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On Some Residual Properties of the Verbal Embeddings of Groups [1]

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Abstract

We consider verbal embedding constructions preserving some residual properties for groups. An arbitrary residually finite countable group H has a V-verbal embedding into a residually finite 2-generator group G for any non-trivial word set V. If in addition H is a residually soluble (residually nilpotent) group, then the group G can be constructed also to be residually soluble (residually nilpotent). The analogs of this embedding also are true without the requirement about residual finiteness: Any residually soluble (residually nilpotent) countable group H for any non-trivial word set V has a V-verbal embedding into a residually soluble (residually nilpotent) 2-generator group G.

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1 Introduction

1.1 The main results

For the word set $V \subseteq F_\infty$ the embedding $\tau : H \to G$ of the group H is called a V-*verbal embedding*, if the isomorphic image $\tau(H)$ lies in the verbal subgroup $V(G)$ of G. Verbal embeddings are economical embeddings in the sense that the image of H occupies "the small part" $V(G)$ inside G: the larger is the variety $\mathfrak{V}$ corresponding to V, the smaller the verbal subgroup $V(G)$. Our main aim is to show that any countable residually free group has a V-verbal embedding into a 2-generator residually finite group, and this can also be combined with residual solubility or residual nilpotence:

Theorem 1.1 *Let H be an arbitrary countable residually finite group and $V \subseteq F_\infty$ be any non-trivial word set. Then there exists a 2-generator residually finite group G admitting a V-verbal embedding $\tau : H \to G$.*

Moreover, if H is a residually finite and residually soluble group (or a residually finite and residually nilpotent group), then G can be chosen to also have that property.

The proof of this theorem occupies Section 2 below. A further modification of the main embedding construction allows one to prove the analogs of Theorem 1.1 for embeddings of countable residually soluble (residually nilpotent) groups into 2-generator residually soluble (residually nilpotent) groups regardless of residual finiteness of the groups. We bring them as Theorem 3.1 and Theorem 3.2 in Section 3.

The fact that every countable residually finite group can be embedded into a 2-generator residually finite group and that this embedding can also preserve residual solubility (residual nilpotence) was established by J.S. Wilson in [23, Theorem A]. Our Theorem 1.1 adds verbality to that embedding.

Since for a trivial word set V the verbal subgroup $V(G)$ is equal to $\{1\}$ for any group G, it is common to restrict to consideration of V-verbal embeddings for non-trivial word sets only. Also, if from the context it is clear which V is assumed, we will just term the embedding "verbal". For background information on varieties of groups we refer to Hanna Neumann's monograph [20]. The details on wreath products can be found in P. Neumanns article [21].

1.2 A brief summary of background and recent development

Before we proceed to the proofs let us give a brief outline of the context in which verbal embeddings occurred, and announce some recent development on them. Since for the given group G its commutator subgroup G', the n-th derived subgroup $G^{(n)}$, the c-th member $\gamma_c(G)$ of the lower central series, the m-th power of G^m and many other commonly used types of subgroups are particular cases of the verbal subgroup $V(G) = \langle \varphi(w) \,|\, w \in V,\ \ \varphi \in \mathrm{Hom}(F_\infty, G) \rangle$ generated by all values in G of the words $w \in V$ for the given words set V, examples of verbal embedding can be found since the early stages of group theory development.

For any group H and for any non-trivial V it is easy to find a group G admitting a V-verbal embedding $\tau : H \to G$. Thus the verbal embeddings are usually considered in connection with some extra properties such as;

- normality or subnormality of embedding (when $\tau(H)$ is a normal or subnormal subgroup in G),
- with conditions making G a "small group" (for example, G is a 2-generator group, when H is countable),
- with requirements on G to be "close" to H (for example, G is abelian, nilpotent, soluble, generalized soluble, generalized nilpotent, ordered group, etc., as long as H has those properties),
- with conditions that make the embedding τ useful for computational purposes for finite groups H and G, etc.

In 1912 W. Burnside proved that a non-abelian group with cyclic center or a non-abelian group the index of whose derived group is p^2 cannot be the commutator group (and thus cannot be a subgroup in the commutator group) of some p-group [2, Theorems on pp. 241 and 242]. In 1957 N. Blackburn has described all the 2-generated p-groups which occur as derived groups of p-groups [1]. Both references, clearly, are examples of the possibility (or impossibility) of normal V-verbal embeddings for the case $V = \{[x_1, x_2]\}$.

