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Groups Factorized by Pairwise Permutable Abelian Subgroups of Finite Rank [1]

Bernhard Amberg — Yaroslav P. Sysak

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Dedicated to Hermann Heineken on the occasion of his 80th birthday

Abstract

It is proved that a group which is the product of pairwise permutable abelian subgroups of finite Prüfer rank is hyperabelian with finite Prüfer rank; in the periodic case the Sylow subgroups of such a product are described. Furthermore, if $G = ABC$ is such a non-periodic product with locally cyclic subgroups A, B and C, then the Prüfer rank of G is at most 8. Moreover, G is soluble of derived length at most 4 and has Prüfer rank at most 6, if $A \cap B \cap C = 1$, and G has a torsion subgroup T such that the factor group G/T is locally cyclic and the Sylow p-subgroups of T are of Prüfer rank at most 2 for odd p and at most 6 for $p = 2$, otherwise.

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1 Introduction

Let the group $G = AB$ be the product of two subgroups A and B, i.e. $G = \{ab \mid a \in A, b \in B\}$. Then the subgroups A and B are permutable, i.e. $AB = BA$, and the structure of G is strongly influenced

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by that of the factors A and B. For instance, if A and B are abelian, then G is metabelian by a well-known theorem of N. Itô [1, Theorem 2.1.1] and almost all known results concerning the structure of the group $G = AB$ with infinite abelian subgroups A and B are based on this theorem.

In this connection it is natural to attempt to extend the theorem of Itô to the case of groups of the form $G = A_1A_2 \dots A_n$ with three or more pairwise permutable abelian subgroups A_i, $1 \leqslant i \leqslant n$. However, it turned out that already in the case $n = 3$ such groups can have a very complicated structure and in particular may contain non-abelian free subgroups. Relevant examples were constructed by the second author in [9], Lemma 2. Nevertheless, if the abelian subgroups A_i satisfy some finiteness conditions for instance are finitely generated, Chernikov or minimax, then some meaningful results in this direction were obtained.

In particular, Heineken and Lennox have shown in [5] that a group $G = A_1A_2 \dots A_n$ with pairwise permutable abelian subgroups A_i is polycyclic if the subgroups A_i are finitely generated. Using this and Huppert's theorem [6] about the supersolubility of finite groups factorized by pairwise permutable cyclic subgroups, they derived also that every group factorized by finitely many pairwise permutable cyclic subgroups is supersoluble and abelian-by-finite. Later in [11] Tomkinson proved that the group $G = A_1A_2 \dots A_n$ is soluble minimax if the subgroups A_i are minimax and G is a locally supersoluble Chernikov group if all A_i are locally cyclic Chernikov subgroups. Some other results can be found in [1], Chapter 7.

The aim of the following is to extend these results to groups factorized by abelian subgroups of finite Prüfer rank. In particular, we extend Tomkinson's result about products of abelian minimax groups to products of abelian groups of finite Prüfer rank and consider in detail non-periodic groups factorized by three pairwise permutable locally cyclic subgroups. It should be observed that the proofs of these and similar results are based on the property that non-trivial groups of the form $G = AB$ with abelian subgroups A and B of finite Prüfer rank have a non-trivial normal subgroup contained in one of the factors A or B (see [1], Theorem 7.1.2).

The notation is standard. In particular, if X and Y are subgroups of a group G and n is a positive number, then $\langle X, Y\rangle$ is the subgroup of G generated by X and Y, X^n is the subgroup of G generated by all n-th powers of the elements of X, X^Y denotes the least Y-invariant subgroup of G containing X, i.e. the subgroup of G generated by all conjugates X^y with $y \in Y$, and X_Y is the largest Y-invariant subgroup of G contained in X, i.e. the intersection of all conjugates X^y with $y \in Y$.

2 Preliminary lemmas

First we recall some known results concerning the structure of groups factorized by two locally cyclic torsion-free subgroups (see [8], Theorem).

