Sub}_G(H)$ of all G-subgroups of H coincides with the set $\mathcal{N}_G(H)$ of all normal subgroups H' of H with $H' \leqslant_G H$. Thus $\mathrm{Sub}_G(H) = \mathcal{N}_G(H)$ is a complete bounded lattice with respect to inclusion. The full subcategory of G-**Grp** whose objects are all abelian G-groups will be denoted by Ab(G-**Grp**). It is an abelian subcategory of G-**Grp**. We say that a G-group H is *uniserial* if the set $\mathrm{Sub}_G(H)$ of G-subgroups of H is linearly ordered under inclusion.

In this paper, we will deal with monomorphisms and epimorphisms in the category G -**Grp**. The monomorphisms in G-**Grp** are exactly the G-group morphisms that are injective mappings, the isomorphisms are precisely the bijective G-group morphisms, and the epimorphisms in G -**Grp** are exactly the surjective mappings. These facts appear in [2]. Since the proof for the epimorphisms in [2, Theorem 2.2] is rather technical, we give here a more direct proof. It is similar to the proof of the analogous result for the category of groups due to Linderholm [13] (also see [14, Exercise 5, p. 21]).

Theorem 2.1 *Let* G *be a group and* (H, φ), (H', φ') *be* G*-groups. A* G*-group morphism* $f\colon H' \to H$ *is an epimorphism in the category* G-**Grp** *if and only if it is a surjective mapping.*

Proof — Clearly, any surjective G-group morphism is an epimorphism in hbox$G$-**Grp**. Conversely, let $f\colon H' \to H$ be an epimorphism in G-**Grp**. Set $A := f(H')$, so that $A \leqslant_G H$.

Step 1: there are G-sets X (via a group morphism $\sigma\colon G \to S_X$, where S_X denotes the symmetric group on X), which are also H-sets (via $\sigma'\colon H \to S_X$), such that the two actions are compatible in the sense that $\sigma'(gh) = \sigma(g) \circ \sigma'(h) \circ \sigma(g^{-1})$ for every $g \in G, h \in H$.

For instance, this is the case of the set $X' = \{*\}$ with one element $*$. A less trivial example is the set $X'' = \{\, hA \mid h \in H\}$ of all left cosets of A in H. In this case, the (well defined!) action of G on X'' is defined by $g(hA) = (gh)A$ and the action of H on X'' is defined by $h_1(hA) = (h_1h)A$ for every $g \in G, h, h_1 \in H$. Let us show that the two actions are compatible. We must show that

$$\sigma'(gh)(h_1A) = (\sigma(g) \circ \sigma'(h) \circ \sigma(g^{-1}))(h_1A)$$

for every $g \in G$, $h, h_1 \in H$. Now $\sigma'(gh)(h_1A) = ((gh)h_1)A$ and

$$(\sigma(g) \circ \sigma'(h) \circ \sigma(g^{-1}))(h_1A) = (g(h(g^{-1}h_1)))A.$$

These two left cosets are equal, because

$$(gh)h_1 = gh \cdot g(g^{-1}(h_1)) = g(h(g^{-1}h_1)).$$

In the rest of this proof, we will need as $X := \{*\} \cup X''$, the disjoint union of X' and X'', which has the properties in the statement of Step 1.

Step 2: there is a permutation $\tau\colon X \to X$ of X such that

$$\tau \circ \sigma(g) = \sigma(g) \circ \tau$$

for every $g \in G$.

In fact, let τ be the transposition $(*\ A)$ (the elements $*$ and the coset $A \in X''$ are swapped and all the other elements of X, that is, all cosets $hA \neq A$, are fixed by τ). In order to prove that

$$(\tau \circ \sigma(g))(x) = (\sigma(g) \circ \tau)(x)$$

for every $g \in G$ and $x \in X$, we must distinguish the three cases $x = hA \neq A$, $x = A$ and $x = *$.

In the first case, we have that $x = hA$ for some $h \in H \setminus A$. The automorphism $\varphi(g) \in \mathrm{Aut}(H)$ induces an endomorphism on the G-subgroup A of H, so that $gA \subseteq A$. Similarly for g^{-1}, so $gA = A$. It follows that if $h \in H \setminus A$, then $gh \in H \setminus A$, so $(gh)A \neq A$. Thus

$$(\tau \circ \sigma(g))(x) = \tau(\sigma(g)(hA)) = \tau((gh)A) = (gh)A$$

and

$$(\sigma(g) \circ \tau)(x) = \sigma(g)(\tau(hA)) = g(hA),$$

and we are done.

In the second case $x = A$, we have that

$$(\tau \circ \sigma(g))(x) = \tau(\sigma(g)(A)) = \tau(A) = *$$

and

$$(\sigma(g) \circ \tau)(x) = \sigma(g)(\tau(A)) = g(*) = *,$$

as we wanted.

In the third case, we have $x = *$, so that

$$(\tau \circ \sigma(g))(x) = \tau(\sigma(g)(*)) = \tau(*) = A$$

is equal to

$$(\sigma(g) \circ \tau)(x) = \sigma(g)(\tau(*)) = g(A) = A.$$

Step 3: for every $h \in H$, one has $\tau \circ \sigma'(h) = \sigma'(h) \circ \tau$ if and only if $h \in A$.

In fact, suppose $h \in A$. We must prove that

$$(\tau \circ \sigma'(h))(x) = (\sigma'(h) \circ \tau)(x)$$

for every $x \in X$. As in Step 2, we have the three cases

$$x = h_1 A \neq A,\ x = A \text{ and } x = *.$$

If $x = h_1 A$ for some $h_1 \in H \setminus A$, then

$$(\tau \circ \sigma'(h))(x) = \tau(\sigma'(h)(h_1 A)) = \tau(hh_1 A).$$

Then $h \in A$ implies $hh_1 \notin A$, so

$$(\tau \circ \sigma'(h))(x) = \tau(hh_1 A) = hh_1 A.$$

Moreover,

$$(\sigma'(h) \circ \tau)(x) = \sigma'(h)(h_1 A) = hh_1 A,$$

as we wanted to prove.

In the second case $x = A$, we have that

$$(\tau \circ \sigma'(h))(x) = \tau(\sigma'(h)(A)) = \tau(A) = *$$

and

$$(\sigma'(h) \circ \tau)(x) = \sigma'(h)(\tau(A)) = h(*) = *,$$

as we wanted.

In the third case, we have $x = *$, so that

$$(\tau \circ \sigma'(h))(x) = \tau(\sigma'(h)(*)) = \tau(*) = A$$

is equal to

$$(\sigma'(h) \circ \tau)(x) = \sigma'(h)(\tau(*)) = hA = A.$$

Finally, suppose $h \in H \setminus A$. We must prove that

$$(\tau \circ \sigma'(h))(x) \neq (\sigma'(h) \circ \tau)(x)$$

for some $x \in X$. In fact, set $x := h^{-1} A$. Then

$$(\tau \circ \sigma'(h))(x) = \tau(\sigma'(h)(h^{-1} A)) = \tau(A) = *$$

and

$$(\sigma'(h)\circ\tau)(x) = \sigma'(h)(\tau(h^{-1}A)) = h(h^{-1}A) = A \neq *.$$

This concludes Step 3.

Now let F_X be the free group on the set X. Thus every bijection $X \to X$ extends to a unique group automorphism $F_X \to F_X$, so that we can view S_X as a subgroup of $K := \mathrm{Aut}(F_X)$. Consider the G-set structure on X given by

$$\sigma\colon G \to S_X \subseteq \mathrm{Aut}(F_X) = K.$$

There is a group morphism $\varphi''\colon G \to \mathrm{Aut}(K)$ defined by

$$\varphi''(g)(a) = \sigma(g)\circ a\circ\sigma(g^{-1})$$

for every $g \in G$ and every $a \in K = \mathrm{Aut}(F_X)$. Also, the H-set structure on X induces a group homomorphism $\sigma'\colon H \to S_X \subseteq K$.

Step 4: because of the compatibility between the action of G on X and the action of H on X (Step 1), the mapping $\sigma'\colon H \to K$ is a G-group morphism.

We must show that $\sigma'(gh) = g\sigma'(h)$ for every $g \in G$, $h \in H$. Now $g\sigma'(h) = \varphi''(g)(\sigma'(h)) = \sigma(g)\circ\sigma'(h)\circ\sigma(g^{-1}) = \sigma'(gh)$.

Now consider the permutation $\tau\colon X \to X$ of Step 2. We have that $\tau \in S_X \subseteq K = \mathrm{Aut}(F_X)$, so that conjugation by τ is a group morphism $\tau'\colon K \to K$, defined by

$$\tau'(k) = \tau\circ k\circ\tau^{-1}$$

for every $k \in K = \mathrm{Aut}(F_X)$.

Step 5: the group morphism $\tau'\colon K \to K$ is a G-group morphism.

