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Splitting Properties of Hyper-(Rank One) Groups [1]

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Hermann Heineken zum 80-sten Geburtstag

Abstract

A group G is said to be an $\mathcal{H}_1$-group (a $\mathcal{P}_1$-group, respectively) if it has an ascending (finte, respectively) normal series whose factors have rank 1. Some splitting and conjugacy theorems for groups with an $\mathcal{H}_1$ (or a $\mathcal{P}_1$)-homomorphic image are proved.

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1 Introduction

Recall that a group G has *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. Thus a group has rank 1 if and only if it is locally cyclic, and hence it is isomorphic either to a subgroup of the additive group of rational numbers or to a subgroup of the multiplicative group of complex roots of unity.

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We shall say that a group G is an $\mathcal{H}_1$-*group* if it has an ascending normal series

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

such that each factor $G_{\alpha+1}/G_\alpha$ has rank 1, and G is called a $\mathcal{P}_1$-*group* if the above normal series can be chosen of finite length. In particular, $\mathcal{H}_1$-groups are hyperabelian and all hypercyclic groups have the $\mathcal{H}_1$-property; moreover, $\mathcal{P}_1$-groups are soluble of finite rank and clearly all supersoluble groups have the $\mathcal{P}_1$-property. It is also clear that the class of $\mathcal{P}_1$-groups and that of $\mathcal{H}_1$-groups are closed with respect to subgroups and homorphic images, and that for a finite group the properties $\mathcal{P}_1$ and $\mathcal{H}_1$ coincide and are equivalent to supersolubility. On the other hand, easy examples show that $\mathcal{P}_1$-groups need not be locally supersoluble. The group classes $\mathcal{P}_1$ and $\mathcal{H}_1$ have been introduced and studied in [1], where it was shown that these classes share many of the important embedding properties already known for nilpotent and supersoluble groups. Recently the wider class of groups having an ascending normal series whose factors are abelian of finite rank has been considered by B.A.F. Wehrfritz (see [8] and [9]).

The aim of this paper is to obtain some splitting and conjugacy theorems for abelian-by-$\mathcal{H}_1$ groups. Splitting properties for abelian-by-(locally nilpotent), abelian-by-(locally supersoluble) and abelian-by-hyperfinite groups have been obtained in [5], [6] and [7], respectively.

Our first main result describes a splitting property over torsion-free divisible abelian normal subgroups of finite rank.

Theorem A *Let* G *be a group, and let* A *be a torsion-free divisible abelian normal subgroup of finite rank of* G *which has no torsion-free divisible* G*-sections of rank* 1*. If* G/A *is either an* $\mathcal{H}_1$*-group or a locally* $\mathcal{P}_1$*-group, then* G *splits over* A *and all complements of* A *in* G *are conjugate.*

It is well-known that results ensuring that a group splits conjugately over an abelian normal subgroup are equivalent to the vanishing of certain cohomology groups of low degree. Although our proofs here are homology free, the above theorem can also be interpreted in the following way.

Corollary *Let* Q *be a group which is either* $\mathcal{H}_1$ *or locally* $\mathcal{P}_1$*, and let* A *be a* Q*-module whose additive group is torsion-free divisible of finite rank. If*

A has no Q-submodules B and C such that $C < B$ and the additive group of B/C is torsion-free divisible of rank 1, then $H^1(Q, A) = H^2(Q, A) = 0$.

Our other main result is a nearly splitting theorem over periodic abelian normal subgroups of finite rank.

Theorem B *Let G be a group, and let A be an abelian normal subgroup of G satisfying the minimal condition on subgroups. If A has no infinite G-sections of rank 1 and G/A is a finitely generated $\mathcal{P}_1$-group, then there exists a finitely generated $\mathcal{P}_1$-subgroup E of G such that $G = AE$ and $A \cap E$ is finite. Moreover, if G splits over A and $\mathfrak{L}$ is the set of all complements of A in G, then A contains a finite G-invariant subgroup B such that $\{XB \mid X \in \mathfrak{L}\}$ is a class of conjugate subgroups of G.*

Most of our notation is standard and can for instance be found in [4].