Proposition 2.1 *Let the group $G = AB$ be the product of two locally cyclic torsion-free subgroups A and B with $A \cap B = 1$. Then, up to transposition of the factors A and B, one of the following statements holds:*

1) $G = A \rtimes B$ *is a semidirect product of A by B and the order of the factor group $B/C_B(A)$ is at most 2;*

2) $G = A \rtimes B$, $C_B(A) = 1$ *and B is cyclic;*

3) $G = (A_0 \times B)\langle a \rangle$, $A = A_0\langle a \rangle$, $a^2 \in A_0$ *and there exists a monomorphism $\phi : B \to A_0$ such that $b^a = b^{-1}b^{\phi}$ for all $b \in B$;*

4) $G = (A_0 \times B_0)\langle a \rangle\langle b \rangle$, $A = A_0\langle a \rangle$ *with* $a^2 \in A_0$ *and* $B = B_0\langle b \rangle$ *with* $b^2 \in B_0$, *where* $ba = a^{-1}b^{-1}$, $a_0^b = a_0^{-1}$ *and* $b_0^a = b_0^{-1}$ *for all* $a_0 \in A_0$ *and all* $b_0 \in B_0$.

As a direct consequence of this proposition we have the following.

Corollary 2.2 *Let the group $G = AB$ be the product of locally cyclic torsion-free subgroups A and B. If $A \cap B = 1$ and G does not satisfy statement 2) of Proposition 2.1, then either all subgroups of A^2 and B^2 are normal in G or, up to transposition of the factors A and B, the subgroup A^2 is central in G and $B_G = 1$.*

The next lemma combines Lemmas 6 and 7 of [8].

Lemma 2.3 *Let the p-group $G = AB$ be the product of locally cyclic subgroups A and B. If there exists a normal subgroup T of G such that $G = T \rtimes A = AB$ and $A \cap B = B \cap T = 1$, then either G is abelian, T is locally cyclic and there exists a monomorphism $\phi : B \to A$ such that $T = \{b^{-1}b^{\phi} \mid b \in B\}$ or G is finite non-abelian with $A = \langle a \rangle$ of order p^m and one of the following statements holds:*

1) $T = \langle x \rangle$ *is cyclic of order* p^n, $B = \langle a^{p^{m-n}} x \rangle$ *and* $x^a = x^{1+p^k}$ *with* $1 \leqslant k < n$;

2) $p = 2$, $T = \langle x, y \mid x^{2^{n-1}} = y^2 = (xy)^2 = 1 \rangle$ *is a dihedral group (in particular, a four-group),* $B = \langle ay \rangle$, $x^a = x^{1+2^k}$ *with* $2 \leqslant k < n$ *and* $y^a = xy$;

3) $p = 2$, $T = \langle x, y \mid x^{2^{n-1}} = 1, x^{2^{n-2}} = y^2 = (xy)^2 = 1 \rangle$ *is a generalized quaternion group,* $B = \langle ay \rangle$, $x^a = x^{1+2^k}$ *with* $2 \leqslant k < n$ *and* $y^a = xy^{-1}$;

We recall also a well-known consequence of Schur's theorem on the finiteness of the derived subgroup of a group that is finite over its center (see [7], Corollary to Theorem 4.12).

Lemma 2.4 *If a group* G *contains a central subgroup* Z *such that the factor group* G/Z *is locally finite, then the derived subgroup of* G *is locally finite.*

3 Products of abelian groups of finite Prüfer rank

The following result is a generalization of the theorem of Tomkinson about groups factorized by abelian minimax subgroups mentioned above.

Theorem 3.1 *Let the group* $G = A_1 A_2 \dots A_n$ *be the product of finitely many pairwise permutable abelian subgroups* $A_1, A_2, \dots, A_n$ *each of which has finite Prüfer rank. Then* G *is a hyperabelian group of finite Prüfer rank.*

Proof — Indeed, if $n = 2$ then G is metabelian by Itô's theorem and if $G \neq 1$ there exists a non-trivial normal subgroup N of G contained in A_1 or A_2 (see [1], Theorem 7.1.2). In the case $n \geqslant 3$ without loss of generality we may assume that A_2 contains a non-trivial cyclic subgroup N which is normal in the subgroup $A_1 A_2$. Then

$$N^G = N^{A_3 \dots A_n} \leqslant A_2 \dots A_n$$

and so N^G is a hyperabelian normal subgroup of finite Prüfer rank by induction on n. Therefore N^G has a non-trivial abelian normal subgroup M which is either finite or torsion-free. If M is finite, then the normal closure M^G is periodic and hypercentral, so that its center Z

is a non-trivial abelian normal subgroup of G. In the other case, M is torsion-free of rank r contained in the r-th hypercenter of N^G by a theorem of Charin (see [7], Lemma 6.37). Therefore the center of N^G may be taken as Z. Passing now to the factor group G/Z and repeating the same arguments, we obtain that G is a hyperabelian group. Since the subgroup $A_2 \dots A_n$ has finite Prüfer rank by induction on n, the Prüfer rank of $G = A_1(A_2 \dots A_n)$ is finite by [1], Theorem 4.3.5. □

As an application, we describe the Sylow subgroups of periodic groups factorized by finitely many pairwise permutable abelian subgroups of finite Prüfer rank. It should be noted that the groups themselves need not be soluble (see [1], Proposition 7.6.3, or [9], Corollary 1).

