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# Generalized Quasinormal Subgroups of Order $p^2$

Stewart Stonehewer

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To Hermann Heineken, for his 80th birthday

**Abstract**

In a recent paper, Cossey and Stonehewer introduced a generalization of the concept of quasinormal subgroups as a consequence of their discovery of a scarcity of the latter in certain finite p-groups of an important universal nature. Cyclic subgroups in this generalized class were shown to possess certain interesting properties, including their invariance under index-preserving projectivities. Naturally the first step in that work was the consideration of the subgroups of prime order. Thus, moving on from cyclic groups, a study of the non-cyclic subgroups of order $p^2$ in this generalized class would seem to be appropriate. It is shown here that they also are invariant under index-preserving projectivities.

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# 1 Introduction

Let $H$ be a subgroup of a group $G$. If $\langle H, K\rangle = HKHK$ for every cyclic subgroup $K$ of $G$, then $H$ is said to be 4-*quasinormal* in $G$ and we write

$$H \ \mathrm{qn}_4 \ G.$$

Clearly the terminology derives from saying that quasinormal subgroups would be 2-quasinormal. In between there would be two types of 3-quasinormal subgroups. But in finite p-groups, which are our main concern, $\langle H, K\rangle = HKH$ or $KHK$ implies $\langle H, K\rangle = HK = KH$. So 3-quasinormal subgroups are 2-quasinormal, i.e. quasinormal here.

In [1], several properties of cyclic 4-quasinormal subgroups of finite p-groups were discovered. It was assumed that

$$p \geqslant 5. \tag{1}$$

For, when H and K are cyclic subgroups of a finite p-group G with $H \text{ qn}_4 G$, the join $J = \langle H, K\rangle$ is always a regular group when $p \geqslant 5$. This follows from [3], III, 10.13 Satz, since $|J/J^p| \leqslant p^3$. The consequences of this are of course highly significant and it makes sense to consider the cases $p = 2$ or $3$ separately. Though in fact it is easy to see that in all nilpotent groups of class at most 2, finite and infinite, *all subgroups are 4-quasinormal*. However, we shall assume (1) in the present work. Again regularity will be significant.

Let $H = \langle h\rangle$ and $K = \langle k\rangle$ be cyclic subgroups generating a finite p-group G ($p \geqslant 5$) and let H be 4-quasinormal in G. It was shown in [1] that

$$G = HCK,$$

where $C = \langle [h, k]\rangle$. Let $|H| = p^m$, $|K| = p^n$ and $|C| = p^\ell$. Then by regularity (see [3], III, 10.5 Hauptsatz)

$$\ell \leqslant m \text{ and } \ell \leqslant n.$$

Moreover

$$H^{p^\ell} \leqslant \zeta_1(G), \tag{2}$$

the centre of G. For, by regularity, $[h, k]^{p^\ell} = 1$ implies $[h^{p^\ell}, k] = 1$ (see [3], III, 10,6 Satz). So (2) holds. Similarly

$$K^{p^\ell} \leqslant \zeta_1(G). \tag{3}$$

We shall show in Section 3 that (3) is also true when H is elementary abelian of rank 2. In [1] it was shown that cyclic 4-quasinormal subgroups of finite p-groups are invariant under index-preserving projectivities. We shall also show in Section 3 that the same is true for elementary abelian 4-quasinormal subgroups of rank 2.

In Section 2 we use regularity to classify all groups that arise in the situations in which we are interested when $|K| = p$. Our notation will be standard. Terms of the lower central series of G are denoted by $\gamma_i(G)$, $i \geqslant 1$, and terms of the upper central series by $\zeta_i(G)$, $i \geqslant 0$. The derived subgroup $\gamma_2(G)$ is also denoted by $G'$. The extraspecial goup $E_p$ of order $p^3$ and exponent $p$ will appear frequently; and as always $C_p$ will be a group of order $p$. A split extension of a group $X$ by a group $Y$ is written as $X \rtimes Y$ and $Y \ltimes X$. The centralizer of a subgroup $H$ in a group $G$ is denoted by $C_G(H)$ and $H^G$ is the normal closure of $H$ in $G$. For elements $x$, $y$ of a group and an integer $n \geqslant 1$, we define inductively the commutator $[x,_{n+1} y] = [[x,_n y], y]$ with $[x,_1 y] = [x, y]$.