By Step 2, we have that $\tau\circ\sigma(g) = \sigma(g)\circ\tau$ for every $g \in G$, so that $\tau\circ\sigma(g)\circ k\circ\sigma(g^{-1})\circ\tau^{-1} = \sigma(g)\circ\tau\circ k\circ\tau^{-1}\circ\sigma(g^{-1})$ for every $k \in K = \mathrm{Aut}(F_X)$. Equivalently, $\tau'(gk) = g\tau'(k)$.

Step 6: for the two G-group morphisms

$$\sigma'\colon H \to K \text{ and } \tau'\circ\sigma'\colon H \to K,$$

we have that $\sigma'\circ f = \tau'\circ\sigma'\circ f$.

We must prove that $\sigma'(f(h')) = \tau'(\sigma'(f(h')))$ for every $h' \in H'$. Equivalently, we must show that $\sigma'(a) = \tau'(\sigma'(a))$ for every $a \in A$, or, equivalently, that $\sigma'(a) = \tau \circ \sigma'(a) \circ \tau^{-1}$. But we have seen in Step 3 that $a \in A$ implies $\tau \circ \sigma'(a) = \sigma'(a) \circ \tau$.

Step 7: proof of the theorem.

Since f is an epimorphism, from what we have verified in Step 6, it follows that $\sigma' = \tau' \circ \sigma'$, that is, $\sigma'(h) = \tau'(\sigma'(h))$ for every $h \in H$. Equivalently, we have that

$$\sigma'(h) = \tau \circ \sigma'(h) \circ \tau^{-1},$$

that is, $\tau \circ \sigma'(h) = \sigma'(h) \circ \tau$ for every $h \in H$. But we have seen in Step 3 that this implies $h \in A$. In other words, $H \subseteq A = f(H')$, which proves that f is a surjective mapping. □

3 The Weak Krull-Schmidt Theorem for biuniform abelian G-groups

Let G be a group and H, H' be two G-groups. We say that H and H' belong to the same *monogeny class*, and write $[H]_m$=$[H']_m$, if there exist a monomorphism $H \to H'$ and a monomorphism $H' \to H$. Dually, we say that H and H' belong to the same *epigeny class*, and write $[H]_e = [H']_e$, if there exist an epimorphism $H \to H'$ and an epimorphism $H' \to H$.

Let G be a group and H an abelian G-group. We say that H is *uniform* if $H \neq 1$ and the intersection of any two non-trivial G-subgroups of H is non-trivial. Dually, we say that H is *couniform* if $H \neq 1$ and the product of any two proper G-subgroups of H is a proper subgroup of H. Finally, we say that H is *biuniform* if it is both uniform and couniform. Clearly, any abelian uniserial G-group $H \neq 1$ is biuniform.

Theorem 3.1 *Let G be a group and $H_1, \dots, H_n, H'_1, \dots, H'_t$ be $n + t$ biuniform abelian G-groups. Then the direct products*

$$H_1 \times \dots \times H_n \textit{ and } H'_1 \times \dots \times H'_t$$

are isomorphic G-groups if and only if $n = t$ and there exist two permutations σ and τ of $\{1, 2, \dots, n\}$ such that $[H_i]_m = [H'_{\sigma(i)}]_m$ and $[H_i]_e = [H'_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.

Proof — Let $\mathbb{Z}G$ denote the group ring of G. Recall that there is a functor $F\colon \mathbb{Z}G\text{-Mod} \to G\text{-}\mathbf{Grp}$, which associates to every left $\mathbb{Z}G$-module (M,λ), where $\lambda\colon \mathbb{Z}G \to \mathrm{End}(M)$ is a ring morphism into the endomorphism ring of the abelian group M, the G-group $(M,\lambda|_G)$, where $\lambda|_G\colon G \to \mathrm{Aut}(M)$ is the restriction of λ and M is viewed in $G\text{-}\mathbf{Grp}$ as a multiplicative group. The corestriction

$$F|^{\mathrm{Ab}(G\text{-}\mathbf{Grp})}\colon \mathbb{Z}G\text{-Mod} \to \mathrm{Ab}(G\text{-}\mathbf{Grp})$$

turns out to be a category equivalence [2, before Remark 2.3]. A subgroup of the abelian group M is a $\mathbb{Z}G$-submodule of (M,λ) if and only if it is a G-subgroup of $(M,\lambda|_G)$, so that $\mathrm{Sub}_G(M)$ is the lattice of all $\mathbb{Z}G$-submodules of M. It follows that, in the category equivalence $F|^{\mathrm{Ab}(G\text{-}\mathbf{Grp})}$, biuniform $\mathbb{Z}G$-modules correspond exactly to biuniform abelian G-groups. The functor $F|^{\mathrm{Ab}(G\text{-}\mathbf{Grp})}$ is the identity on morphisms and, in both categories $\mathbb{Z}G$-Mod and $\mathrm{Ab}(G\text{-}\mathbf{Grp})$, monomorphisms (epimorphisms, resp.) are the injective (surjective, resp.) morphisms respectively. It follows that $F|^{\mathrm{Ab}(G\text{-}\mathbf{Grp})}$ preserves both monogeny classes and epigeny classes. Finally, notice that $F|^{\mathrm{Ab}(G\text{-}\mathbf{Grp})}$ transforms finite direct sums (= products in the additive category $\mathbb{Z}G$-Mod) into finite direct products in the additive category $\mathrm{Ab}(G\text{-}\mathbf{Grp})$. The conclusion now follows from [6, Theorem 9.13]. □

Let $n \geqslant 2$ be an integer. We claim that there exists a group G with n^2 pairwise non-isomorphic biuniform abelian G-groups

$$U_{i,j} \quad (i,j = 1,2,\dots,n)$$

with the following properties:

(a) for every $i,j,k,\ell = 1,2,\dots,n$, $[U_{i,j}]_m = [U_{k,\ell}]_m$ if and only if $i = k$;

(b) for every $i,j,k,\ell = 1,2,\dots,n$, $[U_{i,j}]_e = [U_{k,\ell}]_e$ if and only if $j = \ell$.

From Theorem 3.1, it follows that

$$U_{1,1} \times U_{2,2} \times \dots \times U_{n,n} \simeq U_{\sigma(1),\tau(1)} \times U_{\sigma(2),\tau(2)} \times \dots \times U_{\sigma(n),\tau(n)}$$

for every pair of permutations σ,τ of $\{1,2,\dots,n\}$.

Our example will be constructed adapting an example that appears in [5, Example 2.1] (also see [6, Example 9.20]). In that example, the following ring R was considered. Let $\mathbb{Q}$ be the field of rational numbers and $\mathbf{M}_n(\mathbb{Q})$ be the ring of all $n \times n$-matrices with entries in $\mathbb{Q}$. Let $\mathbb{Z}_p, \mathbb{Z}_q$ be the localizations of the ring $\mathbb{Z}$ at two distinct maximal ideals (p) and (q) of $\mathbf{Z}$ (here $p, q \in \mathbb{Z}$ denote two distinct primes $\geqslant 3$). Let Λ_p be the subring of $\mathbf{M}_n(\mathbb{Q})$ whose elements are the $n \times n$-matrices with entries in $\mathbb{Z}_p$ in and below the diagonal and entries in $p\mathbb{Z}_p$ above the diagonal, that is,

$$\Lambda_p := \begin{pmatrix} \mathbb{Z}_p & p\mathbb{Z}_p & \dots & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p & \dots & p\mathbb{Z}_p \\ \vdots & & \ddots & \\ \mathbb{Z}_p & \mathbb{Z}_p & \dots & \mathbb{Z}_p \end{pmatrix} \subseteq \mathbf{M}_n(\mathbb{Q}).$$

Similarly, set

$$\Lambda_q := \begin{pmatrix} \mathbb{Z}_q & q\mathbb{Z}_q & \dots & q\mathbb{Z}_q \\ \mathbb{Z}_q & \mathbb{Z}_q & \dots & q\mathbb{Z}_q \\ \vdots & & \ddots & \\ \mathbb{Z}_q & \mathbb{Z}_q & \dots & \mathbb{Z}_p \end{pmatrix} \subseteq \mathbf{M}_n(\mathbb{Q}).$$

If

$$R = \begin{pmatrix} \Lambda_p & \mathbf{M}_n(\mathbb{Q}) \\ 0 & \Lambda_q \end{pmatrix},$$

then R turns out to be a subring of the ring $\mathbf{M}_{2n}(\mathbb{Q})$ of all $2n \times 2n$-matrices with rational entries.