In 1959 Hanna Neumann and B.H. Neumann presenting their second construction for embedding of any countable group H into a 2-generator group G stressed that the embedding actually is into the second derived group G'' (the verbal subgroup for $V = \{[[x_1, x_2], [x_3, x_4]]\}$) [19]. Related embeddings can also be found in P. Hall's work [6].

In 1991 widely generalizing these approaches H. Heineken in [7] studied the verbal embeddings for an arbitrary word $w \in F_\infty$. He considered normal verbal embeddings for finite p-groups, and gave the classification of all finite p-groups that for the given non-trivial word w have normal w-verbal embedding into finite p-groups. In [9] verbal embedding were defined in the more general form for any word set V. Heineken's criterion was generalized to classify all the cases when for a given non-trivial word set V the given group H has a normal V-verbal embedding into some group. Both the criteria of [7] and [9] are in terms of automorphisms group Aut (H). See also [15], where the same criterion is proved by other means, and also [8], where the normal embeddings for subgroups were considered.

B. Eick considered the verbal embeddings for finite groups and she generalized Heineken's criterion to classify those finite groups that for the given word w have a normal verbal embedding into finite groups [4]. This embedding also has computational applications, in particular, for finite groups presented by permutations. It is used in computational group theory (see also [5]).

The verbal embeddings were the subject of our research program carried out in 1997–98 at the Universität Würzburg, Germany, under supervision of Prof. Dr. Herman Heineken (project A/97/13683 supported by the DAAD – Deutscher Akademischer Austauschdienst). In [15] we classified the cases, when for the non trivial word set V the soluble (nilpotent) group H has a normal verbal embedding into a soluble (nilpotent) group G. The similar question for abelian groups was solved in [9].

In [11] we proved that, unlike normal embeddings, the subnormal embeddings are possible for any group H and for any non-trivial word set V. Moreover, if H is countable then G can be selected to be 2-generator, which strengthens the embedding theorem of [19] mentioned above. The subnormal verbal embeddings of soluble, generalized soluble and generalized nilpotent groups were considered in [11, 12, 16], and some verbal embeddings for ordered groups were constructed in [13].

Also, we used verbal embeddings to construct some specific classes of groups such as continuously large classes of SI^*-groups, which are not locally soluble [14], continuously large classes of finitely generated soluble non-Hopfian groups [17, 18], etc.

In the current work we wish to combine verbality of the embeddings with properties, which are very different from those we just listed, namely with residual finiteness, residual solubility and residual nilpotence. The arguments are based on the verbal embedding methods we used in cited papers and on the construction by J.S. Wilson in [23], which we find is a very elegant and inspirating generalization of older embedding arguments by wreath products, developed by B.H. Neumann, Hanna Neumann and P. Hall.

2 The proof of Theorem 1.1

2.1 A supersoluble group with finite factors

For technical purposes we need a slightly "stronger" version of the residual finiteness property. Although in a residually finite group H for any element $h \in H$ there is a normal subgroup $N \lhd H$ such that $h \notin N$ and $|H/N|$ is finite, the exponent of h modulo N may not vary much in the sense that it may not be divisible by a pre-given integer (it is easy to find examples of that type). For some residually finite groups, however, this requirement can be added. Recall that the group S is supersoluble if it possesses a series of subgroups

$$S = S_0 \geqslant S_1 \geqslant \cdots \geqslant S_k = \{1\} \tag{1}$$

such that $S_i \lhd S$ and S_{i-1}/S_i is cyclic for all $i = 1, \dots, k$.

Lemma 2.1 *If for the supersoluble group S in the notation above the factors S_{i-1}/S_i are infinite for all $i = 1, \dots, k$, then for any non-trivial element $s \in S$ and for any positive integer n there is a normal subgroup $N = N(s, n) \lhd S$ such that $s \notin N$, the factor S/N is finite and the exponent $\exp(Ns)$ is divisible by n. Moreover, if n is a power of a prime p and G is nilpotent, then N can be selected so that S/N is a p-group.*

Proof — We prove this by induction. For $k = 1$ we have $S \simeq \mathbb{Z}$ is an infinite cycle, and we can take $N(s, n) = \langle sn \rangle = sn\mathbb{Z}$.

