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A Brief History of an Important Classical Theorem

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This note is a by-product of the authors' investigation of relations between the lower and upper central series in a group. There are some famous theorems laid at the foundation of any research in this area. These theorems are so well-known that, by the established tradition, they usually were named in honor of the mathematicians who first formulated and proved them. However, this did not always happened. Our intention here is to talk about one among the fundamental results of infinite group theory which is linked to Issai Schur.

Issai Schur (January 10, 1875 in Mogilev, Belarus — January 10, 1941 in Tel Aviv, Israel) was a very famous mathematician who proved many important and interesting results, first in algebra and number theory. There is a biography sketch of I. Schur wonderfully written by J.J. O'Connor and E.F. Robertson [6]. The authors documented that I. Schur actively and successfully worked in many branches of mathematics, such as for example, finite groups, matrices, algebraic equations (where he gave examples of equations with alternating Galois groups), number theory, divergent series, integral equations, function theory. We cannot add anything important to this, saturated with factual and emotional details, article of O'Connor and Robertson. We just would like to mention that the place of birth of I. Schur, the city of Mogilev in Belarus, was also the birthplace city of another great algebraist *Otto Yu. Schmidt* (September 30, 1891, Mogilev, Belarus — September 7, 1956, Moscow, Russia). There is a project developed by Belarusian algebraists consisting in building up a monument for these two great mathematicians, I. Schur and O. Schmidt, where they will be showing sitting on a bench together. It would be great if this project become a reality.

In many articles and books, the following classical result on infinite groups is also associated with I. Schur.

> (**T**) *If* G *is a group whose center has finite index, then its derived subgroup is finite.*

This result is important and fundamental. It spawned an entire theme, not only in the theory of groups, but also in other areas of algebra. More of its details can be found in the survey paper [4].

As we mentioned above, I. Schur, being an outstanding algebraist, obtained many well-known results in algebra. However, the theorem mentioned above does not belong to him. Nevertheless, in many papers, this theorem is called Schur's theorem, and it is done without accompanying any reference, or with the reference to the paper [7]. We, like many others, also did it automatically. However, there is a natural question: the paper [7] is on finite groups, and the mentioned result is on infinite groups. That is why we decided to read carefully [7]. We have been surprised not to find the above theorem there. In this paper, I. Schur has introduced a concept for a group, which is now called the *Schur multiplicator* or *Schur multiplier* and obtained some properties of this group (only for finite groups!). In the modern terminology, the Schur multiplier $\mathbf{M}(G)$ of a group G is exactly the second cohomology group $H^2(G, \mathbf{U}(C))$. In particular, Schur proved that if G is a finite group, then $[G, G] \cap \zeta(G)$ is an epimorphic image of $\mathbf{M}(G/\zeta(G))$. So we decided to find out how $(\mathbf{T})$ has been named as Schur's theorem.

The first time $(\mathbf{T})$ appeared was ten years after the death of I. Schur in the paper [5] of B.H. Neumann. This latter was a student of Schur and his article was dedicated to Schur's memory, but Neumann did not write anything about Schur's involvement in obtaining this result. At the end of this paper, B.H. Neumann wrote that he received a letter from Reinhold Baer, in which it was mentioned that this result is a corollary of a more general result, which was proved in the paper [1]. In fact, Theorem 3 of this paper proved that if a normal subgroup H of a group G has finite index, then the factor $([G, G] \cap H)/[H, G]$ is also finite. A year later, in the paper [2], Baer brought (**T**) in its standard form, supplying it with a new proof. Baer also wrote nothing about the role of Schur in causing this result. For the first time, the name of I. Schur was associated with (**T**) in the Lectures of Philip Hall [3]. Hall presented (**T**) in a little bit more detailed form, and associated it to I. Schur without accompanying it with any reference. It happened quite often that after a result has been obtained, more proofs, based on different ideas from the original one, and approaches appeared. Thus, for (**T**), we know more than a dozen different proofs. In such an abstract area as mathematics, the author's priority is difficult to trace in a simple and direct way, in contrast to other more popular and less abstract areas of human creativity, such as music, painting, poetry or movies. The authority of Philip Hall was very weighty, so, with his light hand, this theorem, first obtained by B.H. Neumann, was called Schur's theorem. As we noted above, Hall did not provide any specific reference, and the people who followed P. Hall and called (**T**) Schur's theorem needed to link this result with at least one of the works of Schur. So, that is why [7] was misguidedly chosen as a reference, which now traditionally accompanies *Schur's theorem*.

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