## 2 Classification of groups

Let $G$ be a finite $p$-group with $p \geqslant 5$, generated by subgroups $H$ and $K$ with $H$ elementary abelian of rank 2 and $K$ of order $p$. We shall list the groups $G$ for which $H$ is 4-quasinormal together with the embeddings of $H$ and $K$. Thus

$$G = HKHK$$

and for all cyclic subgroups $X$ of $G$ we have $\langle H, X\rangle = HXHX$. Clearly $|G| \leqslant p^5$ and so $G$ has class $\leqslant 4$. Thus $G$ is regular and hence has exponent $p$ (see [3], III, 10.5 Hauptsatz). *If G is abelian, then*

$$G \text{ is elementary abelian of rank 2 or 3.} \tag{4}$$

Next *suppose that G has class* 2. Then it is sufficient to have $G = \langle H, K\rangle$. Since $G' = [H, K]$, it follows that $G'$ is generated by at most 2 elements. We distinguish 2 cases.

(a) *Let* $|G'| = p$. Then $|G/G'| \leqslant p^3$ and so $p^3 \leqslant |G| \leqslant p^4$. If $|G| = p^3$, then

$$G \simeq E_p \tag{5}$$

and $G = HK$. On the other hand, if $|G| = p^4$, then again there is only one possibility, viz.

$$G \simeq E_p \times C_p.$$

(See for example [5], page 197.) More precisely

$$G = \{(\langle h_1\rangle \times \langle [h_1, k]\rangle) \rtimes \langle k\rangle\} \times \langle h_2\rangle, \tag{6}$$

where $H = \langle h_1, h_2\rangle$ and $K = \langle k\rangle$.

(b) *Let* $|G'| = p^2$. In this case $|G| = p^4$ or $p^5$. But groups of order $p^4$ with derived subgroup of order $p^2$ have class 3 (see [5], page 196). Therefore we must have $|G| = p^5$. Then with the notation above, it follows that $G' = \langle [h_1, k] \rangle \times \langle [h_2, k] \rangle$ and so

$$G = (H \times G') \rtimes K. \tag{7}$$

Now *suppose that* G *has class* 3. Then $G'$ is abelian and $|G| = p^4$ or $p^5$. We distinguish these cases. *Suppose first that*

$$|G| = p^4.$$

Here there is just one possibility, viz.

$$G' \simeq C_p \times C_p \text{ and } G \simeq (C_p \times C_p \times C_p) \rtimes C_p$$

with the elementary abelian subgroup indecomposable as $C_p$-module (again see [5]). Thus G is 2-generator and we may suppose that $G = \langle h_1, k \rangle$ (we keep the notation from (a) above). Then without loss of generality we may assume either $HG'$ is abelian, i.e.

$$G = (\langle h_1 \rangle \times \langle [h_1, k] \rangle \times \langle [h_1,_2 k] \rangle) \rtimes \langle k \rangle \text{ with } h_2 \in \langle h_1 \rangle G' \setminus \langle h_1 \rangle; \tag{8}$$

or $KG'$ is abelian, i.e.

$$G = (\langle k \rangle \times \langle [k, h_1] \rangle \times \langle [k,_2 h_1] \rangle) \rtimes \langle h_1 \rangle \text{ with } h_2 \in \langle [k,_2 h_1] \rangle \setminus 1. \tag{9}$$

For any subgroups H, K of any group, we have

$$HKHK = H\{[h, k] | h \in H, k \in K\}K. \tag{10}$$

In (8) we may assume that either $h_2 \in \zeta_1(G)$ or $[h_2, k]$ is a non-trivial element of $\zeta_1(G)$. In (9) we have $h_2 \in \zeta_1(G)$. Thus using (10) it follows that $H \text{ qn}_4 G$ in both (8) and (9).