2 Preliminary results on projectors

Let $\mathfrak{X}$ be an ***H***-closed group class, i.e. a class of groups such that all homomorphic images of $\mathfrak{X}$-groups belong to $\mathfrak{X}$. A subgroup X of a group G is called an *$\mathfrak{X}$-projector* if XN/N is a maximal $\mathfrak{X}$-subgroup of G/N for each normal subgroup N of G. In particular, any $\mathfrak{X}$-projector is a maximal $\mathfrak{X}$-subgroup.

The following reduction lemma is quite useful in order to find projectors of large groups.

Lemma 2.1 *Let $\mathfrak{X}$ be an **H**-closed group class, and let G be a group and N a normal subgroup of G. If S/N is an $\mathfrak{X}$-projector of G/N, then every $\mathfrak{X}$-projector of S is also an $\mathfrak{X}$-projector of G. Moreover, if G/N and S have a unique conjugacy class of $\mathfrak{X}$-projectors, then also the $\mathfrak{X}$-projectors of G are conjugate.*

Proof — Let X be any $\mathfrak{X}$-projector of S. Consider a normal subgroup K of G, and let H/K be an $\mathfrak{X}$-subgroup of G/K containing XK/K. As S/N is an $\mathfrak{X}$-group and X is an $\mathfrak{X}$-projector of S, we have $S = XN$ and hence $SK = XNK \leqslant HN$. Then SK/NK is contained in the $\mathfrak{X}$-subgroup HN/KN of G/KN, and so $SK = HN$ since S/N is an $\mathfrak{X}$-projector of G/N. In particular, H is a subgroup of SK and so

$$H = SK \cap H = K(S \cap H).$$

It follows that

$$H \cap S/K \cap S \simeq H/K$$

is an $\mathfrak{X}$-subgroup of $S/K \cap S$ containing $X(K \cap S)/K \cap S$, and hence $H \cap S = X(K \cap S)$. Therefore $H = K(S \cap H) = KX$ and X is an $\mathfrak{X}$-projector of G. Suppose now that all $\mathfrak{X}$-projectors of G/N are conjugate to S/N and all $\mathfrak{X}$-projectors of S are conjugate to X, and let Y be any other $\mathfrak{X}$-projector of G. Then YN/N is an $\mathfrak{X}$-projector of G/N and so Y is contained in S^g for a suitable element g of G. Clearly, Y and X^g are $\mathfrak{X}$-projectors of S^g and hence there exists $z \in S^g$ such that $Y = X^{gz}$. Therefore G has a unique conjugacy class of $\mathfrak{X}$-projectors. □

Projectors play a special role in the theory of formations. Recall that an $\boldsymbol{H}$-closed group class $\mathfrak{F}$ is a *formation* if it is also $\boldsymbol{R}_0$-closed, i.e. if $G/N_1 \cap N_2$ belongs to $\mathfrak{F}$ whenever N_1 and N_2 are normal subgroups of a group G such that G/N_1 and G/N_2 are $\mathfrak{F}$-groups. Our next result relates in certain cases the existence of projectors with respect to a formation to a splitting theorem.

Lemma 2.2 *Let $\mathfrak{F}$ be a formation, and let G be a group and N an abelian minimal normal subgroup of G. If G is not an $\mathfrak{F}$-group but G/N belongs to $\mathfrak{F}$, then the $\mathfrak{F}$-projectors of G are precisely the complements of N in G.*

Proof — Let X be any $\mathfrak{F}$-projector of G. As G/N is an $\mathfrak{F}$-group, we have $G = XN$, so that N is not contained in X and $X \cap N$ is a proper subgroup of N which is normal in G; thus $X \cap N = \{1\}$ and X is a complement of N in G. Conversely, let X be a complement of N in G, so that in particular $X \simeq G/N$ belongs to $\mathfrak{F}$. Let K be any normal subgroup of G and let H/K be an $\mathfrak{F}$-subgroup of G/K containing XK/K; then

$$H = XN \cap H = X(H \cap N)$$

with $H \cap N$ normal in $G = HN$. If N is contained in H, then $H = G$ and G/K is an $\mathfrak{F}$-group; in this case K must contain N and $XK = G$. Suppose now that N is not contained in H, so that $H \cap N = \{1\}$ and hence $H = X$. Therefore X is an $\mathfrak{F}$-projector of G. □

