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A Note on Formations with the Shemetkov Property

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Abstract

In this paper we describe all hereditary formations with the Shemetkov property $\mathfrak{F}$ such that the intersection of all $\mathfrak{F}$-maximal subgroups coincides with the $\mathfrak{F}$-hypercenter of a finite group.

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1 Introduction and the main results

All groups considered here will be finite. The notation and terminology agree with the books [2],[5]. We refer the reader to these books for the results on formations. Let $\mathfrak{X}$ be a class of groups. Recall that a subgroup U of G is called $\mathfrak{X}$-*maximal* in G provided that: (a) $U \in \mathfrak{X}$, and (b) if $U \leqslant V \leqslant G$ and $V \in \mathfrak{X}$, then $U = V$ (see [5, p.288]); the symbol $\mathrm{Int}_{\mathfrak{X}}(G)$ denotes the intersection of all $\mathfrak{X}$-maximal subgroups of G (see [12]). A chief factor H/K of G is called $\mathfrak{X}$-*central* in G provided $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{X}$ (see [10, pp.127–128] or [6, p.6]). The symbol $Z_{\mathfrak{X}}(G)$ denotes the $\mathfrak{X}$-*hypercenter* of G, that is, the largest normal subgroup of G such that every chief factor H/K of G below it is $\mathfrak{X}$-central. If $\mathfrak{X}$ is a formation, then the $\mathfrak{X}$-hypercenter exists in every group by [10, §14, Lemma 14.1].

Note that the intersection of all maximal abelian subgroups of G is the center of G. According to R. Baer [1], the intersection of all maximal nilpotent subgroups of G coincides with the hypercenter of G. It was shown that the intersection of all maximal supersoluble subgroups of G does not necessary coincide with the supersoluble hypercenter of G (see [12, Example 5.17]). L.A. Shemetkov posed the following question on Gomel Algebraic seminar in 1995: «*For what non-empty (normally) hereditary local (Baer-local) formations* $\mathfrak{X}$ *do the equality* $\mathrm{Int}_{\mathfrak{X}}(G) = Z_{\mathfrak{X}}(G)$ *hold for every group* G?»

The solution to this question for hereditary local formations was obtained by A.N. Skiba in [12] (for the soluble case, see also J.C. Beidleman and H. Heineken [4]) and for the class of all quasi-$\mathfrak{F}$-groups, where $\mathfrak{F}$ is a hereditary local formation, was given in [9]. Note that the methods of [4, 12] are not very useful for non-local or non-hereditary formations. Here we prove the following result.

Theorem. *Let* $\mathfrak{F} \neq (1)$ *be a formation with the Shemetkov property. Assume that* $Z_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ *holds for every group* G. *Then there is a partition* σ *of* $\mathbb{P}$ *such that* $\mathfrak{F}^S$ *is the class of all* σ*-nilpotent groups.*

Corollary. *Let* $\mathfrak{F} \neq (1)$ *be a hereditary formation with the Shemetkov property. Then* $Z_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ *holds for every group* G *if and only if* $\mathfrak{F}$ *is the class of all* σ*-nilpotent groups for some partition* σ *of* $\mathbb{P}$.

It is well known that the class $\mathfrak{F}^S$ of all groups whose all subgroups belong to a formation $\mathfrak{F}$ is a formation. Recall [2, p. 268] that a *Schmidt group* is a non-nilpotent group whose all proper subgroups are nilpotent and a formation $\mathfrak{F}$ has the *Shemetkov property* if all $\mathfrak{F}$-critical groups are either Schmidt groups or groups of prime order (for results about these formations see [2, Chapter 6.4]). There are examples of hereditary non-local formations with the Shemetkov property [3, 7, 14]. According to [7] every hereditary formation with the Shemetkov property is Baer-local [5, Chapter IV, Definition 4.9]. In the universe of all soluble groups this result was earlier obtained by A.N. Skiba [11].

Let $\sigma = \{\pi_i \mid i \in I\}$ be a *partition* of the set of all primes $\mathbb{P}$, i.e. $\cup_{i\in I}\pi_i = \mathbb{P}$ and $\pi_i \cap \pi_j = \emptyset$ for $i \neq j$. Note that from [2, Theorem 6.4.14] it follows that the class of groups all whose Hall π_i-subgroups are normal for all $i \in I$ is a hereditary formation with the Shemetkov property. Groups in this class are called σ*-nilpotent* (see [13]).