Now still with G of class 3, *suppose that* $|G| = p^5$. Here $|G'| = p^2$ or $p^3$ and we consider these cases separately.

(a) *Let* $|G'| = p^2$. Then $|G/G'| = p^3$ and $H \cap G' = K \cap G' = 1$. So $G/\gamma_3(G)$ is the group (6) and we may assume that

$$[h_1, k] \in G' \setminus \gamma_3(G) \text{ and } [h_2, k] \in \gamma_3(G) \setminus 1.$$

Write $z = [h_2, k]$. By the 3-Subgroup Lemma, we have $[h_1, k, h_2] = 1$

and thus $\langle h_2\rangle G'$ is abelian. Again we distinguish 2 cases.

(i) *Suppose that* $HG'$ *is abelian.* Then $HG'$ cannot be indecomposable as K-module, otherwise G would have class 4. Also k cannot centralize $[h_1, k]$, otherwise G would have class 2. Therefore $\langle [h_1, k]\rangle = \langle z\rangle$. Thus we have

$$G = (\langle h_1\rangle \times \langle h_2\rangle \times \langle [h_1, k]\rangle \times \langle [h_1,_2 k]\rangle) \rtimes K \tag{11}$$

and $\langle [h_1,_2 k]\rangle = \langle [h_2, k]\rangle$. Also $G = HKHK$, because it is true modulo $\langle z\rangle$ (where we have group (6)) and it is easy to see that $H \; qn_4 \; G$.

(ii) *Now suppose that* $HG'$ *is not abelian.* In this case we have

$$\langle [h_1, k, h_1]\rangle = \langle z\rangle = \langle [h_2, k]\rangle = \gamma_3(G).$$

So

$$HG' = \langle h_1\rangle \ltimes \langle h_2, [h_1, k], z\rangle \simeq C_p \ltimes (C_p \times C_p \times C_p).$$

Clearly $C_G(G')$ has index p in G and $h_1 \notin C_G(G')$. Therefore without loss of generality we may assume that $k \in C_G(G')$. Then

$$G = H \ltimes (\langle [h_1, k]\rangle \times \langle [h_2, k]\rangle \times K). \tag{12}$$

Here $\langle [h_1, k, h_1]\rangle = \langle [h_2, k]\rangle = \langle z\rangle = \zeta_1(G) = \gamma_3(G)$. Again by considering $G/\langle z\rangle$, it is easy to see that $H \; qn_4 \; G$. The groups (11) and (12) are not isomorphic, because $C_G(G')$ is abelian in (11), but not in (12).

(b) *Still with* G *of order* $p^5$ *and class* 3*, we have to consider the case where* $|G'| = p^3$. We shall see that this possibility cannot occur. For, $G'$ is abelian and therefore

$$HG' \text{ is not abelian.} \tag{13}$$

For, without loss of generality $G = \langle h_1, k\rangle$. Therefore if (13) were false, then $G' = \langle [h_1, k], [h_1,_2 k]\rangle$ of order $p^2$, a contradiction. Hence (13) is true. Also without loss of generality we may assume that $h_2 \in G'$. But $h_2 \notin \zeta_1(G)$. Clearly $\zeta_1(G) \leqslant G'$ and thus $\zeta_1(G)$ must have order p, otherwise we would have $G' = \langle h_2, \zeta_1(G)\rangle$ centralized by H, contradicting (13). It follows that $\gamma_3(G) = \zeta_1(G)$ of order p. But then $G/\gamma_3(G)$ has class 2, order $p^4$ and is generated by 2 elements of order p, a contradiction. Thus there is no group G of class 3 and order $p^5$ with $|G'| = p^3$ in our list.