A normal subgroup H of a group G is said to be $\mathcal{H}_1$*-embedded* in G if it has an ascending G-invariant series

$$\{1\} = H_0 < H_1 < \ldots < H_\alpha < H_{\alpha+1} < \ldots < H_\tau = H$$

whose factors have rank 1. It is easy to show that any group G contains a largest $\mathcal{H}_1$-embedded normal subgroup $\overline{H}(G)$, and $\overline{H}(G)$ coincides with the intersection of all normal subgroups N of G such that G/N does not contain non-trivial normal subgroups of rank 1 (see [4] Part 1, Lemma 1.39.3). The following property of the characteristic subgroup $\overline{H}(G)$ has been proved in [1].

Lemma 2.3 *Let* G *be a group containing a finite normal subgroup* N *such that the factor group* G/N *has the* $\mathcal{H}_1$*-property. Then the subgroup* $\overline{H}(G)$ *has finite index in* G.

Lemma 2.4 *Let* G *be a group, and let* A *be a finite abelian minimal normal subgroup of* G. *If* A *is not cyclic and* G/A *is an* $\mathcal{H}_1$*-group, then* G *splits over* A *and all complements of* A *in* G *are conjugate.*

Proof — Since A is non-cyclic, we have $A \cap \overline{H}(G) = \{1\}$. Moreover, the factor group $\overline{G} = G/\overline{H}(G)$ is finite by Lemma 2.3 and $\overline{G}/\overline{A}$ is supersoluble. Thus the supersoluble projectors of $\overline{G}$ form a unique conjugacy class of complements of $\overline{A}$ in $\overline{G}$. If $K/\overline{H}(G)$ is one such complement, then $G = AK$ and

$$A \cap K \leqslant A \cap \overline{H}(G) = \{1\},$$

so that K is a complement of A in G. Let L be any complement of A in G; then L is a maximal subgroup of G and it has the property $\mathcal{H}_1$. As G is not an $\mathcal{H}_1$-group, it follows that L contains $\overline{H}(G)$ and hence L is a conjugate of K. The lemma is proved. □

Lemma 2.5 *Let* G *be a group containing a finite soluble normal subgroup* N *such that* G/N *has the property* $\mathcal{H}_1$. *Then* G *has* $\mathcal{H}_1$*-projectors and all such projectors are conjugate and self-normalizing.*

Proof — It can obviously be assumed that G is not an $\mathcal{H}_1$-group. Suppose first that N is a minimal normal subgroup of G. As N cannot be cyclic, it follows from Lemma 2.4 that there exists a unique conjugacy class of complements of N in G, and such complements are precisely the $\mathcal{H}_1$-projectors of G; in particular, in this case the $\mathcal{H}_1$-projectors of G are maximal subgroups and hence they are self-normalizing. Assume now that M is a normal subgroup of G such that $\{1\} \neq M < N$. By induction on the order of N, the group G/M has a unique conjugacy class of self-normalizing $\mathcal{H}_1$-projectors; moreover, if S/M is one of such projectors, again by induction we have that also S has a unique conjugacy class of self-normalizing $\mathcal{H}_1$-projectors. It follows from Lemma 2.1 that G itself contains $\mathcal{H}_1$-projectors and these are conjugate. Finally, if T is any $\mathcal{H}_1$-projector of S, then $S = TM$ and hence $N_G(T) \leqslant N_G(S) = S$, so that $N_G(T) = N_S(T) = T$. Therefore all $\mathcal{H}_1$-projectors of G are self-normalizing and the lemma is proved. □

The same argument used in the proof of Lemma 2.5 shows that for $\mathcal{P}_1$-projectors a corresponding result holds.