Assume the lemma is proved for all $1, \dots, k-1$, and take any element $s \in S$ in a group with series (1). If $s \in S \backslash S_1$, then the image of s under natural map $\pi : S \to S/S_1 \simeq \mathbb{Z}$ is an element $S_1 s$ in an infinite cycle, and we can choose as $N = N(s, n)$ the full pre-image $\pi^{-1}(\langle S_1(sn) \rangle)$ of the cyclic subgroup generated in S/S_1 by $S_1(sn)$.

Now assume $s \in S_1$ and by induction hypothesis take $L \lhd S_1$ such that s is not in L, the factor-group S_1/L is finite, and the exponent $\exp(Ls)$ is divisible by n. Set $r = |S_1/L|$ and consider the subgroup

$S_1^r = \langle x^r \mid x \in S_1 \rangle$, which lies in L and which is normal in S, since it is a fully-invariant subgroup in the normal subgroup S_1 of S. The factor group S_1/S_1^r is finite, since it is a supersoluble group of restricted period. The element s is not in S_1^r and, since

$$S_1/L \simeq (S_1/S_1^r)/(L/S_1^r)$$

holds, the exponent $\exp(S_1^r s)$ also is divisible by n.

The factor group $(S/S_1^r)/(S_1/S_1^r)$ is isomorphic to the infinite cycle $S/S_1 \simeq \mathbb{Z}$. Denote the generator of the latter by $S_1 a$. The action of $S_1^r a$ by conjugation defines an automorphism on the normal subgroup S_1/S_1^r of S/S_1^r. Since S_1/S_1^r is finite, $\mathrm{Aut}\left(S_1/S_1^r\right)$ also is finite, and this automorphism must be of some finite exponent h: the element $S_1^r a^h$ acts trivially on S_1/S_1^r, that is, a^h commutes with elements of S_1 modulo S_1^r.

Take $N = N(s,n)$ to be the subgroup generated in S by a^h and S_1^r. It is easy to check that N is normal in S: all powers of a commute with a^h and normalize S_1^r. And the elements of S_1 normalize S_1^r, and they commute with a^h modulo S_1^r. The factor S/N clearly is finite.

Taking into account the normalizing effect of the powers of the element a^h on S_1^r, it is easy to check that an element of N belongs to S_1 only if it belongs to S_1^r. This means that for any $s \in S_1$ the exponent of s modulo N is equal to the exponent of s modulo S_1^r and is divisible by n.

Finally, assume n is a power of the prime p. By the procedure above we can find $N \triangleleft S$ such that $\exp(Ns)$ is divisible by n. Being a finite nilpotent group, the factor S/N is a direct product of its Sylow subgroups. Denote by Q the direct product of those Sylow subgroups, which correspond to primes other than p. Let N' be the full pre-image of Q under the natural map $S \to S/N$. Clearly, s is not in N', and N' is normal in S. It suffices to take N' as N. The lemma is proved. □

If H is a residually finite and residually soluble group, then by refining the subgroups H_i (where necessary) it is possible to get such a system of normal subgroups $H_i \triangleleft H$, $i \in I$, $\bigcap_{i\in I} H_i = \{1\}$ that each factor H/H_i is finite and soluble simultaneously. And if H is a residually finite and residually nilpotent group, then it is possible to construct the H_i so that each factor H/H_i is finite group of prime-power order. We will use this below without references.

2.2 An auxiliary verbal embedding

The group S of Lemma 2.1 is a constructive block for the proof of the following verbal embedding:

Lemma 2.2 *Let* H *be a countable residually finite group and* $U \subseteq F_\infty$ *be any non-trivial word set. Then there exists a countable residually finite group* $R = R(H, U)$ *admitting a* U*-verbal embedding* $\rho : H \hookrightarrow R$.

Moreover, if H *is a residually soluble or a residually nilpotent group, then* R *can be constructed to also have the same properties.*

Proof — Take a (finite or countably infinite) system of normal subgroups $H_i \triangleleft H$, $i = 1, 2, \ldots$, such that $\bigcap_i H_i = \{1\}$. Denote $n_i = |H/H_i|$, $i = 1, 2, \ldots$