Let $G := U(R)$ be the group of the invertible elements of R. First of all, we will determine the elements of G. Let $U(\mathbb{Z}_p), U(\mathbb{Z}_q)$ be the groups of invertible elements of the rings $\mathbb{Z}_p, \mathbb{Z}_q$, respectively. Observe that $U(\mathbb{Z}_p) = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, and that $U(\mathbb{Z}_p)$ is the free abelian group with free set of generators the set of all primes distinct from p. Let G_p be the subgroup of $GL_n(\mathbb{Q})$ whose elements are the $n \times n$-matrices with entries in $U(\mathbb{Z}_p)$ in the diagonal, entries in $p\mathbb{Z}_p$ above the diagonal, and entries in $\mathbb{Z}_p$ below the diagonal, so that

$$G_p := \begin{pmatrix} U(\mathbb{Z}_p) & p\mathbb{Z}_p & \dots & p\mathbb{Z}_p \\ \mathbb{Z}_p & U(\mathbb{Z}_p) & \dots & p\mathbb{Z}_p \\ \vdots & & \ddots & \\ \mathbb{Z}_p & \mathbb{Z}_p & \dots & U(\mathbb{Z}_p) \end{pmatrix} \subseteq \mathbf{GL}_n(\mathbb{Q}).$$

Similarly, let G_q be the subgroup of $GL_n(\mathbb{Q})$ whose elements are the $n \times n$-matrices with entries in $U(\mathbb{Z}_q)$ in the diagonal, entries in $q\mathbb{Z}_q$ above the diagonal, and entries in $\mathbb{Z}_q$ below the diagonal, so that

$$G_q := \begin{pmatrix} U(\mathbb{Z}_q) & q\mathbb{Z}_q & \dots & q\mathbb{Z}_q \\ \mathbb{Z}_q & U(\mathbb{Z}_q) & \dots & q\mathbb{Z}_q \\ \vdots & & \ddots & \\ \mathbb{Z}_q & \mathbb{Z}_q & \dots & U(\mathbb{Z}_q) \end{pmatrix} \subseteq \mathbf{GL}_n(\mathbb{Q}).$$

It is easy to see that

$$G = \begin{pmatrix} G_p & \mathbf{M}_n(\mathbb{Q}) \\ 0 & G_q \end{pmatrix}. \tag{1}$$

To see it, recall that the Jacobson radicals of Λ_p and R are

$$J(\Lambda_p) = \begin{pmatrix} p\mathbb{Z}_p & p\mathbb{Z}_p & \dots & p\mathbb{Z}_p \\ \mathbb{Z}_p & p\mathbb{Z}_p & \dots & p\mathbb{Z}_p \\ \vdots & & \ddots & \\ \mathbb{Z}_p & \mathbb{Z}_p & \dots & p\mathbb{Z}_p \end{pmatrix}$$

and

$$J(R) = \begin{pmatrix} J(\Lambda_p) & \mathbf{M}_n(\mathbb{Q}) \\ 0 & J(\Lambda_q) \end{pmatrix}$$

respectively [5, Example 2.1], so that $R/J(R)$ is isomorphic to the direct product $(\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})^n$ of $2n$ fields. The canonical projection $\pi\colon R \to R/J(R)$ is a local morphism, i.e., an element of R is invertible in R if and only if its image in $R/J(R)$ is invertible in $R/J(R)$. Thus we have a local morphism

$$R \to (\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})^n,$$

which associates to each matrix in R the residues of its elements in the diagonal. Hence a matrix in R belongs to G if and only if its first n elements in the diagonal are in $U(\mathbb{Z}_p)$ and the last n elements in the diagonal are in $U(\mathbb{Z}_q)$.

Let e_{n+1} be the idempotent matrix with 1 in the $(n+1,n+1)$-entry and 0 in the other entries. The left ideal

$$Re_{n+1} = \begin{pmatrix} 0 & \dots & 0 & \mathbb{Q} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \mathbb{Q} & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathbb{Z}_q & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \mathbb{Z}_q & 0 & \dots & 0 \end{pmatrix}$$

of R is uniserial.

The modules $U_{i,j}$ constructed in [5, Example 2.1] are submodules of homomorphic images of the left ideal Re_{n+1} of R. Thus, in order to show that the G-groups $U_{i,j}$ are biuniform as G-groups, it suffices to show that Re_{n+1} is uniserial as a G-group. To this end, it is sufficient to show that the G-subgroups of Re_{n+1} coincide with the R-submodules of Re_{n+1}, and for this it is enough to show that the canonical mapping $\mathbb{Z}G \to R$ is onto, that is, that every element of R is a finite sum of invertible elements of R. Because of (1), it suffices to show that in the ring direct product $(\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})^n$ of $2n$ rings, every element is a finite sum of invertible elements. In fact, every element of $(\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})^n$ is a sum of two invertible elements, because, for every element

$$(a_1, \dots, a_n, b_1, \dots, b_n)$$

of $(\mathbb{Z}/p\mathbb{Z})^n \times (\mathbb{Z}/q\mathbb{Z})^n$, we have that

$$(a_1, \dots, a_n, b_1, \dots, b_n) =$$

$$(a_1 \pm 1, \dots, a_n \pm 1, b_1 \pm 1, \dots, b_n \pm 1) - (\pm 1, \dots, \pm 1, \pm 1, \dots, \pm 1),$$

where ± 1 is $+1$ if a_i is $\equiv 0, 1, 2, \dots, p-2 \pmod{p\mathbb{Z}_p}$, and ± 1 is -1 if $a_i \equiv p-1 \pmod{p\mathbb{Z}_p}$, and similarly for the b_i's (here we are using the fact that $p, q \geqslant 3$, so that every element in $\mathbb{Z}_p, \mathbb{Z}_q$ is a sum of two invertible elements). Thus we have proved that Re_{n+1} is a uniserial G-group, so that all the n^2 G-groups $U_{i,j}$ are uniserial, hence biuniform.

Thus the G-groups $U_{i,j}$, $i, j = 1, 2, \dots, n$, have the required properties. In fact, morphisms, monomorphisms and epimorphisms in the two categories $\mathrm{Ab}(G\text{-}\mathbf{Grp})$ and R-Mod coincide, so that two R-modules are in the same monogeny class as R-modules if and only if they are in the same monogeny class as G-groups, and similarly for epigeny classes. This concludes the proof of our claim.

4 The category G-Sdp of all G-semidirect products

The notion of suddirect product was introduced by Birkhoff in 1944, and is particularly convenient in the study of G-groups. Let us be more precise. We will now consider semidirect product of two groups. The construction of semidirect product can also be carried out for any pair of monoids. If M and M' are monoids and $\varphi\colon M' \to \mathrm{End}(M)$ is a monoid morphism, the *semidirect product* of M and M' via φ, denoted $M \rtimes_\varphi M'$, is the cartesian product $M \times M'$ with the multiplication defined by

$$(a, a')(b, b') = (a\varphi(a')(b), a'b'),$$

where $a, b \in M$ and $a', b' \in M'$ [12, p. 425]. In this paper, we will be concerned only about semidirect product of groups.

Recall that a group P is a semidirect product of its normal subgroup H and its subgroup G, written $P = H \rtimes G$, if and only if there exists a group morphism $P \to G$ with kernel H which is the identity on G. For a fixed group G, we will consider the category G-**Sdp** of G-*semidirect products* (G-**Sdp** is sometimes also called the *category of pointed objects over the* G*-group* G). Its objects are the triples (P, α, β), where P is a group and $\alpha\colon G \to P$, $\beta\colon P \to G$ are group morphisms such that the composite mapping $\beta\alpha$ is the identity automorphism id_G of the group G. The morphisms from (P, α, β) to (P', α', β') in G-**Sdp** are the group morphisms $a\colon P \to P'$ for which the diagram

$$\begin{array}{ccccc} G & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & G \\ \| & & \downarrow a & & \| \\ G & \xrightarrow{\alpha'} & P' & \xrightarrow{\beta'} & G \end{array}$$

commute, that is, such that $a\alpha = \alpha'$ and $\beta' a = \beta$.

There is a category equivalence $F\colon G$-**Grp** $\to$ G-**Sdp** [2, 4.1]. It assigns to any object (H, φ) of G-**Grp** the object (P, α, β) of G-**Sdp**, where P is the semidirect product $H \rtimes G$, which we had denoted by $H \rtimes_\varphi G$ above, that is, the cartesian product $H \times G$ with the operation defined by

$$(h_1, g_1)(h_2, g_2) = (h_1\varphi(g_1)(h_2), g_1g_2),$$

$\alpha\colon G \to P$ is defined by $\alpha(g) = (1, g)$ and $\beta\colon P \to G$ is defined by $\beta(h, g) = g$.

If (H, φ) and $(H^{'}, \varphi^{'})$ are G-groups and $(P, \alpha, \beta), (P', \alpha', \beta')$ are the corresponding objects of G-**Sdp**, the functor

$$F\colon G\text{-}\mathbf{Grp} \to G\text{-}\mathbf{Sdp}$$

associates to any G-group morphism

$$f\colon (H, \varphi) \to (H', \varphi')$$

the group morphism $\widetilde{f}\colon P = H \rtimes G \to P' = H' \rtimes G$ defined by

$$\widetilde{f}(h, g) = (f(h), g)$$

for every $(h, g) \in P = H \rtimes G$.