Lemma 2.6 *Let G be a group containing a finite soluble normal subgroup N such that G/N has the property $\mathcal{P}_1$. Then G has $\mathcal{P}_1$-projectors and all these projectors are conjugate and self-normalizing.*

3 Splitting theorems

Let G be a group. If g is any element of G and m, n are integers, we consider the mapping

$$g(m,n) : x \in G \longmapsto x^{-m}(x^n)^g \in G.$$

We have obviously that $g(0,0)$ is the zero map, $g(1,0)$ is the inversion map and $g(0,1)$ is the conjugation by g. Moreover, $g(1,1)$ is the commutator map $x \mapsto [x,g]$ and if G is a nilpotent group there is a positive integer k such that $xg(1,1)^k = 1$ for all x and g in G. We shall say that the group G has the $(*)$-*property* if for all elements x, g of G there exist a positive integer k and non-zero integers $m_1, \dots, m_k, n_1, \dots, n_k$ (depending on x and g) such that

$$x\Big(\prod_{i=1}^{k} g(m_i, n_i)\Big) = 1.$$

Observe that, if A is an abelian normal subgroup of a group G such that $[A, G'] = \{1\}$, then each $g(m,n)$ induces on A a G-endomorphism and such G-endomorphisms pairwise commute; this applies in particular when A is a normal subgroup of rank 1. In this special case, if a is any element of A and $g \in G$, as $\langle a \rangle \cap \langle a^g \rangle \neq \{1\}$, there exist non-zero integers m, n such that $a^m = (a^g)^n$ and hence $ag(m,n) = 1$. From this remark we can deduce the following result.

Lemma 3.1 *Every $\mathcal{H}_1$-group G has the $(*)$-property.*

Proof — Let

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

be an ascending normal series whose factors have rank 1, and assume that $\mu \leqslant \tau$ is a non-zero ordinal number such that the $(*)$-property holds for all pairs (y, g), with $y \in G_\alpha$ for some $\alpha < \mu$ and $g \in G$. If μ is a limit ordinal, then

$$G_\mu = \bigcup_{\alpha<\mu} G_\alpha$$

and hence $(*)$ also holds for the pairs in $G_\mu \times G$. Suppose now that μ is not a limit, so that we can consider $\mu - 1$ and $G_\mu/G_{\mu-1}$ is a group of rank 1. If x is any element of G_μ and $g \in G$, there are non-zero integers m, n such that $xg(m, n)$ lies in $G_{\mu-1}$; then for some positive integer k there exist non-zero integers $m_1, \ldots, m_k, n_1, \ldots, n_k$ such that

$$x\Big(g(m, n) \prod_{i=1}^{k} g(m_i, n_i)\Big) = 1.$$

Therefore by transfinite induction it follows that $(*)$ holds for all pairs of elements of G, and hence G has the $(*)$-property. □

As the class of groups with the $(*)$-property is obviously local, we have the following consequence.

Corollary 3.2 *Let* G *be a locally* $\mathcal{P}_1$*-group. Then* G *has the* $(*)$*-property.*

Lemma 3.3 *Let* G *be a group and let* A *be an abelian normal subgroup of* G*. If* g *is an element of* G *such that* $g(m, n)$ *induces the zero map on* A *for some non-zero integers* m *and* n*, then* g *normalizes all divisible subgroups of* A*.*

Proof — Let B be a divisible subgroup of A, and let b be any element of B. Then $b = u^n$ for some $u \in B$; as $ug(mn, n) = 1$, we have

$$b^g = (u^n)^g = u^m \in B.$$

Similarly, it follows from the identity $(u^m)^{g^{-1}} = u^n$ that also $b^{g^{-1}}$ belongs to B. Therefore $B^g = B$ and g normalizes B. □

Let G be a group and M a G-module. Recall that a *derivation* from G into M is a map $\delta : G \longrightarrow M$ such that

$$(xy)\delta = (x\delta)y + y\delta$$

for all x, y in G. Then $1\delta = 0$ and the set

$$\ker\delta = \{x \in G \mid x\delta = 0\}$$

is a subgroup of G, which is called the *kernel* of δ.