Since the word set U is non-trivial, the corresponding variety

$$\mathfrak{U} = \operatorname{var}(F_\infty / U(F_\infty))$$

is distinct from the variety of all groups $\mathfrak{O}$. The set of all nilpotent groups generates $\mathfrak{O}$, thus there is a group S, which does not belong to $\mathfrak{U}$ and satisfies the conditions of Lemma 2.1. One may take a free nilpotent group S of large enough class and of large enough finite rank: the finitely generated nilpotent groups are supersoluble, and the corresponding factors in (1) can be constructed to be infinite. Since $S \notin \mathfrak{U}$, the verbal subgroup $U(S)$ contains a non-trivial element $s \in U(S)$. By Lemma 2.1 for the integers n_i there are normal subgroups

$$N_i = N(s, n_i) \triangleleft S, \quad i = 1, 2, \ldots$$

such that $s \notin N_i$, the factor group S/N_i is finite and $\exp N_i s$ is divisible by n_i. Consider the direct wreath products

$$W_i = H/H_i \operatorname{wr} S/N_i, \quad i = 1, 2, \ldots \tag{2}$$

Denote by W the Cartesian product of all the wreath products of (2), and denote by M the Cartesian product of the base subgroups M_i of these wreath products. We are going to construct the group R by a set of its generators singled out in W.

For any $h \in H$ and any $i = 1, 2, \ldots$ define an element $\gamma_{h,i}$ in M_i:

$$\gamma_{h,i}(t \cdot N_i s^j) = H_i h^j, \quad j = 0, 1, \ldots, |N_i s| - 1,$$

where t is an element in the transversal T_i of the cyclic subgroup

$\langle N_i s \rangle$ in the group S/N_i. Also define an element $\Gamma_h \in M$: its coordinates on elements of S/N_i are equal to the coordinates of $\gamma_{h,i}$ respectively:

$$\Gamma_h(t \cdot N_i s^j) = \gamma_{h,i}(t \cdot N_i s^j) \text{ for any } t, i, j.$$

Denote by δ the injective embedding of S into the Cartesian product W, which assigns to each $l \in S$ the vector with coordinates $N_i l$, $i = 1, 2, \dots$ Since $s \in U(S)$, we get that

$$\delta(s) \in U\left(\prod S/N_i\right) \leqslant U(W).$$

Since a verbal subgroup is characteristic, $U(W)$ contains conjugations of $\delta(s)$ by any elements of W. Thus $U(W)$ contains the elements

$$\pi_h = \delta(s)^{\Gamma_h}(\delta(s))^{-1}$$

for any $h \in H$. We have

$$\pi_h = \Gamma_h^{-1}\delta(s)\Gamma_h(\delta(s))^{-1} = \Gamma_h^{-1}\Gamma_h^{(\delta(s))^{-1}}.$$

From here the values of π_h can be computed taking into account the shifting effect of $N_i s$ on coordinates of $\gamma_{h,i}$ in the base subgroup M_i for each i. Clearly,

$$\Gamma_h^{-1}(t \cdot N_i s^j) = H_i h^{-j}$$

and

$$\Gamma_h^{(\delta(s))^{-1}}(t \cdot N_i s^j) = \Gamma_h(t \cdot N_i s^{j+1}) = H_i h^{j+1}.$$

Therefore for any $j = 0, 1, \dots, (\exp N_i s) - 2$ we have $\pi_h(t \cdot N_i s^j) = H_i h$.

To compute the value of π_h on the last coordinates with index

$$t \cdot N_i s^{(\exp N_i s)-1}$$

we use the fact that the order of $N_i s$ (that is, the number of coordinates in the substrings of $\gamma_{h,i}$ corresponding to each coset $t \cdot N_i$) is divisible by the order of the element $H_i h$. Namely, the conjugation of the first element $H_i h^0 = H_i$, standing at index $t \cdot N_i s^0$, by the element $(\delta(s))^{-1}$ (that is, by the element $N_i s^{-1}$) shifts it to the last index

$$t \cdot N_i s^{-1} = t \cdot N_i s^{(\exp N_i s)-1}.$$

So at the last index we have the following value for π_h:

$$H_i h^{-[(\exp N_i s)-1]} \cdot H_i h^0 = H_i h.$$

Thus π_h is a constant function with the value $H_i h$ inside each M_i. Since all H_i intersect trivially, we have an injective embedding $\pi : h \to \pi_h$ of H into W. Although we also have $\pi(H) \leqslant U(W)$, this is not yet the desired embedding, since W may not be a countable group.

Since $s \in U(S)$, there is a finite system $\{s_1, \dots, s_u\}$ of elements in S such that $s \in U(\langle s_1, \dots, s_u \rangle)$ also holds. Denote

$$R = \langle \Gamma_h, \delta(s_i) \mid h \in H; i = 1, \dots, u \rangle.$$

Then R contains the element $\delta(s)$ with the desired properties. Thus by arguments above it contains all the elements π_h. There only remains to denote by ρ the restriction of π from H to R.