If we restrict the category equivalence $F\colon G\text{-}\mathbf{Grp} \to G\text{-}\mathbf{Sdp}$ to the abelian objects, we get a category equivalence between the category Ab(G-**Grp**) and the category Ab(G-**Sdp**). But we have see in the proof of Theorem 3.1 that the category Ab(G-**Grp**) is equivalent to the category $\mathbb{Z}G$-Mod. Thus the category Ab(G-**Sdp**) is abelian. Its objects are the semidirect products $P = H \rtimes G$ with H an abelian group. In particular, if $(P, \alpha, \beta), (P', \alpha', \beta')$ are two objects of G-**Sdp** with H,H' abelian, then

$$\mathrm{Hom}_{G\text{-}\mathbf{Sdp}}(P,P')$$

is an abelian additive group. The addition in $\mathrm{Hom}_{G\text{-}\mathbf{Sdp}}(P, P')$ is defined as follows. If $a, b \in \mathrm{Hom}_{G\text{-}\mathbf{Sdp}}(P, P')$, then

$$(a + b)(p) = a(p)b(p)(\alpha'\beta(p))^{-1}$$

for every $p \in P$. The zero element of the additive group

$$\mathrm{Hom}_{G\text{-}\mathbf{Sdp}}(P, P')$$

is $\alpha'\beta$. The opposite of a is $-a$ defined by

$$(-a)(p) = (a(p))^{-1}(\alpha'\beta(p))^2$$

for every $p \in P$.

As we have already said at the beginning of this section, products in the category G-**Sdp** are very nice examples of subdirect products in the sense of Birkhoff. Let $(P_\lambda, \alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$, be a family of objects in the category G-**Sdp**, and construct the pullback of the group morphisms $\beta_\lambda \colon P_\lambda \to G$. We denote it by $\prod^G_{\lambda\in\Lambda} P_\lambda$. It is the subgroup of the direct product $\prod_{\lambda\in\Lambda} P_\lambda$ consisting of all the λ-tuples

$$(p_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} P_\lambda$$

such that

$$\beta_\lambda(p_\lambda) = \beta_\mu(p_\mu)$$

for every $\lambda, \mu \in \Lambda$. Let $\alpha \colon G \to \prod^G_{\lambda\in\Lambda} P_\lambda$ be the mapping defined by $\alpha(g) = (\alpha_\lambda(g))_{\lambda\in\Lambda}$ for every $g \in G$ and $\beta \colon \prod^G_{\lambda\in\Lambda} P_\lambda \to G$ be the mapping defined by $\beta(p_\lambda)_{\lambda\in\Lambda} = \beta_{\lambda_0}(p_{\lambda_0})$, where λ_0 is any element of Λ. Then $(\prod^G_{\lambda\in\Lambda} P_\lambda, \alpha, \beta)$ is the product in G-**Sdp** of the family $(P_\lambda, \alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$. The projections $\pi_\mu \colon \prod^G_{\lambda\in\Lambda} P_\lambda \to P_\mu$ are the restrictions of the canonical projections $\prod_{\lambda\in\Lambda} P_\lambda \to P_\mu$.

For a finite family $(P_\lambda, \alpha_\lambda, \beta_\lambda)$, $\lambda \in \Lambda$, of G-semidirect products

$$P_\lambda = H_\lambda \rtimes G$$

with the groups H_λ abelian groups (Λ finite), product and coproduct coincide, because the category Ab(G-**Sdp**) is additive. In this case, if we consider the mappings $\varepsilon_\mu \colon P_\mu \to \prod^G_{\lambda\in\Lambda} P_\lambda$, defined by

$$\varepsilon_\mu(p_\mu) = (q_\lambda)_{\lambda\in\Lambda},$$

where $q_\mu = p_\mu$ and $q_\lambda = \alpha_\lambda\beta_\mu(p_\mu)$ for $\lambda \neq \mu$, then $\pi_\mu\varepsilon_\mu = \mathrm{id}_{P_\mu}$ and $\sum_{\lambda\in\Lambda} \varepsilon_\lambda\pi_\lambda = \mathrm{id}_{\prod^G_{\lambda\in\Lambda} P_\lambda}$.

Let us pay specific attention to the abelian category Ab(G-**Sdp**). The monomorphisms (epimorphisms) in this abelian category are the injective (surjective) morphisms. Thus, for any two semidirect products $P = H \rtimes G$ and $P' = H' \rtimes G$ with H, H' abelian groups, we say that P and P' belong to the same *monogeny class*, and write $[P]_m = [P']_m$, if there exist injective morphism $P \to P'$ and $P' \to P$ in the category G-**Sdp**. Dually, we say that P and P' belong to the same *epigeny class*, and write $[P]_e = [P']_e$, if there exist surjective morphisms $P \to P'$ and $P' \to P$ in G-**Sdp**.

Let $P = H \rtimes G$ be a semidirect product with H an abelian group. We say that P is *biuniform* if H is a biuniform G-group with respect to the G-action defined by conjugation. From Theorem 3.1, we immediately obtain that:

Theorem 4.1 *Let* G *be a group and*

$$P_i = H_i \rtimes G \quad (i = 1, 2, \dots, n),\ P'_j = H'_j \rtimes G \quad (j = 1, 2, \dots, t)$$

be $n + t$ *biuniform semidirect products with all the groups* H_i, H'_j *abelian groups. Then the subdirect products*

$$\prod_i^G P_i \text{ and } \prod_j^G P'_j$$

are isomorphic (in the category G-**Sdp***) if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \dots, n\}$ *such that* $[P_i]_m = [P'_{\sigma(i)}]_m$ *and* $[P_i]_e = [P'_{\tau(i)}]_e$ *for every* $i = 1, 2, \dots, n$.

5 Biuniform abelian normal subgroups of a group

We are now ready to generalize the results in Section 3 to abelian biuniform normal subgroups of an arbitrary group P. Consider the following problem. Let P be any group and H an abelian normal subgroup of P. Then H is a P-group via conjugation, so that it is an abelian P-group. Thus H turns out to be a left $\mathbb{Z}P$-module via the left scalar multiplication $p \cdot h = php^{-1}$ for every $p \in P$, $h \in H$. If H, H' are abelian normal subgroups of P, the set $\mathrm{Hom}_P(H, H')$ of all P-normal morphisms from H to H', that is, the set of all group morphisms $f\colon H \to H'$ such that $f(php^{-1}) = pf(h)p^{-1}$) ([9, Section 4] and [2, Introduction]), is an additive group. We say that two normal subgroups H, H' of P are P-*isomorphic* if there exists a P-normal group isomorphism $H \to H'$. The set $\mathrm{End}_P(H)$ of all P-normal endomorphisms of H is a ring. If the ring $\mathrm{End}_P(H)$ is local, then H is indecomposable as a P-group, that is, it cannot be written as a direct product of two non-trival normal subgroups of P. If $\mathrm{End}_P(H)$ is semiperfect, then H is a direct product of finitely many indecomposable normal subgroups of P [6, Proposition 3.14], and the Krull-Schmidt Theorem applies, so that any two direct-product decompositions of H into indecomposable normal subgroups of P are P-*isomorphic,* that is, such a direct-product decomposition is unique up to bijective P-normal morphisms.

Thus let P be a group and $\mathcal{N}(P)$ the set of all normal subgroups of P, so that $\mathcal{N}(P)$ is a bounded complete modular lattice under inclusion. Let $\mathcal{N}_{ab}(P) \subseteq \mathcal{N}(P)$ be the set of all abelian normal subgroups of P. Then $\mathcal{N}_{ab}(P)$ is a partially ordered subset of $\mathcal{N}(P)$, and $\mathcal{N}_{ab}(P)$ is a union of intervals $[1, H]$ of $\mathcal{N}(P)$, $H \in \mathcal{N}_{ab}(P)$. Here by the interval $[1, H]$ we mean the set of all $H' \in \mathcal{N}(P)$ with $H' \subseteq H$. Thus $\mathcal{N}_{ab}(P)$ is a union of modular lattices, but is not a sublattice of $\mathcal{N}(P)$, because if $H, L \in \mathcal{N}_{ab}(P)$, then $H \vee L = HL$ is not necessarily abelian. Thus $H \vee L$ does not necessarily exist in $\mathcal{N}_{ab}(P)$, but it does exist when $H \wedge L = H \cap L = 1$. More generally, if $\{H_i \mid i \in I\}$ is a set of join-independent abelian groups in the modular lattice $\mathcal{N}(P)$ [6, p. 51], then $\bigvee_{i \in I} H_i$ (the upper bound in the complete lattice $\mathcal{N}(P)$) belongs to $\mathcal{N}_{ab}(P)$. Notice that, applying Zorn's Lemma, every element of $\mathcal{N}_{ab}(P)$ is contained in a maximal element of $\mathcal{N}_{ab}(P)$.

When the group P is finite, the situation is very simple. For any abelian normal subgroup H of P, H is a finite left $\mathbb{Z}P$-module, hence a left $\mathbb{Z}P$-module of finite composition length. Thus Fitting's Lemma applies, so that, by the Krull-Schmidt Theorem [6, Corollary 2.23], any two direct-product decompositions of H into indecomposable normal subgroups of P are P-isomorphic.