Next lemma is essentially the metabelian case of our main results.

Lemma 3.4 *Let G be a metabelian group, and let A be a divisible abelian normal subgroup of G with finite rank $r > 1$ such that $[A, G'] = \{1\}$. Suppose that A does not contain proper non-trivial divisible normal subgroups of G and G/A has the property $(*)$. Then the following hold:*

(a) *if A is torsion-free, then G splits over A and all complements of A in G are conjugate;*

(b) *if A is a p-group for some prime number p and G/A is finitely generated, then there is a subgroup S of G such that $G = AS$ and $A \cap S$ is finite. Moreover, if G splits over A and $\mathfrak{L}$ is the set of all complements of A in G, there exists a finite G-invariant subgroup B of A such that $\{XB \mid X \in \mathfrak{L}\}$ is a conjugacy class of subgroups of G.*

Proof — Since A has no proper non-trivial divisible subgroups, it follows from Lemma 3.3 that G contains an element g such that $g(m, n)$ induces a non-zero G-endomorphism on A for all non-zero integers m and n. In particular, $Ag(m, n) = A$ for all $m, n \neq 0$. Let x be any element of G. As G/A has the $(*)$-property, there exists a positive integer k and non-zero integers $m_2, \dots, m_k, n_2, \dots, n_k$ such that the element

$$[x, g] \prod_{i=2}^{k} g(m_i, n_i)$$

belongs to A (where $[x, g] \in A$ if $k = 1$); put also $m_1 = n_1 = 1$. Thus

$$[x, g] \prod_{i=2}^{k} g(m_i, n_i) = a \prod_{i=2}^{k} g(m_i, n_i),$$

for some $a \in A$, because

$$A \prod_{i=2}^{k} g(m_i, n_i) = A.$$

On the other hand, G' is abelian and $[A, G'] = \{1\}$, so that

$$(xa^{-1})\prod_{i=1}^{k} g(m_i, n_i) = [xa^{-1}, g]\prod_{i=2}^{k} g(m_i, n_i)$$
$$= ([x, g][a, g]^{-1})\prod_{i=2}^{k} g(m_i, n_i)$$
$$= \Big([x, g]\prod_{i=2}^{k} g(m_i, n_i)\Big)\Big([a, g]^{-1}\prod_{i=2}^{k} g(m_i, n_i)\Big)$$
$$= \Big(x\prod_{i=1}^{k} g(m_i, n_i)\Big)\Big(a\prod_{i=1}^{k} g(m_i, n_i)\Big)^{-1} = 1.$$

Thus xa^{-1} belongs to the set $S_G(g)$ consisting of all elements y of G such that $y\alpha = 1$ for some function

$$\alpha = g(1, 1)g(m_2, n_2)\dots g(m_r, n_r)$$

and suitable non-zero integers $m_2, \dots, m_r, n_2, \dots, n_r$. Therefore $G = S_G(g)A$. Since G' is abelian, again the identity

$$y\big(g(1, 1)g(m_2, n_2)\dots g(m_r, n_r)\big) = [y, g]\big(g(m_2, n_2)\dots g(m_r, n_r)\big)$$

yields that $S_G(g)$ is a subgroup of G.

We distinguish the two cases. Suppose first that A is torsion-free. Then the map $g(m, n)$ induces an automorphism on A for all non-zero integers m and n, so that $A \cap S_G(g) = \{1\}$ and $S_G(g)$ is a complement of A in G. Let L be any complement of A in G, so that

$$G = LA = L[A, g],$$

and there exist elements y of L and a of A such that $g = y[a, g]$. Thus $y = g^a$ and it follows easily that

$$L = S_G(y) = S_G(g)^a.$$

Therefore all complements of A in G are conjugate.