Lemma 3.2 *Let the periodic group* $G = A_1A_2 \dots A_n$ *be the product of finitely many pairwise permutable abelian subgroups* $A_1, A_2, \dots, A_n$ *each of which has finite Prüfer rank. If* P_i *is the Sylow* p*-subgroup of* A_i *and* Q_i *is the* p*-complement to* P_i *in* A_i *for each* $1 \leqslant i \leqslant n$ *and a prime* p*, then* $P_1P_2 \dots P_n$ *is a Sylow* p*-subgroup of* G *and* $Q_1Q_2 \dots Q_n$ *is a* p*-complement to* $P_1P_2 \dots P_n$ *in* G*.*

Proof — If $n = 2$, the lemma is a special case of [4], Proposition 2.6. Therefore $P_iP_j = P_jP_i$ and $Q_iQ_j = Q_jQ_i$ for every $1 \leqslant i \neq j \leqslant n$. Thus $P = P_1P_2 \dots P_n$ and $Q = Q_1Q_2 \dots Q_n$ are subgroups of G. Moreover, since G is a hyperabelian group by Lemma 3.1, P is a p-subgroup and Q is a p'-subgroup by [1], Corollary 3.2.7. By induction on n, we may assume that $R = P_1P_2 \dots P_{n-1}$ is a Sylow p-subgroup of $A_1A_2 \dots A_{n-1}$ and $S = Q_1Q_2 \dots Q_{n-1}$ is a p-complement to R in $A_1A_2 \dots A_{n-1}$. Then $A_1A_2 \dots A_{n-1} = RS$ and so

$$G = RSA_n = R(SQ_n)P_n = RQP_n.$$

Therefore, if G_p is a Sylow p-subgroup of G containing P, then

$$G_p = RQP_n \cap G_p = R(Q \cap G_p)P_n = RP_n = P.$$

Similarly, if $G_{p'}$ is a Sylow p'-subgroup of G containing Q, then

$$G_{p'} = SRA_n \cap G_{p'} = SPQ_n \cap G_{p'} = S(P \cap G_{p'})Q_n = SQ_n = Q.$$

Since G has finite Prüfer rank, we have $G = PQ$ by Corollary 3.1.6 of [3]. □

For groups $G = ABC$ factorized by three pairwise permutable abelian subgroups A, B and C of rank 1 more can be said. The next two lemmas consider in more detail the structure of the Sylow subgroups of some periodic groups of this form which will be needed later.

Lemma 3.3 *Let the* p*-group* $G = ABC$ *be the product of three pairwise permutable locally cyclic subgroups* A, B *and* C. *If there exists a normal subgroup* T *of* G *such that the factor group* G/T *is locally cyclic and* $A \cap T = B \cap T = C \cap T = 1$, *then up to a permutation of the factors* A, B *and* C *we have* $T = PQ$ *with* $P = AB \cap T$ *and* $Q = AC \cap T$.

Proof — Since G/T is a locally cyclic p-group, without loss of generality we may assume that $AT \geqslant BT \geqslant CT$. Then $G = ABC = AT$, so that

$$AP = A(AB \cap T) = AB \text{ and } AR = A(AC \cap T) = AC.$$

This implies

$$\begin{aligned} A(PQ) = (AP)Q = (AB)Q = (BA)Q = B(AQ) \\ = B(AC) = ABC = G \end{aligned}$$

and hence $T = (A \cap T)(PQ) = PQ$, as desired. □

Lemma 3.4 *Let the periodic group* $G = ABC$ *be the product of three pairwise permutable locally cyclic subgroups* A, B *and* C. *If* G *contains a normal subgroup* T *such that the factor group* G/T *is locally cyclic and* $A \cap T = B \cap T = C \cap T = 1$, *then the Sylow* p*-subgroups of* T *are of Prüfer rank at most* 2 *for odd* p *and at most* 6 *for* $p = 2$.