Among the elements $H \in \mathcal{N}_{ab}(P)$, we are particularly interested in the abelian normal subgroups H of P for which the interval $[1, H]$ has Goldie dimension 1 and dual Goldie dimension 1. We will say that these abelian normal subgroups H of P are *biuniform*. Thus an abelian normal subgroup H of P is biuniform if and only if $H \neq 1$, the intersection of any two non-trivial normal subgroups of P contained in H is non-trivial and the product of any two normal subgroups of P properly contained in H is a proper subgroup of H.

For every biuniform abelian normal subgroup H of P, the endomorphism ring $\mathrm{End}_P(H)$ has at most two maximal right ideals [6, Theorem 9.1]. If $\{H_1, \dots, H_n\}$ is a join-independent set of abelian biuniform normal subgroups of P, we can apply the results in Section 3 about the Weak Krull-Schmidt Theorem for P-groups (Theorem 3.1) and describe the direct-product decompositions of the abelian normal subgroup

$$H := H_1 \vee \ldots \vee H_n = H_1 \times \ldots \times H_n$$

of P.

We say that two normal subgroups H, H' of P belong to the same *monogeny class*, and write $[H]_m = [H']_m$, if there exist a P-normal injective group morphism $H \to H'$ and a P-normal injective group morphism $H' \to H$. Dually, we say that H and H' belong to the same *epigeny class*, and write $[H]_e = [H']_e$, if there exist a P-normal surjective group morphism $H \to H'$ and a P-normal surjective group morphism $H' \to H$. From Theorem 3.1, we immediately get that:

Theorem B *Let* P *be any group. Let*

$$H_1, \ldots, H_n, H'_1, \ldots, H'_t$$

be $n + t$ *biuniform abelian normal subgroups of* P*. Suppose that the products* $H_1 \ldots H_n, H'_1 \ldots H'_t$ *are direct, that is,* $H_1 \ldots H_n = H_1 \times \ldots \times H_n$ *and* $H'_1 \ldots H'_t = H'_1 \times \ldots \times H'_t$*. Then the normal subgroups* $H_1 \times \ldots \times H_n$ *and* $H'_1 \times \ldots \times H'_t$ *of* P *are* P*-isomorphic if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \ldots, n\}$ *such that* $[H_i]_m = [H'_{\sigma(i)}]_m$ *and* $[H_i]_e = [H'_{\tau(i)}]_e$ *for every* $i = 1, 2, \ldots, n$*.*

Remark 5.1 Notice that the automorphisms of the G-group G in the category of G-groups are exactly the central automorphisms of G [2, paragraph after Remark 3.2]. Coherently, the classical Theorem of Krull-Schmidt-Remak concerns the existence of a *central* automorphism of the group G of which we study the direct-product decompositions. In Theorem B, in a similar way, we get a P*-isomorphism*, that is, an isomorphism in the category of P-groups. More precisely:

Proposition 5.2 *Let* $G = H_1 \times \ldots \times H_n = H'_1 \times \ldots \times H'_n$ *be two direct-product decompositions of a group* G*. The following conditions are equivalent:*

(a) *There exists a central automorphism* φ *of* G *such that* $\varphi(H_i) = H'_i$ *for every* $i = 1, 2, \ldots, n$*.*

(b) *There exists a* G*-isomorphisms* $\varphi_i \colon H_i \to H'_i$ *for every* $i = 1, 2, \ldots, n$*.*

Proof — If there is a central automorphism φ of G such that $\varphi(H_i) = H'_i$ for every $i = 1, 2, \ldots, n$, the matrix representation of φ

with respect to the two direct-product decompositions

$$G = H_1 \times \ldots \times H_n = H'_1 \times \ldots \times H'_n$$

of G [9, at the end of Section 4] is diagonal:

$$\begin{pmatrix} \psi_{H'_1,H_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_{H'_n,H_n} \end{pmatrix}.$$

Since φ is a central automorphism, it is a normal automorphism [19, 3.3.6], namely, a G-group automorphism of the G-group G. It follows now easily that all the morphisms

$$\varphi_i := \psi_{H'_n,H_n} \colon H_i \to H'_i$$

are G-isomorphisms. The converse is now easy. □

As an example of the situation studied in this section, we can take as our group P the semidirect product

$$(U_{1,1} \oplus \ldots \oplus U_{n,n}) \rtimes G,$$

where the multiplicative group G and the additive groups

$$U_{i,j}\ (i, j = 1, 2, \ldots, n)$$

are those of the example constructed in Section 2. The group G acts on the groups $U_{i,j}$ via left multiplication. Thus in

$$P := (U_{1,1} \oplus \ldots \oplus U_{n,n}) \rtimes G$$

the operation is defined by

$$(h_1, \ldots, h_n, g)(h'_1, \ldots, h'_n, g') = (h_1 + gh'_1, \ldots, h_n + gh'_n, gg').$$

The normal P-subgroup $U_{1,1} \oplus \ldots \oplus U_{n,n}$ of P decomposes in exactly $n!$ non-isomorphic ways as a direct product (direct sum) of normal subgroups of P.

Notice that any indecomposable direct summand of a finite direct sum of uniserial left R-modules is a uniserial submodule [18]. It follows that, in the previous example, if H is any normal indecomposable subgroup of P that is a direct summand of the normal subgroup

$$U_{1,1} \oplus \ldots \oplus U_{n,n}$$

of P, then H is uniserial as a P-group.

6 The G-set Hom(H,A)

We use the standard additive notation for the mappings between the multiplicative groups H and A. Thus if $f, f' \colon H \to A$ are any two mappings, then $f + f' \colon H \to A$ is the mapping defined by

$$(f + f')(x) = f(x)f'(x)$$

for every $x \in H$. If f and f' are group homomorphisms, then $f+f'$ is a group homomorphism if and only if $f(x)$ commutes with $f'(y)$ for all $x, y \in H$, that is, if and only if the subgroups $f(H)$ and $f'(H)$ of A centralize each other. The identity endomorphisms and the trivial homomorphisms will be denoted by id and 0, respectively, so that $\mathrm{id}\, f = f\, \mathrm{id} = f$ and $f + 0 = 0 + f = f$. Thus, on the set $\mathrm{Hom}(H, A)$, there is a partially defined operation $+$, as follows. Set

$$S := \{ (f, f') \in \mathrm{Hom}(H, A) \times \mathrm{Hom}(H, A) \mid [f(H), f'(H)] = 1 \}$$

and, for every $(f, f') \in S$, set

$$(f + f')(h) = f(h)f'(h)$$

for every $h \in H$. Then we have a mapping $+ \colon S \to \mathrm{Hom}(H, A)$.

Let G and A be groups and let H be a G-group. Let

$$\lambda_g \colon H \to H$$

denote left multiplication by g for every $g \in G$. Consider the set $\mathrm{Hom}(H, A)$ of all group morphisms of H into A. Then $\mathrm{Hom}(H, A)$ becomes a G-set if we define as left scalar multiplication the mapping

$$\cdot \colon G \times \mathrm{Hom}(H, A) \to \mathrm{Hom}(H, A)$$

defined by $gf = f \circ \lambda_{g^{-1}}$ for every $g \in G$, $f \in \mathrm{Hom}(H, A)$.

For every $f \in \mathrm{Hom}(H, A)$, the subgroup $f(H)$ of A is a G-group in a natural way, and we say that the morphism f is G-*uniserial* if $f(H)$ is a uniserial G-group. Thus a morphism $f\colon H \to A$ is G-uniserial if and only if for every $h, h' \in H$ there exists $g \in G$ such that either $f(gh) = f(h')$ or $f(gh') = f(h)$.

Recall that if $N_1, \ldots, N_n$ are normal proper subgroups of a group H, then $N_1, \ldots, N_n$ are said to be *coindependent* if

$$N_i\left(\bigcap_{j\neq i} N_j\right) = H$$

for every $i = 1, 2, \ldots, n$. Equivalently, if the canonical mapping

$$H \to H/N_1 \times \ldots \times H/N_n, \quad h \mapsto (hN_1, \ldots, hN_n),$$

is an onto mapping [9, Lemma 3.7].

We say that a finite family $f_1, \ldots, f_n\colon H \to A$ of group morphisms is *independent* if:

(1) $f_i \neq 0$ for every $i = 1, 2, \ldots, n$;

(2) the finite family of subgroups $f_i(H)$ of A is independent (that is, if the subgroup of A generated by

$$f_1(H) \cup \ldots \cup f_n(H)$$

is the direct product $f_1(H) \times \ldots \times f_n(H)$);

(3) the normal subgroups $\ker f_1, \ldots, \ker f_n$ of H are coindependent.