Assume now that A is a p-group for some prime number p and G/A is finitely generated. Then $G = AE$ for some finitely generated subgroup $E = \langle y_1, \dots, y_n\rangle$, and since G/A has the $(*)$-property, for each $j = 1, \dots, n$ there exist a positive integer $k(j)$ and non-zero integers

$$m_{j,1}, \dots, m_{j,k(j)}, n_{j,1}, \dots, n_{j,k(j)}$$

such that the element

$$[y_j, g] \prod_{i=1}^{k(j)} g(m_{j,i}, n_{j,i})$$

belongs to A. The mapping

$$\delta = g(1,1) \prod_{j=1}^{n} \prod_{i=1}^{k(j)} g(m_{j,i}, n_{j,i})$$

is a derivation from G into the abelian group G', and $E\delta$ is contained in A because the $g(m,n)$'s commute on G'. As $A\delta = A$, for each element y of E we have $y\delta = a\delta$ for some $a \in A$ and ya^{-1} belongs to the subgroup

$$\ker\delta = \{x \in G \mid x\delta = 1\}.$$

Therefore $G = A\ker\delta$. Moreover, $B = A \cap \ker\delta$ is a proper G-invariant subgroup of A and hence it is finite.

Suppose finally that G splits over A, and let X be any complement of A in G. Then $g = x[a,g]$ for suitable elements x of X and a of A, so that $x = g^a$ and $xA = gA$. Let δ^* be the derivation of G into G' obtained by replacing g by x in the definition of δ. Then X is contained in $\ker\delta^* = (\ker\delta)^a$ and

$$\ker\delta^* = X(A \cap \ker\delta^*) = X(A \cap \ker\delta) = XB,$$

and hence XB is conjugate to $\ker\delta$. The proof is complete. □

Proof of Theorem A — Suppose that B is a non-trivial G-invariant subgroup of A such that A/B is not periodic, and let B^*/B be the subgroup consisting of all elements of finite order of A/B. Then B^* is a proper G-invariant subgroup of A and A/B^* is torsion-free. By induction on the rank of A, it can be assumed that A/B^* has a complement X/B^* in G/B^*. Clearly, B^* has no torsion-free divisible X-sections. Moreover, B^* is a pure subgroup of A, so that it is divisible and again by induction it has a complement Y in X. Then Y is also a complement of A in G. It is also easy to show that if the complements of A/B^* in

G/B^* are conjugate and the complements of B^* in X are conjugate, then also the complements of A in G form a single conjugacy class. Therefore it can be assumed without loss of generality that G acts rationally irreducibly on A. In particular, A cannot contain proper non-trivial divisible G-invariant subgroups.

Since G/A is an $\mathcal{H}_1$-group (or a locally $\mathcal{P}_1$-group), its commutator subgroup K/A is hypercentral (or locally nilpotent, respectively), and so there exists a unique conjugacy class of self-normalizing hypercentral (or locally nilpotent, respectively) supplements of A in K (see [2], Theorem 7). If S is one such supplement, application of the Frattini argument yields that $G = AN_G(S)$, so that $A \cap N_G(S) = A \cap S$ is a normal subgroup of G. Suppose first that $[A, K] \neq \{1\}$. Since the intersection $A \cap Z(K)$ is a normal subgroup of G, and G acts rationally irreducibly on its torsion-free normal subgroup A, it follows that $A \cap Z(K) = \{1\}$. In particular,

$$A \cap N_G(S) = A \cap S = \{1\},$$

so that $N_G(S)$ is a complement of A in G. Let U be any other complement of A in G. Then $K \cap U$ is a complement of A in K and of course $U \leqslant N_G(K \cap U)$. On the other hand,

$$A \cap N_G(K \cap U) = A \cap C_G(K \cap U) \leqslant A \cap Z(K) = \{1\}.$$

It follows that $N_G(K \cap U) = U$, and so U is conjugate to $N_G(S)$ as $K \cap U$ is conjugate to S. Therefore we may assume that $[A, K] = \{1\}$, so that in particular $[A, G'] = \{1\}$.