We conclude by considering *the case where* G *has class* 4. Then G has order $p^5$ and is a group of maximal class. Again $G'$ is abelian, because $[\gamma_2(G), \gamma_3(G)] \leqslant \gamma_5(G) = 1$ and $\gamma_2(G)/\gamma_3(G)$ has order $p$. Of course the upper and lower central series of $G$ coincide. Also $G$ is generated by 2 elements, so we may assume that

$$H \cap G' = \langle h_2 \rangle \text{ of order } p \text{ and } G = \langle h_1, k \rangle.$$

Clearly $G' = \langle [h_1, k] \rangle^G$ and

$$G = (G' \rtimes \langle h_1 \rangle) \rtimes \langle k \rangle. \tag{14}$$

Since $h_2$ cannot belong to $\zeta_1(G)$, $G'$ (as an elementary abelian subgroup of rank 3) has at least 2 fixed points under the $h_1$-action. The fact that $G'$ is abelian means that $G$ is not an ausnahmegruppe, as defined in [3], III, 14.5 (see [3], III, 14.11 Satz). Then it follows that

$$\langle h_1, G' \rangle = C_G(\gamma_2(G)/\gamma_4(G)) = C_G(\gamma_3(G)). \tag{15}$$

Thus

$$[h_1, k] \in \gamma_2(G) \setminus \gamma_3(G), \quad [h_1,_2 k] \in \gamma_3(G) \setminus \gamma_4(G)$$

and $[h_1,_3 k] \in \gamma_4(G) \setminus 1$. Again we distinguish 2 cases depending on where $h_2$ lies.

(a) *Suppose that* $h_2 \in \gamma_3(G)$. Then $[h_2, k] \in \zeta_1(G) \setminus 1$ and $G/\langle [h_2, k] \rangle$ is one of the groups (8) or (9). Therefore $G = HKHK$. Also $H \text{ qn}_4 G$ here. For, let $g \in G$. If $g \in HG'$, then $\langle H, g \rangle$ has class $\leqslant 2$ and so $H \text{ qn}_4 \langle H, g \rangle$. On the other hand, if $g \notin HG'$, then $\langle [h_2, g] \rangle = \langle [h_2, k] \rangle$ and our results above applied to $G$ modulo this central subgroup show that again $H \text{ qn}_4 \langle H, g \rangle$. It follows that in the group (14) we have

$$G' = \langle [h_1, k] \rangle \times \langle h_2 \rangle \times \langle z \rangle,$$

an indecomposable $K$-module with $z \in \zeta_1(G)$. There are 2 possibilities depending on whether $HG'$ is abelian or not. There is just one group in the first case and it is not difficult to see that there is also just one group in the second case. We refer to them as (14A) and (14B) respectively.

(b) *Finally suppose that* $h_2 \notin \gamma_3(G)$. Then, by (15), $HG'$ is abelian and indecomposable as $K$-module. Again we show that $H \text{ qn}_4 G$. It is easy to to see that it suffices to show that

$$G = HKHK. \tag{16}$$

Let $w_1 = [h_1, k]$, $w_2 = [h_{1,2}\, k]$, $w_3 = [h_{1,3}\, k]$. Then we may assume that $h_2 = w_1 w_2^i w_3^j$. Since it suffices to show that

$$G = H\{[h, k^\ell] | h \in H, \ell \in \mathbb{Z}\}K,$$

it also suffices to show that, for any $\alpha$, $\beta$,

$$w_2^\alpha w_3^\beta = h_2^\lambda [h_1^\mu h_2^\nu, k^\sigma] \tag{17}$$

for some $\lambda$, $\mu$, $\nu$ and $\sigma$. For elements $x$, $y$ of a group and a positive integer $n$,

$$[x, y^n] \equiv [x, y]^n [x,_2 y]^{\binom{n}{2}} [x,_3 y]^{\binom{n}{3}} \ldots [x,_n y] \bmod N,$$

where $N$ is the normal closure in $\langle x, y \rangle$ of the group generated by the set of all commutators in $\{y, [x, y]\}$ of weight $\geqslant 2$ in $[x, y]$ (see [4], Corollary 1.1.7). Applying this formula to the commutator in (17), we obtain $N = 1$ and the following congruences modulo p:-

$$\lambda + \sigma\mu \equiv 0,$$

$$\alpha \equiv -i\sigma\mu + \mu\binom{\sigma}{2} + \sigma\nu,$$

$$\beta \equiv -j\sigma\mu + \mu\binom{\sigma}{3} + \nu\binom{\sigma}{2} + i\sigma\nu.$$