Proposition. *Let $\mathfrak{F}$ be a formation such that $\mathfrak{F} \cap \mathfrak{S} = \mathfrak{N}$. Then $\mathfrak{F}^S = \mathfrak{N}$. In particular $\mathfrak{F}$ has the Shemetkov property.*

It follows from the above result that the class $\mathfrak{N}^*$ of all quasinilpotent groups and the class $\mathfrak{N}_{ca}$ of all groups whose chief factors are either central or simple non-abelian are formations with the Shemetkov property. It was shown in [9] that

$$\mathrm{Int}_{\mathfrak{N}^*}(G) = Z_{\mathfrak{N}^*}(G)$$

holds for every group G and there is a group H with

$$\mathrm{Int}_{\mathfrak{N}_{ca}}(H) \neq Z_{\mathfrak{N}_{ca}}(H).$$

Thus the converse of Theorem is false.

2 Proof of the main results

A.F. Vasil'ev and the author [15] developed a method for studying formations with the Shemetkov property. Recall [15] that a Schmidt (p, q)-group is a Schmidt group with a normal Sylow p-subgroup. An N-*critical graph* $\Gamma_{Nc}(G)$ of a group G [15, Definition 1.3] is a directed graph on the vertex set $\pi(G)$ of all prime divisors of $|G|$ and (p, q) is an edge of $\Gamma_{Nc}(G)$ iff G has a Schmidt (p, q)-subgroup. An N-*critical graph* $\Gamma_{Nc}(\mathfrak{X})$ of a class of groups $\mathfrak{X}$ [15, Definition 3.1] is a directed graph on the vertex set

$$\pi(\mathfrak{X}) = \bigcup_{G \in \mathfrak{X}} \pi(G)$$

such that $\Gamma_{Nc}(\mathfrak{X}) = \cup_{G \in \mathfrak{X}} \Gamma_{Nc}(G)$.

Proof of Theorem — Let prove that if $Z_{\mathfrak{F}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ holds for every group G, then there is a partition $\sigma = \{\pi_i \mid i \in I\}$ of the set of all primes $\mathbb{P}$ such that $\mathfrak{F}^S$ is the class $\mathfrak{N}_\sigma$ of all σ-nilpotent groups.

1) $\mathfrak{F}^S \neq (1)$.

Assume that $\mathfrak{F}^S = (1)$. It means that every abelian simple group does not belong $\mathfrak{F}$. Note that the quotient group of some group by its maximal normal subgroup is a simple group. Since $\mathfrak{F} \neq (1)$ is a formation, a non-abelian simple group G belongs $\mathfrak{F}$. Let p be a prime, $H = G \wr Z_p$ be a regular wreath product of G and the cyclic group Z_p of order p and N be the base of H. Since G is a simple non-abelian group, N is a minimal normal subgroup of H. Let K be an $\mathfrak{F}$-maximal subgroup of H. Then $KN/N \simeq K/(K \cap N) \in \mathfrak{F}$ is a subgroup of $H/N \simeq Z_p \notin \mathfrak{F}$. Hence $K/(K \cap N) \simeq 1$. It means that $K \leqslant N$. Therefore N is the unique $\mathfrak{F}$-maximal subgroup of H. So $N = Z_{\mathfrak{F}}(H)$. Now $N \rtimes (H/C_H(N)) \simeq N \rtimes H \in \mathfrak{F}$. So $H \in \mathfrak{F}$. Thus $H/N \simeq Z_p \in \mathfrak{F}$, the contradiction.

2) $\pi(\mathfrak{F}^S) = \mathbb{P}$.

Assume that there is a prime $q \notin \pi(\mathfrak{F}^S)$. It means that $Z_q \notin \mathfrak{F}$. Since $\mathfrak{F}^S \neq (1)$ by 1), there is a prime p with $Z_p \in \mathfrak{F}^S \subseteq \mathfrak{F}$. There exists a simple $\mathbb{F}_p Z_q$-module V which is faithful for Z_q by [5, Chapter B, Theorem 10.3]. Let $G = G(p,q) = V \rtimes Z_q$. Now V is a normal Sylow p-subgroup of G. Note that V is the unique $\mathfrak{F}$-maximal subgroup of G. Hence $V = Z_{\mathfrak{F}}(G)$ by our assumption. Since V is a chief factor of G, $G \simeq V \rtimes (G/C_G(V)) \in \mathfrak{F}$. Since $\mathfrak{F}$ is a formation, it follows that $G/V \simeq Z_q \in \mathfrak{F}$, a contradiction.