Proof — For each prime p let A_p, B_p and C_p denote the Sylow p-subgroups of A, B and C, respectively. These subgroups are pairwise permutable and $G_p = A_pB_pC_p$ is a Sylow p-subgroup of G by Lemma 3.2. Since G is a hyperabelian group of finite Prüfer rank, the intersection $T_p = G_p \cap T$ is a Sylow p-subgroup of T which is normal in G_p. Clearly the factor group G_p/T_p is locally cyclic, so that G_p satisfies the hypothesis of Lemma 3.3. Therefore among the intersections $A_pB_p \cap T_p$, $A_pC_p \cap T_p$ and $B_pC_p \cap T_p$ there are two P and Q such that $T_p = PQ$. If p is odd, then P and Q are locally cyclic by Lemma 2.3 and so the Prüfer rank of T_p does not exceed 2. In the other case by the same lemma each of the 2-subgroups P and Q is either locally cyclic or dihedral or generalized quaternion. Thus the Prüfer rank of the subgroup $T_p = PQ$ does not exceed 6, as follows from [2], Lemma 5.6, and [10], Lemma 3.3. □

4 Non-periodic groups factorized by three locally cyclic subgroups

We describe first the structure of groups factorized by two abelian subgroups one of which is periodic and the other is locally cyclic torsion-free.

Lemma 4.1 *Let* $G = AB$ *be the product of a periodic subgroup* A *and a locally cyclic subgroup* B*. If the factor group* A/A_G *is abelian and* B *is torsion-free, then* A/A_G *is of order at most* 2*.*

Proof — Passing to the factor G/A_G, we may assume that the subgroup A is abelian. Then the derived subgroup G' of G is abelian by Itô's theorem. Furthermore, if T is a periodic normal subgroup of G, then $AT = A(B \cap AT) = A$ and so $T = 1$. Therefore G' is torsion-free and so the intersection $B \cap G'$ is contained in the center of the subgroup BG'. Clearly $BG' = A_1B$ with $A_1 = A \cap BG'$. If $A_1 \neq 1$, then $B \cap G' \neq 1$ and so the factor group $BG'/B \cap G'$ is periodic. But then the derived subgroup of BG' is periodic by Lemma 2.4 and so trivial. Therefore the subgroup BG' is abelian and torsion-free, so that $A_1 = 1$. Hence $G' \leqslant B$ and thus A induces on B a periodic group of automorphims. Since the automorphism group of any locally cyclic torsion-free group is embedded in the multiplicative group of rational numbers, the order of A does not exceed 2, as desired. □

In the next two lemmas we consider a group G factorized by three pairwise permutable locally cyclic torsion-free subgroups and determine conditions under which one of these subgroups contains a nontrivial normal subgroup of G.

Lemma 4.2 *Let* $G = ABC$ *be the product of three pairwise permutable locally cyclic torsion-free subgroups* A, B *and* C*. If* $A_G = B_G = C_G = 1$*, then none of the subgroups* AB, AC *and* BC *satisfies statement* 2) *of Proposition 2.1.*

Proof — Suppose the contrary and assume that the subgroup AB satisfies this statement. Then $AB = A \rtimes B$, $C_B(A) = 1$ and the subgroup B is cyclic. Therefore A is non-cyclic and $A_C = 1$, because otherwise

$$(A_C)^G = (A_C)^{ACB} = (A_C)^B \leqslant A,$$

contrary to the hypothesis of the lemma. Thus the subgroup C^2 is central in AC by Corollary 2.2 and so $AC = (C^2 \times A)\langle c\rangle$ with $c \in C$ and $a^c = a^{-1}a^{\phi}$ for each $a \in A$ and some monomorphism $\phi : A \to C^2$ by Proposition 2.1. Hence the subgroup C is non-cyclic and

$$C^2_G = C^2_{ACB} = C^2_B$$

is contained in C and normal in G. Thus $C^2_B = 1$ and so $C_B = 1$. For the above reason $BC = (B^2 \times C)\langle b\rangle$ with $b \in B$ and $c^b = c^{-1}c^{\psi}$ for each $c \in C$ and some monomorphism $\psi : C \to B^2$. Since the subgroup C is non-cyclic and B is cyclic, this gives a contradiction and completes the proof. □

Lemma 4.3 *Let $G = ABC$ be the product of three pairwise permutable locally cyclic torsion-free subgroups A, B and C. If none of these subgroups contains a non-trivial normal subgroup of G, then*

$$A \cap B = A \cap C = B \cap C = 1$$

and the subgroups A^2, B^2 and C^2 generate a maximal abelian normal subgroup of G.