Suppose $\sigma = 1$. Then $\alpha \equiv -i\mu + \nu$ and $\beta \equiv -j\mu + i\nu$. Thus eliminating $\nu$, we have $(j - i^2)\mu \equiv i\alpha - \beta$. Therefore if $j \not\equiv i^2$, we have values for $\lambda$, $\mu$, $\nu$ and $\sigma$. Now suppose that $\sigma = 2$. Then

$$\alpha \equiv (-2i + 1)\mu + 2\nu \text{ and } \beta \equiv -2j\mu + (1 + 2i)\nu.$$

Again eliminating $\nu$, we have$(1 + 4j - 4i^2)\mu \equiv (1 + 2i)\alpha - 2\beta$. Therefore if $j \equiv i^2$, then again we have values for $\lambda$, $\mu$, $\nu$ and $\sigma$. So (17) holds and (16) is true.

This final group is isomorphic to (14A), but with $h_2 \in G' \setminus \gamma_3(G)$. We shall refer to it as group (14C) with regard to this embedding of H.

Our classification is now complete. We have 10 non-isomorphic groups, but for 2 of them there are 2 different embeddings of H and K, viz. (8) and (9) and (14A) and (14C).

# 3 Properties of 4-quasinormal subgroups

We begin by using the results obtained in Section 2 to prove that (3) in Section 1 is also true when $\ell = 1$ and the 4-quasinormal subgroup $H$ is elementary abelian of rank 2.

**Theorem 3.1** *Let* $G = \langle H, K \rangle$ *be a finite* $p$*-group* ($p \geqslant 5$) *with* $H$ *an elementary abelian subgroup of rank* 2 *and* $K$ *a cyclic subgroup. Suppose that* $H$ *is* 4*-quasinormal in* $G$*. Then* $K^p$ *is contained in the centre of* $G$*.*

Proof — We show first that

$$K^p \lhd G. \tag{18}$$

When $p \geqslant 7$, this is easy to see. For, $|G/G^p| \leqslant p^5$ and so by [3], III, 10.13 Satz, G is regular. Then $G^p = K^p \lhd G$. However, using the classification in Section 2, we can include the case $p = 5$.

Let $|K| = p^n$. It will be sufficient to prove (18) when $n = 2$. For then, when $n \geqslant 3$, we obtain $K_1 = K^{p^{n-1}} \lhd G$, since

$$H \; qn_4 \; \langle H, K^{p^{n-2}} \rangle.$$

If $K_1 \leqslant H$, then (18) follows from (3). Otherwise induction on $n$ gives $K^p/K_1 \lhd G/K_1$, i.e. (18).

Therefore let $n = 2$, so $K_1 = K^p$. We argue by induction on $|G|$ . When $|G| = p^3$, (18) is clearly true. Thus by induction we may assume that

$$K_1 \leqslant \zeta_2(G) \tag{19}$$

(if a minimal normal subgroup of G lies in H, then (19) follows from (3)). Thus $[H, K_1] \leqslant \zeta_1(G)$ and so $J = \langle H, K_1 \rangle$ has class $\leqslant 2$. Since we may assume that J is not abelian, it must be one of the 3 groups in Section 2 of class 2. In the first case (given by (5)) we have $J \simeq E_p$. Then with $K = \langle k \rangle$ and $k_1 = k^p$, we can assume that $H = \langle h_1 \rangle \times \langle h_2 \rangle$ with $h_2 = [h_1, k_1] \in \zeta_1(G)$ (by (19)). Therefore $|G| \leqslant p^5$ and G has class $\leqslant 4$, i.e. G is regular. Then $G^p = K^p \lhd G$.