Turning to the final two statements of the lemma notice that, if the residually finite group H also is residually soluble, then all the factor groups H/H_i can be assumed to be soluble. And, since the factors S/N_i are nilpotent, all the wreath products (2) also are soluble, and the group W is residually soluble together with its subgroup R.

If the residually finite group H is residually nilpotent, we may assume each H/H_i is a finite p_i-group for some prime p_i. Then by the last statement of Lemma 2.1 the suitable $N_i \triangleleft S$ can be selected so that S/N_i is a p_i-group, and the wreath product W_i in (2) is nilpotent, since it is a finite p_i-group. Thus W and R are residually nilpotent. □

We will apply Lemma 2.2 to the group H of Theorem 1.1 for the case, when U is the word set corresponding to the product variety $\mathfrak{V}\mathfrak{A}$, where $\mathfrak{A}$ is the variety of all abelian groups, and $\mathfrak{V}$ is the variety corresponding to the non-trivial word set V of Theorem 1.1. Clearly, $\mathfrak{V}\mathfrak{A} \neq \mathfrak{O}$, and U is a non-trivial word set.

2.3 Construction of the 2-generator group

The final step of the construction is based on a simplification of the important argument of J.S. Wilson in subsections 2.2 and 2.3 in [23]. Denote by R_i, $i = 1, 2, \dots$, a series of normal subgroups of the residually finite (residually finite and residually soluble, residually finite and residually nilpotent) group $R = R(H, V)$ of Lemma 2.2, for which $\bigcap_i R_i = \{1\}$ and the factors R/R_i are finite (are finite soluble, prime-power order) groups. In some way order the elements of the countable group R:

$$r_1, \dots, r_i, \dots \tag{3}$$

and denote by α_n the number of the first members of (3) such that

$$\{R_n r_1, \dots, R_n r_{\alpha_n}\}$$

contains all the cosets in R/R_n. Let $\beta_n = 2(\alpha_1 + \dots + \alpha_n)$. We need two variants of a countable series of additively written finite cyclic groups

$$Z_n = \langle z_n \rangle, \quad n = 1, 2, \dots \tag{4}$$

and of some "sparse" subsets of integer indices. Each variant will be used for one segment one the proof.

Variant 1) When R is a residually finite or a residually finite and residually soluble group, set the order $|Z_n| = 2^{\beta_n + 1}$, and define the indices $t_i = 2^i$, $i = 1, 2, \dots$

Variant 2) When R is a residually finite and residually nilpotent group (and if without loss of generality the order of each factor R/R_n is a power of some prime p_n), define $|Z_n|$ as follows. Take $\sigma_1 = 1$ and set $|Z_1| = p_1^{\sigma_1 + 2\alpha_1 + 1}$. If $p_2^{\sigma_2}$ is the least power of p_2 greater than $p_1^{\sigma_1 + 2\alpha_1}$, then set $|Z_2| = p_2^{\sigma_2 + 2\alpha_2 + 1}$. By induction, if $p_n^{\sigma_n}$ is the least power of p_n greater than $p_{n-1}^{\sigma_{n-1} + 2\alpha_{n-1}}$, then set $|Z_n| = p_{n-1}^{\sigma_{n-1} + 2\alpha_{n-1} + 1}$. For this case the indices t_i can be defined as:

$$t_1 = p_1^{\sigma_1 + 1}, t_2 = p_1^{\sigma_1 + 2}, \dots, t_{\beta_1} = p_1^{\sigma_1 + 2\alpha_1};$$

$$t_{\beta_1 + 1} = p_2^{\sigma_2 + 1}, t_{\beta_1 + 2} = p_2^{\sigma_2 + 2}, \dots, t_{\beta_1 + 2\alpha_2} = t_{\beta_2} = p_2^{\sigma_2 + 2\alpha_2};$$

....

$$t_{\beta_{n-1} + 1} = p_n^{\sigma_n + 1}, t_{\beta_{n-1} + 2} = p_n^{\sigma_n + 2}, \dots, t_{\beta_{n-1} + 2\alpha_n} = t_{\beta_n} = p_n^{\sigma_n + 2\alpha_n};$$

....

In both cases the indices $t_1, \dots, t_{\beta_n}$ can be considered to be some elements of the "first part" of the cyclic group Z_n. If needed, we will denote these elements of the additive group Z_n not by t_i but by $t_i \cdot z_n$ in order to stress to which one of the cycles (4) it belongs.