Notice the analogy between this notion of independent finite family of morphisms $H \to A$ and the notion of complete family of orthogonal idempotents in $\mathrm{End}(H)$. For any group H, there is a bijection between the set of all n-tuples $(H_1, \ldots, H_n)$ of normal subgroups of H with $H = H_1 \times \ldots \times H_n$ and the set of all n-tuples $(e_1, \ldots, e_n)$ of normal endomorphisms of H with $e_1 + \ldots + e_n = \mathrm{id}_H$ and $e_i e_j = 0$ for $i \neq j$ [19, 3.3.3]. Any family $e_1, \ldots, e_n$ of non-zero normal endomorphisms of H with $e_1 + \ldots + e_n = \mathrm{id}_H$ and $e_i e_j = 0$ for $i \neq j$ is clearly independent. Conversely, it follows from the next Lemma that if $e_1, \ldots, e_n$ is an independent family of endomorphisms of H with $e_1 + \ldots + e_n = \mathrm{id}_H$, then $H = e_1(H) \times \ldots \times e_n(H)$.

Lemma 6.1 *If* $f_1, \dots, f_n \colon H \to A$ *is an independent family of group morphisms, then:*

(a) *The sum* $f_1 + \dots + f_n \colon H \to A$ *is a group morphism.*

(b) *The image of* $f_1 + \dots + f_n$ *is* $f_1(H) \times \dots \times f_n(H)$.

Proof — Statement (a) easily follows from condition (2) of the definition of independent family. For (b), the image of the morphism

$$f_1 + \dots + f_n$$

is contained in the subgroup of A generated by the union

$$f_1(H) \cup \dots \cup f_n(H),$$

which is equal to $f_1(H) \times \dots \times f_n(H)$.

Conversely, let a be an element in $f_1(H) \times \dots \times f_n(H)$, so that

$$a = f_1(h_1) \dots f_n(h_n)$$

for suitable $h_1, \dots, h_n \in H$. Since the kernels are coindependent, there exists $h \in H$ such that $h \equiv h_i \pmod{\ker f_i}$ for every $i = 1, \dots, n$. Thus $f_i(h) = f_i(h_i)$ for all indices i. Therefore

$$a = f_1(h_1) \dots f_n(h_n) = f_1(h) \dots f_n(h) =$$

$$(f_1 + \dots + f_n)(h) \in (f_1 + \dots + f_n)(H). \qquad \square$$

Remark 6.2 In our previous paper [2], we stressed the several analogies and the differences between the categories R-Mod and G-**Grp**, for instance the analogies between the regular objects ${}_RR$ and ${}_GG$. One of the differences concernes idempotents endomorphisms. Namely, if ${}_RM$ is a left module and $E := \mathrm{End}({}_RM)$ is its endomorphism ring, then there is a one-to-one correspondence φ between the set I of all idempotent elements of E and the set

$$\{(A, B) \mid A, B \leqslant {}_RM,\ {}_RM = A \oplus B\}$$

of all pairs (A, B) of submodules of ${}_RM$ whose sum is direct and equal to ${}_RM$. If $e \in I$, the corresponding pair is

$$\varphi(e) = (\ker e, e({}_RM)).$$

The analogous result in the category **Grp** of groups is the following. Let H be a group and I be the set of all idempotent endomorphisms of H. There is a one-to-one correspondence φ between I and the set

$$\{(A,B) \mid A,B \leqslant H,\ H = A \rtimes B\}$$

of all pairs (A,B) of subgroups of H such that H is the semidirect product of its normal subgroup A and its subgroup B. If $e \in I$, the corresponding pair is $\varphi(e) = (\ker e, e(H))$.

If I' is the set of all idempotent normal endomorphisms of H, then the one-to-one correspondence φ of the previous paragraph restricts to a one-to-one correspondence between I' and the set

$$\{\,(A,B) \mid A,B \leqslant H,\ H = A \times B\,\}$$

of all pairs (A,B) of normal subgroups of H such that H is the direct product of its normal subgroups A and B.

Now let G be a group and H be a G-group. If $e\colon H \to H$ is an idempotent G-group endomorphism, then both the kernel and the image of e are G-subgroups of H and $H = \ker e \rtimes e(H)$. Thus the correspondence φ restricts to a one-to-one correspondence between the set of all idempotent endomorphisms of H in G-**Grp** and the set of all pairs (A,B) of G-subgroups of H such that H is a semidirect product (as a group) of its normal subgroup A and its subgroup B.

In the particular case of a group P and a normal subgroup H of P, if $e\colon H \to H$ is an idempotent P-normal endomorphism, then both the kernel and the image of e are normal subgroups of P and $H = \ker e \times e(H)$. Thus the correspondence φ restricts to a one-to-one correspondence between the set of all idempotent P-normal endomorphisms of H and the set of all pairs (A,B) of normal subgroups of P such that H is a direct product $H = A \times B$ (direct product as a group).

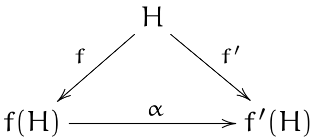
Let G and A be groups and let H be a G-group. We say that

$$f, f' \in \mathrm{Hom}(H,A)$$

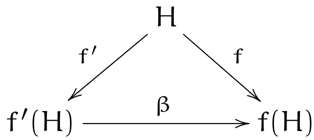
belong to the same *monogeny class*, and write $[f]_m = [f']_m$, if the G-groups $f(H)$ and $f'(H)$ belong to the same monogeny class. Equivalently, if there exist an injective group morphism

$$\alpha\colon f(H) \to f'(H)$$

and an injective group morphism $\beta\colon f'(H) \to f(H)$ such that the diagrams

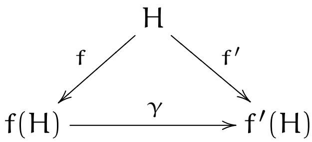


and



commute. Dually for $f, f' \in \mathrm{Hom}(H, A)$ in the same *epigeny class*.

Finally, we say that $f, f' \in \mathrm{Hom}(H, A)$ are *equivalent*, and write $f \sim f'$, if the G-groups $f(H)$ and $f'(H)$ are isomorphic, that is, if there exists a group isomorphism $\gamma\colon f(H) \to f'(H)$ which makes the diagram



commute.

Theorem 6.3 *Let* G *and* A *be groups and let* H *be a* G*-group. Suppose either* A *abelian or* H *abelian. Let* $f_1, \dots, f_n$ *and* $f'_1, \dots, f'_t$ *be two independent families of* G*-uniserial morphisms in* $\mathrm{Hom}(H, A)$*. Then*

$$f_1 + \ldots + f_n \sim f'_1 + \ldots + f'_t$$

if and only if $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \dots, n\}$ *such that* $[f_i]_m = [f'_{\sigma(i)}]_m$ *and* $[f_i]_e = [f'_{\tau(i)}]_e$ *for every* $i = 1, 2, \dots, n$.

Proof — Set $f := f_1 + \ldots + f_n$ and $f' := f'_1 + \ldots + f'_t$. We have that $f \sim f'$ if and only if the G-groups $f(H)$ and $f'(H)$ are isomorphic. Since the families are independent, we know that

$$f(H) = f_1(H) \times \ldots \times f_n(H)$$

and $f'(H) = f'_1(H) \times \ldots \times f'_t(H)$ (Lemma 6.1), where these direct-product decompositions are direct-product decompositions in the abelian category $\mathrm{Ab}(G\text{-}\mathbf{Grp})$. Thus $f \sim f'$ if and only if the direct products $f_1(H) \times \ldots \times f_n(H)$ and $f'(H) = f'_1(H) \times \ldots \times f'_t(H)$ of uniserial abelian G-groups are isomorphic. It is now easy to conclude from Theorem 3.1. □

The most interesting case of G-set Hom(H,A) is when H=G♭A, as follows. The construction of the G-group G♭A has been studied with deep results in [3, pp. 245–248], [4, p. 45], [15, p. 2571] and [16], and the properties we present here are taken from those four articles. Let G and A be any two groups and let $G * A$ be their free product. The identity morphism $\mathrm{id}_G\colon G \to G$ and the trivial morphism

$$0\colon A \to G$$

define a group morphism $\mathrm{id}_G *0\colon G * A \to G$. Let $\varepsilon_G\colon G \to G * A$ be the canonical embedding of G into the free product, so that we get a G-semidirect product (pointed object)

$$G \xrightarrow{\varepsilon_G} G * A \xrightarrow{\mathrm{id}_G *0} G.$$

Let G♭A be the kernel of $\mathrm{id}_G *0\colon G * A \to G$. Then $G * A$ splits as a semidirect product $G * A = (G \flat A) \rtimes G$. The group morphism

$$\mathrm{id}_G *0\colon G * A \to G$$

maps a word

$$g_1 a_1 g_2 a_n \dots g_n a_n \in G * A$$

to

$$g_1 g_2 \cdots g_n$$

. Thus G♭A consists of all words

$$g_1 a_1 g_2 a_n \dots g_n a_n \in G * A$$

with $g_1 g_2 \dots g_n = 1_G$. By induction on n, it is easy to see that a word $g_1 a_1 g_2 a_n \dots g_n a_n \in G * A$ is in G♭A if and only if it can be written as a product of finitely many words of the form

$$gag^{-1} \quad (g \in G, a \in A).$$

In particular, G♭A is a G-group (G acts on G♭A via conjugation).