Proof — If $A_G = B_G = C_G = 1$, then $A \cap B \cap C = 1$ and so one of the intersections $A \cap B$, $A \cap C$ or $B \cap C$ must be trivial. Clearly, up to a permutation of the factors A, B and C, we may assume $A \cap B = 1$. In accordance with Lemma 4.2 and Corollary 2.2, the subgroup AB satisfies one of the following two conditions: either all subgroups of A^2 and B^2 are normal in AB or, up to transposition of the factors A and B, the subgroup A^2 is central in AB and $B_A = 1$.

If in the first case one of the subgroups A^2_C or B^2_C is normal in AC or BC, respectively, then it is normal in G, contrary to the assumption. Therefore $A_C = B_C = 1$ and so the subgroup C^2 must be normal in both AC and BC by Corollary 2.2. But then C^2 will be normal in G, contrary to the condition $C_G = 1$.

Consider now the case when the subgroup AB satisfies the second condition, i.e. the subgroup A^2 is central in AB and $B_A = 1$. Then the subgroup A^2_C is central in AB and so normal in G. As $A_G = 1$, it follows that $A_C = 1$ and in particular $A \cap C = 1$. Therefore C^2 is central in AC by Corollary 2.2 and hence the subgroup C^2_B is normal in G. Thus the condition $C_G = 1$ implies $C_B = 1$ and in particular $B \cap C = 1$. As above, then B^2 is central in BC and this means that the subgroups A^2, B^2 and C^2 are pairwise centralized and disjoint.

Since each of the subgroups AB, AC and BC satisfies statement 3) of Proposition 2.1, it follows that the subgroup $A^2 \times B^2 \times C^2$ is a maximal abelian normal subgroup of G, as desired. □

The following two results describe the general structure of non-periodic groups factorized by three pairwise permutable locally cyclic subgroups. The first theorem concerns the case when the intersection of these subgroups is trivial and the second theorem deals with the situation when this is not so.

Theorem 4.4 *Let the non-periodic group* $G = ABC$ *be the product of three pairwise permutable locally cyclic subgroups* A, B *and* C. *If* $A \cap B \cap C = 1$, *then* G *is a soluble group of derived length at most* 4 *and of Prüfer rank at most* 6.

Proof — Clearly in what follows we may assume that the subgroup C is torsion-free. The proof is divided into several steps.

1) Consider first the case when the subgroup A is periodic and B is torsion-free. Then the factor groups A/A_C and A/A_B are of order at most 2 by Lemma 4.1 and so, up to a transposition, $A_B \leqslant A_C$. Thus the subgroup A_B is C-invariant which means that $A_B = A_G$ and $A_G BC$ is a subgroup of index at most 2 in G. Since the subgroup BC is metabelian with Prüfer rank at most 3, the subgroup $A_G BC$ is soluble with derived length at most 3 and has Prüfer rank at most 4. Therefore G is soluble with derived length at most 4 and with Prüfer rank at most 5.

2) Next let the subgroups A and B be periodic. Then the factor groups A/A_C and B/B_C are of order at most 2 by Lemma 4.1, so that the subgroup $\langle A_C, B_C\rangle$ is of index at most 4 in AB. Since

$$\langle A_C, B_C\rangle^G = \langle A_C, B_C\rangle^{CAB} = \langle A_C, B_C\rangle^{AB} \leqslant AB,$$

the normal closure $\langle A_C, B_C\rangle^G$ is of index at most 4 in AB. Therefore the factor group $AB/\langle A_C, B_C\rangle^G$ is abelian and so the factor group $(AB)/(AB)_G$ is of order at most 2 by Lemma 4.1. As above, this implies that G is soluble of derived length at most 4 and its Prüfer rank does not exceed 5.