3) *If* $(p,q) \in \Gamma_{Nc}(\mathfrak{F}^S)$, *then all Schmidt* (p,q)-*groups belong* $\mathfrak{F}$.

Recall that $\mathfrak{F}^S$ is a formation such that the classes of all $\mathfrak{F}$-critical and $\mathfrak{F}^S$-critical groups coincides. Hence $\mathfrak{F}^S$ is a hereditary formation with the Shemetkov property. Now 3) directly follows from Lemma 4 of [11].

4) *Let* $r \neq p \neq q$ *be primes. If* $(r,p) \in \Gamma_{Nc}(\mathfrak{F}^S)$ *and* $(r,q) \in \Gamma_{Nc}(\mathfrak{F}^S)$ *or* $r = q$, *then* $(p,q) \in \Gamma_{Nc}(\mathfrak{F}^S)$.

Assume that $r \neq p \neq q$ are primes, (r,p) belongs to $\Gamma_{Nc}(\mathfrak{F}^S)$, $(p,q) \notin \Gamma_{Nc}(\mathfrak{F}^S)$ and $(r,q) \in \Gamma_{Nc}(\mathfrak{F}^S)$ or $r = q$. So $G(p,q)$ is an $\mathfrak{F}^S$-critical group by 2) and our assumption. So $G(p,q) \notin \mathfrak{F}$.

There exists a simple $\mathbb{F}_r G(p,q)$-module T which is faithful for $G(p,q)$ by [5, Chapter B, Theorem 10.3]. Let

$$H = H(r,p,q) = T \rtimes G(p,q).$$

From $G(p,q) \notin \mathfrak{F}$ it follows that if K is a $\mathfrak{F}$-maximal subgroup of H, then $KT < H$. Let show that every maximal subgroup M of H which contains T belongs to $\mathfrak{F}$. Note that T is a normal Sylow r-subgroup of M and M/T is isomorphic to a maximal subgroup of $G(p,q)$. So it is either a direct product V of groups of order p or a group Z_q of order q.

Assume that $M \notin \mathfrak{F}$. Then it contains an $\mathfrak{F}$-critical subgroup. Since T is normal in M and $\mathfrak{F}$ is a formation with the Shemetkov property, it can be Z_n for $n \in \{r, p, q\}$ or a Schmidt (r, n)-group for $n \in \{p, q\}$. From $\pi(\mathfrak{F}^S) = \mathbb{P}$ by 2) it follows that the first case is impossible. According to our initial assumption, 1) and 3) all Schmidt (r, p)-groups belong to $\mathfrak{F}$ and either all Schmidt (r, q)-groups belong $\mathfrak{F}$ or $r = q$. Note that Schmidt (r, q)-groups exist only for $r \neq q$. It means that the second case is impossible too, a contradiction. Thus $M \in \mathfrak{F}$.

Therefore the sets of all $\mathfrak{F}$-maximal subgroups of H and maximal subgroups of H which contain T coincide. Thus $T \leqslant \mathrm{Int}_{\mathfrak{F}}(H)$. It follows from $Z_{\mathfrak{F}}(H) = \mathrm{Int}_{\mathfrak{F}}(H)$ that $T \leqslant Z_{\mathfrak{F}}(H)$.

Note that T is a chief factor of H, so that

$$T \rtimes (H/C_H(T)) \simeq T \rtimes G(p, q) \in \mathfrak{F}.$$

It means that $G(p, q) \in \mathfrak{F}$. Since $G(p, q)$ is a Schmidt (p, q)-group whose proper subgroups belong $\mathfrak{F}^S$, $G(p, q) \in \mathfrak{F}^S$ and $(p, q) \in \Gamma_{Nc}(\mathfrak{F}^S)$, the contradiction.