As is easy to check (we omit the calculation, since such arguments are repeatedly used in the literature), for each $d \in Z_n$ there is at most one pair of non-negative integers k, l, both not exceeding β_n, such that $t_k d = t_l$.

Denote by B_n the Cartesian product of the copies of R/R_n, indexed by the elements of Z_n (the base subgroup of the wreath product $(R/R_n) \,\mathrm{wr}\, Z_n$), and let $B = \prod_{n=1,2,\dots} B_n$ be their Cartesian product. We can think of an element of B as of an infinite vector of coordinates, which are laid out as an infinite column of finite rows, and which are indexed by elements of the disjoint union of all cycles Z_n. Define an element $\theta \in B$ by the following steps for each $n = 1, 2, \dots$:

1. for "even" indices t_{2k}, $k = 1, \dots, \beta_n/2$, define

$$\theta(t_{2k}) = \theta(t_{2k \cdot z_n}) = R_n r_k;$$

2. for "odd" indices t_{2k-1}, $k = \beta_{n-1}/2 + 1, \dots, \beta_n/2$, define

$$\theta(t_{2k-1}) = \theta(t_{2k-1 \cdot z_n}) = R_n r_{k-\beta_{n-1}/2};$$

3. for all other indices t_j define $\theta(t_j) = \theta(t_{j \cdot z_n}) = R_n$.

Thus for each n in the n-th row of θ, besides trivial coordinates, we have selected two collections of coordinates. The first collection has $\beta_n/2$ coordinates $R_n r_1, \dots, R_n r_{\beta_n/2}$, which are distributed "uniformly" for all n in the sense that their indices $t_2, t_4, \dots, t_{\beta_n}$ are the same for all rows (as long as Z_n accommodates them). And the second, smaller collection has α_n coordinates $R_n r_1, \dots, R_n r_{\alpha_n}$, which are distributed "uniquely" for each n in the sense that for any other $m \neq n$ the m-th row of θ no longer contains non-trivial coordinates at the indices $t_{\beta_{n-1}+1}, \dots, t_{\beta_n - 1}$.

Denote by ψ the automorphism of B defined by its shifting actions on coordinates in each B_n: for any $\delta \in B$ we have $\delta^{\psi}(i \cdot z_n) = \delta((i+1) \cdot z_n)$ for $i = 0, \dots, |Z_n| - 2$, and $\delta^{\psi}(i \cdot z_n) = \delta(0 \cdot z_n)$ for $i = |Z_n| - 1$. This defines a split extension E of W by $\langle \psi \rangle$, in which we select the subgroup $G = \langle \theta, \psi \rangle$.

Lemma 2.3 *In the above notation:*

1) *for an arbitrary pair of indices* k, l *there is a positive integer* n_0 *and an element* $\mu = \mu_{k,l} \in G$ *such that for any* $n \geqslant n_0$ *we have* $\mu(0) = \mu(0 \cdot z_n) = R_n[r_k, r_l]$, *and* $\mu(i) = \mu(i \cdot z_n) = R_n$ *for* $i = 1, \dots, |Z_n| - 1$.

2) *for an arbitrary pair of indices* k, l *and for any positive integer* n *the group* G *contains an element* $\rho = \rho_{k,l,n}$ *in which* $\rho(0) = \rho(0 \cdot z_n) = R_n[r_k, r_l]$*, and all the other coordinates of* ρ *are trivial.*

Notice that the lemma states nothing about the coordinates of $\mu_{k,l}$ in the n-th row for any $n < n_0$.

Proof — Let $\mu = [\theta^{\psi^{t_{2k}}}, \theta^{\psi^{t_{2l}}}]$ and take any integer n_0 for which $\beta_{n_0} > k, l$. For any $n \geqslant n_0$ the first coordinate $\mu(0)$ in the n-th row clearly is equal to

$$\mu(0) = \mu(0 \cdot z_n) = [\theta(t_{2k}), \theta(t_{2l})] = R_n[r_k, r_l].$$

All the remaining coordinates in the n-th row of μ are trivial, since by selection of the indices t_i for each $d \in Z_n$ there is at most one pair of indices $k, l \leqslant \beta_n$ for which $t_k d = t_l$. For small values $n < n_0$, however, the effect of shiftings $\psi^{t_{2k}}$ and $\psi^{t_{2l}}$ upon θ may be different: the first rows of μ may look differently, but they will still consist of some (finitely many) commutators.