3) We turn now to the case when the subgroups A and B are torsion-free and $A \cap B \neq 1$. Then $A \cap C = B \cap C = 1$ and one of the subgroups A, B or C contains a non-trivial normal subgroup of G by Lemma 4.3. But if one of the subgroups A_G or B_G is non-trivial and, for instance, $A_G \neq 1$, then the images of A and B in the

factor group G/A_G are periodic and the image of C modulo A_G is torsion-free. Therefore it follows from step (2) that the normal subgroup $(AB)_G$ has index at most 2 in AB and so G is soluble of derived length at most 4. On the other hand, if $C_G \neq 1$, then the image of C in the factor group G/C_G is periodic and the images of A and B modulo C_G are torsion-free. It was shown by step (1), that C/C_G is of order at most 2 and so the subgroup ABC_G is of index at most 2 in G. Thus the derived length of G does not exceed 4 and the Prüfer rank of G is at most 5, as in (1).

4) Finally, if A and B are torsion-free and $A \cap B = A \cap C = B \cap C = 1$, then either at least one of the subgroups A, B or C contains a non-trivial normal subgroup of G or $A^2 \times B^2 \times C^2$ is an abelian normal subgroup of G. If for instance $A_G \neq 1$ in the first case, then the factor group A/A_G is periodic and so the factor groups A/A_C and A/A_B are of order at most 2 by Lemma 4.1. Repeating arguments from step 1), we conclude that the group G is soluble with derived length at most 4 and with Prüfer rank at most 5. In the second case the factor group $G/(A^2 \times B^2 \times C^2)$ is elementary abelian of order 8 and so G is a metabelian group of Prüfer rank at most 6. □

Theorem 4.5 *Let the group* $G = ABC$ *be the product of three pairwise permutable locally cyclic torsion-free subgroups* A, B *and* C. *If*

$$A \cap B \cap C \neq 1,$$

then G *has a torsion subgroup* T *such that the factor group* G/T *is locally cyclic and the Sylow* p*-subgroups of* T *are of Prüfer rank at most* 2 *for odd* p *and at most* 6 *for* $p = 2$. *In particular,* G *is a hyperabelian group of Prüfer rank at most* 8.

Proof — Clearly the intersection $D = A \cap B \cap C$ is a central subgroup of G and the factor group G/D is the product of pairwise permutable periodic locally cyclic subgroups A/D, B/D and C/D. Hence G/D is a periodic hyperabelian group by Theorem 3.1. Since D is locally cyclic and torsion-free, the group G has torsion-free rank 1 and the derived subgroup of G is locally finite by Lemma 2.4. Therefore the set of all elements of finite order of G is the torsion subgroup T and the factor group G/T is torsion-free with torsion-free rank 1, so that G/T is locally cyclic.

It is easy to see that the periodic factor group G/D satisfies the hypothesis of Lemma 3.4. Indeed, if bars are used for denoting homomorphic images in G/D, then $\overline{T}$ is a normal subgroup of $\overline{G}$ and the factor group $\overline{G}/\overline{T}$ is locally cyclic, because it is isomorphic with G/T. Furthermore,

$$\overline{A} \cap \overline{T} = \overline{B} \cap \overline{T} = \overline{C} \cap \overline{T} = 1,$$

since $A \cap T = B \cap T = C \cap T = 1$. Finally, $\overline{G} = \overline{ABC}$ is the product of pairwise permutable locally cyclic subgroups, as was mentioned above. Thus it follows from Lemma 3.4 that the Sylow p-subgroups of $\overline{T}$ are of rank at most 2 for odd p and at most 6 for $p = 2$. Since $\overline{T}$ is isomorphic to T, this implies that every finite subgroup F of T is soluble and the Sylow subgroups of F are generated by at most 6 elements. Therefore F is at most 7-generated by a theorem of Kovacs (see [1], Theorem 4.2.1), so that the Prüfer rank of T does not exceed 7. As G/T is locally cyclic, the Prüfer rank of G is at most 8.□

In conclusion, we mention two open questions about a group $G = ABC$ factorized by three pairwise permutable locally cyclic subgroups A, B and C.

Question 1. *Is* G *locally supersoluble, provided that the group* G *is periodic*?

Question 2. *Is* G *soluble if the subgroups* A, B *and* C *are torsion-free and* $A \cap B \cap C \neq 1$?

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Bernhard Amberg
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
55099 Mainz (Germany)
e-mail: amberg@uni-mainz.de

Yaroslav P. Sysak
Institute of Mathematics
Ukrainian National Academy of Sciences
01601 Kiev (Ukraine)
e-mail: sysak@imath.kiev.ua