Every element of G♭A can be written in a *unique* way as a product

$$(g_1 a_1 g_1^{-1})(g_2 a_2 g_2^{-1}) \dots (g_n a_n g_n^{-1})$$

with $g_i \neq g_{i+1}$ for every $i=1,\dots,n-1$ and $a_i \neq 1$ for every $i=1,\dots,n$.

More precisely, $G \flat A$ is the free product of $|G|$ copies of A, as the following proposition shows.

Proposition 6.4 *Let* G, A, L *be groups. For every family of group morphisms* $\psi_g \colon A \to L$ *(*$g \in G$*), there exists a unique group morphism*

$$\psi \colon G \flat A \to L$$

such that $\psi(gag^{-1}) = \psi_g(a)$ *for every* $g \in G,\ a \in A$.

Proof — If such a group morphism ψ exists, it is unique because the elements gag^{-1} generate $G \flat A$. In order to show that ψ is well defined, it suffices to notice that $(gag^{-1})(ga'g^{-1}) = gaa'g^{-1}$ and $\psi_g(a)\psi_g(a') = \psi_g(aa')$. □

If $F_{G\times A}$ is the free group on the cartesian product $G \times A$, there is a canonical group epimorphism

$$F_{G\times A} \to G \flat A, \quad (g, a) \mapsto gag^{-1},$$

whose kernel is the normal subgroup of $F_{G\times A}$ generated by the subset

$$\{(g, a)(g, a')(g, aa')^{-1} \mid g \in G,\ a, a' \in A\}$$

of $F_{G\times A}$ (show by induction on n that if

$$(g_1, a_1)^{\pm 1} \dots (g_n, a_n)^{\pm 1} \in F_{G\times A}$$

is in the kernel, then $(g_1, a_1)^{\pm 1} \dots (g_n, a_n)^{\pm 1}$ belongs to the subgroup generated by the elements $(g, a)(g, a')(g, aa')^{-1}$).

As we have already said, $G \flat A$ is a G-group. It has the following property. If we fix any mapping $\varphi \colon G \to \mathrm{End}(A)$, we can apply Proposition 6.4 to the groups G, A, A and the family of group morphisms

$$\varphi(g) \colon A \to A,$$

getting a unique group morphism $\psi \colon G \flat A \to A$ such that

$$\psi(gag^{-1}) = \varphi(g)(a)$$

for every $g \in G,\ a \in A$. Since $G \flat A$ is the coproduct of $|G|$ copies of A, it is easily seen that this assignment $\varphi \mapsto \psi$ is a bijection

$$\mathrm{Hom}_{\mathbf{Set}}(G, \mathrm{End}(A)) \to \mathrm{Hom}_{\mathbf{Grp}}(G \flat A, A).$$

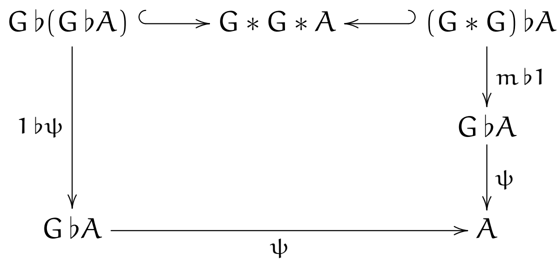
Let $\lambda_g \colon G \to G$ denote left multiplication by g, for every $g \in G$. Endow the set $\mathrm{Hom}_{\mathbf{Set}}(G, \mathrm{End}(A))$ with a G-set structure defining as left scalar multiplication the mapping:

$$\cdot \colon G \times \mathrm{Hom}_{\mathbf{Set}}(G, \mathrm{End}(A)) \to \mathrm{Hom}_{\mathbf{Set}}(G, \mathrm{End}(A))$$

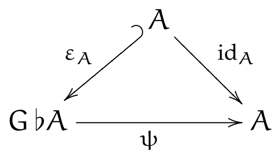
where $gf = f \circ \lambda_{g^{-1}}$ for every $g \in G$, $f \in \mathrm{Hom}_{\mathbf{Set}}(G, \mathrm{End}(A))$. Then the bijection above becomes a G-set isomorphism

$$\mathrm{Hom}_{\mathbf{Set}}(G, \mathrm{End}(A)) \simeq \mathrm{Hom}_{\mathbf{Grp}}(G \flat A, A).$$

In this G-set isomorphism, the elements $\varphi \in \mathrm{Hom}_{\mathbf{Set}}(G, \mathrm{End}(A))$ that are group homomorphisms $\varphi \colon G \to \mathrm{Aut}(A)$ corresponds to the group morphisms $\psi \in \mathrm{Hom}_{\mathbf{Grp}}(G \flat A, A)$ such that the two diagrams



and



commute. The commutativity of the two diagrams expresses the fact that the mapping $\varphi \colon G \to \mathrm{End}(A)$ is a monoid morphism, i.e., preserves multiplication and identity, respectively.

If $\varphi \colon G \to \mathrm{Aut}(A)$ is any group morphism, so that we have a G-group structure on A, then the G-group A is a homomorphic image of the G-group $G \flat A$. In fact, the corresponding $\psi \colon G \flat A \to A$, defined by $\psi(gag^{-1}) = \varphi(g)(a)$ for every $g \in G$, $a \in A$, is a G-group epimorphism.

Now suppose that G is a group and A is an abelian G-group that is a direct product $A = A_1 \times \ldots \times A_n$ of finitely many uniserial (or biuniform) G-subgroups A_i. Let $\psi \colon G \flat A \to A$ be the corresponding G-epimorphism and $e_i \colon A \to A$ the idempotent endomorphisms corresponding to the direct-product decomposition

$$A = A_1 \times \ldots \times A_n.$$

Then $f_1 := e_1\psi, \ldots, f_n := e_n\psi$ is an independent family of morphisms with $f_1 + \ldots + f_n = \psi$, and Theorem 6.3 applies.

7 Final remarks

We conclude this article with three remarks.

Remark 7.1 Let us recall a construction that appears in our previous paper [2, Section 2]. Let $\mathbf{C}$ be any category and M any monoid. Define the category $\mathbf{C}^M$ of all M-objects in $\mathbf{C}$. It is the category of all pairs (C, φ), where C is an object in $\mathbf{C}$ and $\varphi\colon M \to \mathrm{End}(C)$ is a monoid morphism. There is a forgetful functor $U\colon \mathbf{C}^M \to \mathbf{C}$, which associates to every object (C, φ) of $\mathbf{C}^M$ the object C of $\mathbf{C}$. If $\mathbf{C}$ has M-indexed coproducts, the forgetful functor $U\colon \mathbf{C}^M \to \mathbf{C}$ has a left adjoint $F\colon \mathbf{C} \to \mathbf{C}^M$, which is defined on objects by $F(C) = M \cdot C$, where $M \cdot C$ is the M-indexed coproduct of $|M|$ copies of C.

Now suppose that $\mathbf{C}$ has M-indexed products. This occurs when M is a group G and $\mathbf{C} = \mathbf{Grp}$, where, for any family of groups H_g indexed by the elements g of G, the M-indexed product is the direct product of the groups H_g. The forgetful functor $U\colon \mathbf{C}^M \to \mathbf{C}$ then also has a right adjoint $L\colon \mathbf{C} \to \mathbf{C}^M$. The functor L is defined on objects by $L(C) = (C^M, \varphi)$, where C^M is the M-indexed product of $|M|$ copies of C, and $\varphi\colon M \to \mathrm{End}(C^M)$ is constructed as follows. For every $m \in M$, let $\pi_m\colon C^M \to C$ be the projection. Now if $m \in M$ is fixed and we consider the family of morphisms

$$f_{m'} := \pi_{m'm}\colon C^M \to C, \quad m' \in M,$$

we get a unique morphism $f := \varphi(m)\colon C^M \to C^M$ such that

$$\pi_{m'}\varphi(m) = \pi_{m'm}$$

for every $m' \in M$. Thus we get a mapping

$$\varphi\colon M \to \mathrm{End}(C^M)$$

such that

$$\pi_{m'}\varphi(m) = \pi_{m'm}$$

for every $m, m' \in M$, and it is easily checked that φ is a monoid morphism.

The action of L on the morphisms of **C** is defined as follows. Let C and C' be two objects in **C** and $\alpha\colon C \to C'$ be a morphism. Consider $L(C) = (C^M, \varphi)$ and $L(C') = ({C'}^M, \varphi')$. For each $m \in M$, let

$$\pi_m\colon C^M \to C \quad \text{and} \quad \pi'_m\colon {C'}^M \to C'$$

be the projections. Given the family of morphisms

$$\alpha_m = \alpha \circ \pi_m\colon C^M \to C',$$

$L(\alpha)$ is the unique morphism $L(\alpha)\colon C^M \to {C'}^M$ such that

$$\pi'_m L(\alpha) = \alpha_m = \alpha\pi_m$$

for every $m \in M$, so

$$\varphi'(m)L(\alpha) = L(\alpha)\varphi(m)$$

for every $m \in M$, i.e., $L(\alpha)$ is a morphism $L(C) \to L(C')$ in $\mathbf{C}^M$.