In the second case (given by (6)) we have

$$J = \{(\langle h_1 \rangle \times \langle [h_1, k_1] \rangle) \rtimes \langle k_1 \rangle\} \times \langle h_2 \rangle.$$

Again $[h_1, k_1] \in \zeta_1(G)$ and $N = \langle [h_1, k_1], k_1 \rangle \lhd G$ by induction. Also $N \leqslant \zeta_1(G'K)$ since $[k_1, G'] = 1$ (by (19)). Since G/N has class $\leqslant 4$, it follows that $G'K/N$ has class $\leqslant 3$. Hence $G'K$ has class $\leqslant 4$ and so $G'K$ is regular. All the groups classified in Section 2 have exponent $p$, therefore $G'$ (=$[H, K]$) has exponent $p$. Thus $(G'K)^p = K^p \lhd G$.

In the third case when $J$ is the group given by (7), we have 2 distinct minimal normal subgroups of $G$, viz. $\langle [h_1, k_1] \rangle$ and $\langle [h_2, k_1] \rangle$. Since $K_1$ is central in $G$ modulo each one, we have $K_1 \lhd G$.

We have now proved (18). It remains to show that with $|K| = p^n$,

$$K^p \leqslant \zeta_1(G). \tag{20}$$

Let $h \in H$. Using the commutator collection formula (see [4], Proposition 1.1.32 (ii)), we have

$$[k^p, h] \equiv [k, h]^p [k, h, k]^{\binom{p}{2}} \dots [k, h,_{p-1} k] \bmod X, \tag{21}$$

where $X$ is the normal closure in $\langle h, k \rangle$ of the set of all basic commutators in $\{k, [k, h]\}$ of weight $\geqslant p$ and of weight $\geqslant 2$ in $[k, h]$, together with the $p$-th powers of all basic commutators in $\{k, [k, h]\}$ of weight $< p$ and of weight $\geqslant 2$ in $[k, h]$. From the classification in Section 2, $G'$ is abelian modulo $K^p$. Therefore $X = 1$. Also $[k, h,_3 k] \in K^p$ and hence $[k, h,_4 k] = 1$. Again from Section 2 we see that $H^G$ has exponent $p$. Thus (21) gives $[k^p, h] = 1$. Therefore $K^p \leqslant \zeta_1(G)$. □

In [1] it was shown that cyclic 4-quasinormal subgroups of finite $p$-groups ($p \geqslant 5$) are invariant under index-preserving projectivities. Using the classification in Section 2 we can show that this is also true for elementary abelian 4-quasinormal subgroups of rank 2.

**Theorem 3.2** *In finite $p$-groups ($p \geqslant 5$), elementary abelian subgroups of rank 2 that are 4-quasinormal are invariant under index-preserving projectivities.*

Proof — Let $G = \langle H, K \rangle$ be a finite $p$-group ($p \geqslant 5$) with

$$H = \langle h_1 \rangle \times \langle h_2 \rangle \simeq C_p \times C_p,$$

$K = \langle k \rangle$ cyclic of order $p^n$ and $H \operatorname{qn}_4 G$. Let $\overline{G}$ be a second group and

$$\sigma : G \to \overline{G}$$

an index-preserving projectivity. We show that

$$\overline{G} = H^\sigma K^\sigma H^\sigma K^\sigma. \tag{22}$$

We consider first the case $n = 1$. The fact that $G$ has exponent $p$, forcing $\overline{G}$ also to have exponent $p$, is particularly relevant. Then $G$ is one of the 10 groups listed in Section 2, with 2 different embeddings of $H$ and $K$ for two of these groups. In fact it is easy to see in each case that $G \simeq \overline{G}$, but of course projectivities are not always induced by isomorphisms. Clearly $H^\sigma \simeq C_p \times C_p$ and $K^\sigma \simeq C_p$. The first 5 possibilities for $G$ all have class $\leqslant 2$ and $\overline{G} \simeq G$. So all subgroups of $\overline{G}$ are 4-quasinormal.