To prove the second statement take $\rho = [\theta^{\psi^{t_u}}, \theta^{\psi^{t_v}}]$ for $u = \beta_{n-1} + 2k - 1$, $v = \beta_{n-1} + 2l - 1$. Then in the n-th row

$$\rho(0) = \rho(0 \cdot z_n) = [\theta(t_u), \theta(t_v)] = R_n[r_k, r_l].$$

All the other coordinates in the n-th row are trivial for the reason concerning the singe pair we just mentioned. And all the coordinates in other rows are trivial, since θ may contain non-trivial coordinates at indices t_u, t_v in the n-th row only. □

Lemma 2.3 allows to construct an injective image of the commutator subgroup $R' = [R, R]$ in G. Any non-trivial element $c \in R'$ is a product of finitely many commutators of type $[r_k, r_l]$. By statement (1) of Lemma 2.3 for each such pair the group G contains an element $\mu_{k,l}$ in which each n-th row for all $n \geqslant n_0$ begins with $R_n[r_k, r_l]$ followed by trivial cosets R_n. If m is the maximum of all such integers n_0 occurring, we get that G contains an element μ in which for all rows $n \geqslant m$ the first coordinate is $R_n c$, and it is followed by trivial cosets R_n for $i = 1, \ldots, |Z_n| - 1$.

Whatever the coordinates of μ in rows $n < m$ may be, by multiplying μ to some finitely many elements of type $\rho = \rho_{k,l,n}$ (provided by statement (2) of Lemma 2.3) and their conjugates ρ^{ψ^j}, $j = -1, -2, \ldots,$

one will get a product $\omega_c \in G$ for which

$$\omega_c(0 \cdot z_n) = R_n c, \quad \omega_c(i \cdot z_n) = R_n \text{ for } i = 1, \dots, |Z_n| - 1$$

holds for an *arbitrary* row. Since the subgroups R_n, $n = 1, 2, \dots$, intersect trivially, the map $\nu : c \to \omega_c$ is an embedding of R' into G.

Consider the composition $\tau = \rho \cdot \nu$ of the embedding ρ of Lemma 2.2 with ν. Since the variety $\mathfrak{U} = \mathfrak{V}\mathfrak{A}$ contains $\mathfrak{A}$ as a subvariety, the word set $U \subseteq F_\infty$ is a subset of the commutator subgroup $F'_\infty = [F_\infty, F_\infty]$, and thus the image $\rho(H)$ is contained in the commutator R'. Therefore τ is an embedding of the initial group H into the two-generator group G.

Both the extension E and its subgroup G are residually finite groups. If R is a residually finite and residually soluble group, then E and G also have this property, since all the wreath products $R/R_n \operatorname{wr} Z_n$ are finite and soluble. And when R is a residually finite and residually nilpotent group, the above wreath products also can be nilpotent, since, without loss of generality, the factors R/R_n can be selected to be finite p_n-groups, and the finite cycles Z_n can be selected to be groups of orders of powers of p_n. Thus τ preserves all the residual properties of Theorem 1.1, and it remains to show that the embedding τ is V-verbal.

Since the embedding ρ is U-verbal, we have $\rho(H) \subseteq U(R)$. By [20, 21,12] the word set $U \leqslant F_\infty$, which corresponds to $\mathfrak{U} = \mathfrak{V}\mathfrak{A}$, is generated by words of type $v(w_1, \dots, w_k) \in F_\infty$, where $v \in V$ is an identity of $\mathfrak{V}$, and where $w_1, \dots, w_k \in F'_\infty$ are commutator words (i.e., some identities of $\mathfrak{A}$). Thus $U(R) \subseteq V(R')$ holds, and ρ actually is a V-verbal embedding of H into the commutator R'. Since $\nu(R') \leqslant G$, we get that τ is a V-verbal embedding of H into G.

All the steps of the proof of Theorem 1.1 are now completed. □

3 Some other verbal embedding constructions with residual properties

A simplified version of the construction of Section 2 can be used for embeddings of residually soluble or residually nilpotent groups (without requirement of residual finiteness).