Thus the forgetful functor $U\colon \mathbf{C}^M \to \mathbf{C}$ has both a right adjoint and a left adjoint, provided that **C** has M-indexed coproducts and products.

We can apply this construction to the category $\mathbf{C} = \mathrm{Ab}$ of abelian groups, which has arbitrary products and coproducts, and to an arbitrary monoid M. Then the category Ab^M is clearly equivalent to the category $\mathbb{Z}M$-Mod, so that it is easy to adapt Theorem A to this case of the category of all M-objects in Ab.

Notice that the Ω-groups considered in [19, 3.3.6] are exactly the objects of the category $\mathbf{C}^M$, where M is the free monoid on the set Ω and **C** is the category **Grp**.

Remark 7.2 A behaviour similar to that studied in this paper takes place in the setting of Hopf algebras [2, Subsection 4.2]. Let k be a field fixed once and for all. Let $(A, m_A, u_A, \Delta_A, \varepsilon_A, S_A)$ be a Hopf algebra. Recall that a Hopf algebra

$$(M, m_M, u_M, \Delta_M, \varepsilon_M, S_M)$$

is a *left A-module Hopf algebra* [2, Definition 4.2] if

(a) M is a left A-module, i.e., a left module over the algebra (A, m_A, u_A), via

$$A \otimes M \to M, \quad a \otimes x \mapsto a \cdot x.$$

(b) m_M, u_M, Δ_M and ε_M are left A-module morphisms.

Let A-ModH be the category of all left A-module Hopf algebras. Here the morphisms are the mappings that preserve both the left A-module structure and the Hopf algebra structure. For any G-group H, the group algebras kG and kH are Hopf algebras and, extending by k-bilinearity the left scalar multiplication $G \times H \to H$ to a left scalar multiplication $kG \otimes kH \to kH$, the group algebra kH becomes a left kG-module Hopf algebra. Now the category of commutative and cocommutative Hopf algebras is an abelian category ([20, Corollary 4.16], or [17, Theorem 4.3]). More generally, the category of cocommutative Hopf algebras over a field of characteristic zero is semi-abelian [11], and the abelian objects in this semi-abelian category are the commutative and cocommutative Hopf algebras [21]. It would be therefore natural to restrict our attention to commutative and cocommutative Hopf algebras and consider, for a fixed Hopf algebra A, the left A-module Hopf algebras M that are commutative and cocommutative, seeing if a weak form of the Krull-Schmidt Theorem holds in this case. Notice that if M, M' are commutative and cocommutative Hopf algebras, then the sum of two morphisms

$$f, f' \in \mathrm{Hom}(M, M')$$

is given by the convolution product $*$ defined by

$$(f * g)(x) = m_{M'} \circ (f \otimes f')(\Delta_M x)$$

for all $x \in M$. Also notice that monomorphisms in the category of cocommutative Hopf algebras are injective mappings, epimorphisms are surjective mappings and coproducts are tensor products over the base field [20, proof of Theorem 4.4]. For instance, from our example in Section 2, we get n^2 Hopf algebras $kU_{i,j}$, $i, j = 1, \dots, n$, which are left kG-module Hopf algebras, such that

$$kU_{1,1} \otimes kU_{2,2} \otimes \dots \otimes kU_{n,n}$$

$$\simeq kU_{\sigma(1),\tau(1)} \otimes kU_{\sigma(2),\tau(2)} \otimes \ldots \otimes kU_{\sigma(n),\tau(n)}$$

as left kG-module Hopf algebras for every pair of permutations σ, τ of $\{1, 2, \ldots, n\}$. For every $i, j, k, \ell = 1, 2, \ldots, n$, we have that

$$[kU_{i,j}]_m = [kU_{k,\ell}]_m$$

if and only if $i = k$, and $[kU_{i,j}]_e = [kU_{k,\ell}]_e$ if and only if $j = \ell$.

Remark 7.3 There are versions of the Weak Krull-Schmidt Theorem not only for biuniform modules, but also for several other classes of modules, like cyclically presented modules over a local ring [1], or kernels of morphisms between indecomposable injective modules [8]. For the general categorical pattern, see [10], and for a survey about these topics, see [7]. For instance, let us describe the behaviour of kernels of morphisms between indecomposable injective modules.

For a right module A_R over a ring R, let $E(A_R)$ denote the injective envelope of A_R. Two modules A_R and B_R are said to *have the same upper part*, denoted by $[A_R]_u = [B_R]_u$, if there exist a morphism

$$\varphi\colon E(A_R) \to E(B_R)$$

and a morphism

$$\psi\colon E(B_R) \to E(A_R)$$

such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$.

Theorem 7.4 (Weak Krull-Schmidt Theorem [8])
Let

$$\varphi_i\colon E_{i,0} \to E_{i,1} (i = 1, 2, \ldots, n)$$

and

$$\varphi'_j\colon E'_{j,0} \to E'_{j,1} \ (j = 1, 2, \ldots, t)$$

be $n + t$ *non-injective morphisms between indecomposable injective right modules* $E_{i,0}, E_{i,1}, E'_{j,0}, E'_{j,1}$ *over an arbitrary ring* R*. Then the direct sums* $\oplus_{i=0}^{n} \ker \varphi_i$ *and* $\oplus_{j=0}^{t} \ker \varphi'_j$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ, τ *of* $\{1, 2, \ldots, n\}$ *such that*

$$[\ker \varphi_i]_m = [\ker \varphi'_{\sigma(i)}]_m \text{ and } [\ker \varphi_i]_u = [\ker \varphi'_{\tau(i)}]_u$$

for every $i = 1, 2, \ldots, n$.

It is therefore possible to modify the results in this paper substituting biuniform and uniserial modules with these kernels of morphisms between indecomposable injective modules, getting very similar results.

REFERENCES

[1] B. Amini – A. Amini – A. Facchini: "Equivalence of diagonal matrices over local rings", *J. Algebra* 320 (2008), 1288–1310.

[2] M.J. Arroyo Paniagua – A. Facchini: "Category of G-Groups and its Spectral Category", *Comm. Algebra* (2016), to appear; DOI:10.1080/00927872.2016.1222409.

[3] F. Borceux – G. Janelidze – G.M. Kelly: "Internal object actions", *Comment. Math. Univ. Carolin.* 46 (2005), 235–255.

[4] D. Bourn – G. Janelidze: "Protomodularity, descent, and semidirect products", *Theory Appl. Categ.* 4 (2) (1998), 37–46.

[5] A. Facchini: "Krull-Schmidt fails for serial modules", *Trans. Amer. Math. Soc.* 348 (1996), 4561–4575.

[6] A. Facchini: "Module Theory. Endomorphism Rings and Direct Sum Decompositions in some Classes of Modules", *Birkhäuser*, Basel (2010).

[7] A. Facchini: "Direct-sum decompositions of modules with semilocal endomorphism rings", *Bull. Math. Sci.* 2 (2012), 225–279.

[8] A. Facchini – Ş. Ecevit – M. T. Koşan: "Kernels of morphisms between indecomposable injective modules", *Glasgow Math. J.* 52 (2010), 69–82.

[9] A. Facchini – A. Lucchini: "The Krull-Schmidt Theorem holds for biuniform groups", to appear.

[10] A. Facchini – P. Příhoda: "The Krull-Schmidt Theorem in the case two", *Algebr. Represent. Theory* 14 (2011), 545–570.

[11] M. Gran – G. Kadjo – J. Vercruysse: "A torsion theory in the category of cocommutative Hopf algebras", *Appl. Categ. Structures* (2016), published online.

[12] A.V. Jategaonkar: "A counter-example in ring theory and homological algebra", *J. Algebra* 12 (1969), 418–440.

[13] C.E. Linderholm: "A group epimorphism is surjective", *Amer. Math. Monthly* 77 (1970), 176–177.

[14] S. Mac Lane: "Categories for the Working Mathematician", Second Edition, *Springer*, New York (1998).

[15] S. Mantovani – G. Metere: "Normalities and commutators", *J. Algebra* 324 (2010), 2568–2588.

[16] S. Mantovani – G. Metere: "Internal crossed modules and Peiffer condition", *Theory Appl. Categ.* 23 (2010), 113–135.

[17] K. Newman: "A correspondence between bi-ideals and sub-Hopf algebras in cocommutative Hopf algebras", *J. Algebra* 36 (1975), 1–15.

[18] P. Příhoda: "Weak Krull-Schmidt theorem and direct sum decompositions of serial modules of finite Goldie dimension", *J. Algebra* 281 (2004), 332–341.

[19] D.J.S. Robinson: "A Course in the Theory of Groups", Second Edition, *Springer*, New York (1996).

[20] M. Takeuchi: "A correspondence between Hopf ideals and sub-Hopf algebras", *Manuscripta Math.* 7 (1972), 251-270.

[21] C. Vespa – M. Wambst: "On some properties of the category of cocommutative Hopf algebras", to appear.

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