Next suppose that $G$ has class 3 and order $p^4$. In the 2 isomorphic cases (8) and (9), $G$ is the split extension of an elementary abelian group of rank 3 by $C_p$ with $C_p$ acting indecomposably. So $\overline{G}$ must be the same. Also when $G$ is presented in terms of $H$ and $K$ as in (8), then $\overline{G}$ is presented in terms of $H^\sigma$ and $K^\sigma$ in the same way. Similarly in case (9). Thus again we have (22).

Next let $G$ have class 3 and order $p^5$. So $G'$ has order $p^2$ and in the first case (11) $HG'$ is abelian. Then again $\overline{G} \simeq G$ and $\overline{G}$'s presentation in terms of $H^\sigma$ and $K^\sigma$ is the same as $G$'s presentation in terms of $H$ and $K$ and we have (22). Similarly in (12) when $HG'$ is not abelian, then $\overline{G} \simeq G$ and the above arguments apply.

Finally let $G$ have class 4, i.e. maximal class. When $h_2 \in \gamma_3(G)$ (i.e. in groups (14A) and (14B)), then

$$\langle h_2, k\rangle^\sigma \simeq E_p \ \text{ and } \ [\langle h_2\rangle^\sigma, \langle k\rangle^\sigma] = \langle z\rangle^\sigma = \zeta_1(\overline{G}).$$

So $\overline{G} \simeq G$ and $H^\sigma$ $qn_4$ $\overline{G}$ by the argument in Section 2. When $h_2 \notin \gamma_3(G)$ (i.e. in group (14C)) the same arguments apply. Thus we have proved (22) when $n = 1$.

Now assume that $n \geqslant 2$. Then $K^p \leqslant \zeta_1(G)$, by Theorem 1. Also $G/K^p$ has exponent $p$ and hence $G^p = K^p$. Thus $(G^p)^\sigma = (K^p)^\sigma$, i.e. $(\overline{G})^p = (K^\sigma)^p \lhd \overline{G}$. Therefore, by (22), $H^\sigma$ is 4-quasinormal in $\overline{G}$ modulo $(K^p)^\sigma$. Hence $H^\sigma$ $qn_4$ $\overline{G}$. □

In conclusion we make two remarks. In [1], a subgroup $H$ of a group $G$ was said to be *strongly 4-quasinormal* in $G$ if

$$\langle H, K\rangle = HKHK$$

for *all* subgroups $K$ of $G$. It was shown that cyclic 4-quasinormal subgroups of finite $p$-groups ($p \geqslant 5$) are strongly 4-quasinormal. However, this is not the case for $qn_4$-subgroups that are elementary abelian of rank 2. For example, let $p$ be any odd prime and consider a free nilpotent group of class 2 and rank 4, with free generators $h_1$,

$h_2$, $k_1$ and $k_2$, factored by its p-th power; and impose the additional relations

$$[h_1, h_2] = [k_1, k_2] = 1.$$

Call this group G and let $H = \langle h_1, h_2 \rangle$, $K = \langle k_1, k_2 \rangle$. Then $G = \langle H, K \rangle$, $|G| = p^8$, and every subgroup of G is 4-quasinormal. But H is not strongly 4-quasinormal in G, because $|HKHK| < p^8$.

Also in [2] it was shown that if p is an odd prime, then every quasinormal subgroup of order $p^2$ in a finite p-group G contains a quasinormal subgroup of G of order p. However, *quasinormal* cannot be replaced by 4-*quasinormal* here. For, in the group (14C) in Section 2, the join of any subgroup of H of order p with K has class $\geqslant 3$. Therefore such a subgroup cannot be 4-quasinormal in G.

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Stewart Stonehewer
University of Warwick
Coventry CV4 7AL, England
e-mail: s.e.stonehewer@warwick.ac.uk