5) *There is a partition* $\sigma = \{\pi_i \mid i \in I\}$ *of* $\mathbb{P}$ *such that* $\Gamma_{Nc}(\mathfrak{F}^S)$ *is the disjoint union of complete graphs* Γ_i *where* $V(\Gamma_i) = \pi_i$.

Let $\sim$ be a relation on the set of all primes $\mathbb{P}$ such that $p \sim p$ and $p \sim q$ if $(p, q) \in \Gamma_{Nc}(\mathfrak{F}^S)$ for $p \neq q$. It is clear that $\sim$ is reflexive. From 4) for $r = q$ it follows that if $(q, p) \in \Gamma_{Nc}(\mathfrak{F}^S)$, then $(p, q) \in \Gamma_{Nc}(\mathfrak{F}^S)$. Hence $\sim$ is symmetric. From this and 4) it follows that if $(p, r), (r, q) \in \Gamma_{Nc}(\mathfrak{F}^S)$, then $(p, q) \in \Gamma_{Nc}(\mathfrak{F}^S)$. So $\sim$ is transitive. Thus $\sim$ is an equivalence relation. Let π_i be the i-th equivalence class under $\sim$. Then $\sigma = \{\pi_i \mid i \in I\}$ is a partition of $\mathbb{P}$. Let Γ_i be an induced subgraph of $\Gamma_{Nc}(\mathfrak{F}^S)$ on the vertex set π_i. It is clear that Γ_i is a complete directed graph on π_i.

6) $\mathfrak{F}^S = \mathfrak{N}_\sigma$ *is the class of all* σ*-nilpotent groups.*

From 2), 3) and 5) it follows that all Schmidt π_i-groups and cyclic groups of order $p \in \pi_i$ belong $\mathfrak{F}^S$ for all $i \in I$. Since $\mathfrak{F}^S$ is a formation with the Shemetkov property, we see that the class $\mathfrak{G}_{\pi_i}$ of all π_i-groups is a subset of $\mathfrak{F}^S$ for all $i \in I$. Note that every σ-nilpotent group G is a direct product of all its Hall π_i-subgroups for all i in $\{j \mid \pi_j \cap \pi(G) \neq \emptyset\}$. Since $\mathfrak{F}^S$ is a formation, $\mathfrak{N}_\sigma \subseteq \mathfrak{F}^S$.

Let $\mathfrak{X}$ be a class of groups. According to [15, Theorem 5.4] if $\Gamma_{Nc}(\mathfrak{X})$ is a disjoint union of graphs Γ_i where $V(\Gamma_i) = \pi_i$, then every group in $\mathfrak{X}$ has a normal Hall π_i-subgroup. It means that $\mathfrak{F}^S \subseteq \mathfrak{N}_\sigma$. Thus $\mathfrak{F}^S = \mathfrak{N}_\sigma$. □

Proof of Corollary — From Theorem it follows that there is a partition $\sigma = \{\pi_i \mid i \in I\}$ of $\mathbb{P}$ such that $\mathfrak{F} = \mathfrak{F}^S = \mathfrak{N}_\sigma$.

Assume that $\mathfrak{F} = \mathfrak{N}_\sigma$. Now $Z_\mathfrak{F}(G) = \mathrm{Int}_\mathfrak{F}(G)$ holds for every group G by [12, Theorem A and Proposition 4.2] or [8, Corollary 1].□

Proof of Proposition — From $\mathfrak{F} \cap \mathfrak{S} = \mathfrak{N}$ is a hereditary formation it follows that $\mathfrak{N} \subseteq \mathfrak{F}^S$. Assume that $\mathfrak{F}^S \neq \mathfrak{N}$. Then there is a non-soluble group in $\mathfrak{F}^S$. Since $\mathfrak{F}^S$ is hereditary, it contains an $\mathfrak{S}$-critical group G. Now all maximal subgroups of G belong $\mathfrak{F}^S \cap \mathfrak{S} = \mathfrak{N}$. Hence G is a Schmidt group. Thus G is soluble, the contradiction. Therefore $\mathfrak{F}^S = \mathfrak{N}$. Since the sets of all $\mathfrak{F}$-critical groups and $\mathfrak{F}^S$-critical groups coincide, we see that $\mathfrak{F}$ is the formation with the Shemetkov property. □

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