Returning to the proof of Lemma 2.2 recall that for any non trivial word set U in the free nilpotent group S (of sufficiently large rank and class) we took an element $s \in S$ so that $s \in U(S)$. Clearly, s is of infinite order. If $H_i \triangleleft H$, $i = 1, 2, \ldots$, such that all H/H_i are soluble and $\bigcap_i H_i = \{1\}$, then in case of finite factors H/H_i take the same wreath product as in (2), and in case of infinite factors H/H_i take $H/H_i \,\mathrm{wr}\, S$, and define $\gamma_{h,i}(t \cdot s^j) = H_i s^j$ for any integer $j \in \mathbb{Z}$ and for any t from the transversal T of $\langle s \rangle$ in S. Then the elements Γ_h can be re-defined taking into account this modification for infinite factors H/H_i, and the homomorphism δ can be altered so that for infinite H/H_i it identically maps s to the copy of that element in the active group S of $H/H_i \,\mathrm{wr}\, S$. Then in the proof of Lemma 2.3 we may have some of the factors R/R_n infinite. For them we take the $Z_n = Z$, and we will have an infinite n-th row in $\theta \in B$. A slight adaptation of the arguments following Lemma 2.3 allows one to add verbality also to the embedding of Theorem C in [23]:

Theorem 3.1 *Let H be an arbitrary countable residually soluble group and $V \subseteq F_\infty$ be an arbitrary non-trivial word set. Then there exists a 2-generator residually soluble group G admitting a V-verbal embedding $\tau : H \to G$.*

Embedding of countable soluble groups in 2-generator soluble groups was established in [19, Corollary 5.2], and we added verbality to this embedding in [11, statement C in Theorem 1].

The situation is different for residually nilpotent and for nilpotent groups. Not every countable nilpotent group can be embedded into a 2-generator nilpotent group, since a subgroup of a finitely generated nilpotent group is finitely generated due to the maximal condition on subgroups. And not every countable residually nilpotent group can be embedded into a 2-generator residually nilpotent group, because by theorem of A.I. Mal'cev [10] a finitely generated residually nilpotent group (together with its subgroups) has to be residually finite. So residual finiteness is a necessary condition and, as Corollary A1 in [23] shows, it also is sufficient. The arguments in Section 2 add verbality in this case also:

Theorem 3.2 *Let H be an arbitrary countable residually nilpotent group. Then for an arbitrary non-trivial word set $V \subseteq F_\infty$ there exists a 2-generator residually nilpotent group G admitting a V-verbal embedding $\tau : H \to G$ if and only if H is residually finite.*

[23] also considers embeddings of periodic residually finite groups. An adaptation of the arguments in Lemma 2.1 and Lemma 2.2 ex-

tends to cover this case also, but we prefer not to consider that case here.

Another direction for generalization is to consider Lemma 2.2 not only for countable but for any infinite group H. If for an index set I there exists a system of normal subgroups H_i, $i \in I$, such that $\bigcap_{i \in I} H_i = \{1\}$ and each factor H/H_i is a finite (finite soluble or finite nilpotent) group then, taking into account that the group S of Lemma 2.1 is countable, a slight modification in the proof of Lemma 2.2 allows one to prove:

Proposition 3.3 *Let H be an arbitrary infinite residually finite group of any cardinality. Then for an arbitrary non-trivial word set $V \subseteq F_\infty$ there exists a residually finite group G of the same cardinality as H admitting a V-verbal embedding $\tau : H \to G$. Moreover, if H is a residually soluble or a residually nilpotent group, then G can be constructed to also have that property.*

This is an amendment to [11] in which we proved that any infinite group for any non-trivial V has a verbal embedding into a group of the same cardinality, with some properties preserved by the embedding. Notice that if H is finite, then the group G of Proposition 3.3 can be constructed to also be finite, but this is the insignificant trivial case, since residual finiteness is just finiteness here.

One of the properties that can be added to the embedding τ of a countable group into a 2-generator group is *subnormaliy* of the embedding: the image $\tau(H)$ of the given countable group H is a subnormal subgroup of the 2-generator group G. This was first established by R. Dark in [3], and most of the embeddings of countable groups into 2-generator groups (with additional properties) we constructed in [11, 12, 13, 15, 16, 17, 18] are subnormal. The constructions in Theorem 1.1, Theorem 3.2 and Theorem 3.2 do not provide subnormality for the embeddings τ. We would like to complete this work by suggesting a question:

Problem 3.4 *Let H be a residually finite (residually soluble, residually nilpotent) countable group.*

1) *Does H admit a subnormal embedding into a residually finite (residually soluble, residually nilpotent) 2-generator group?*

2) *If yes, then can those subnormal embeddings be V-verbal for an arbitrary non-trivial word set $V \subseteq F_\infty$?*

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