Suppose that G/A is an $\mathcal{H}_1$-group. As A is contained in the centre of K, we have that $K/Z(K)$ is $\mathcal{H}_1$-embedded in $G/Z(K)$, and hence K' is an $\mathcal{H}_1$-embedded normal subgroup of G (see [1], Lemma 5). It follows that

$$G'' \leqslant K \leqslant \overline{H}(G),$$

so that the factor group $G/\overline{H}(G)$ is metabelian. Clearly, $A \cap \overline{H}(G) = \{1\}$, and so Lemma 3.4 yields that $A\overline{H}(G)/\overline{H}(G)$ has a complement $V/\overline{H}(G)$ in $G/\overline{H}(G)$ and all such complements are conjugate. Then $G = AV$ and

$$A \cap V = A \cap \big(A\overline{H}(G) \cap V\big) = A \cap \overline{H}(G) = \{1\},$$

so that V is a complement of A in G. Let W be any complement of A in G. As W is an $\mathcal{H}_1$-group, also the product $W\overline{H}(G)$ belongs to the class $\mathcal{H}_1$. On the other hand, $A \cap W\overline{H}(G)$ is a normal subgroup of G,

so that
$$A \cap W\overline{H}(G) = \{1\}$$
and hence $\overline{H}(G)$ is contained in W. Therefore all complements of A in G are conjugate.

Assume finally that the group G/A is locally $\mathcal{P}_1$. As $[A, G'] = \{1\}$, the mapping $g(m, n)$ induces on A a G-endomorphism for every element g of G and for all integers m, n. Moreover, such G-endomorphism is either the zero map or an automorphism of A, because G acts rationally irreducibly on A. For each $g \in G$, consider the subgroup $S_G(g)$ consisting of all elements x of G such that

$$x \prod_{i=1}^{k} g(m_i, n_i) = 1$$

for suitable non-zero integers $m_1, \dots, m_k, n_1, \dots, n_k$ As in the proofs of Lemma 4 and Lemma 5 of [2], it can be proved that

$$G'' \leqslant \bigcap_{g \in G} S_G(g).$$

It follows that for every $g \in G$ there exist non-zero integers $m = m(g)$ and $n = n(g)$ such that the function $g(m, n)$ acts as the zero map on $A \cap G''$. Let a be any element of $A \cap G''$. Then for each $g \in G$, we have $(a^n)^g = a^m$, where $m = m(g)$ and $n = n(g)$, and hence the normal closure $\langle a \rangle^G$ has rank 1. Since A has no non-trivial G-invariant subgroups of rank 1, we deduce that $A \cap G'' = \{1\}$. It follows from Lemma 3.4 that AG''/G'' has a complement in G/G'' and all its complements are conjugate. If V/G'' is a complement of AG''/G'' in G/G'', then $G = AV$ and

$$A \cap V = A \cap (AG'' \cap V) = A \cap G'' = \{1\}.$$

Thus V is a complement of A in G. On the other hand, if W is any complement of A in G, we have

$$G' \leqslant C_G(A) = AW \cap C_G(A) = A\big(W \cap C_G(A)\big),$$

and so
$$G'' \leqslant W \cap C_G(A) \leqslant W.$$

It follows that W/G'' is a complement of AG''/G'' in G/A, and hence the complements of A in G form a single conjugacy class. The theorem is proved. □

A famous theorem of P. Hall proves that if N is a nilpotent normal subgroup of a group G and the factor group G/N' is nilpotent, then G itself is nilpotent. In order to prove our second main result, we need the following lemma of "Hall type" for the $\mathcal{P}_1$-property.

Lemma 3.5 *Let G be a group, and let N be a nilpotent normal subgroup of G such that the factor group G/N' has the $\mathcal{P}_1$-property. Then G is a $\mathcal{P}_1$-group.*

Proof — Let c be the nilpotency class of N. The statement is obvious if $c \leqslant 1$, and so we may suppose that $c > 1$. By induction on c, it can be assumed that the $\mathcal{P}_1$-property holds for the factor group $G/Z(N)$. Then $N/Z(N)$ is $\mathcal{P}_1$-embedded in $G/Z(N)$, and so N' is $\mathcal{P}_1$-embedded in G (see [1], Lemma 2). Therefore G is a $\mathcal{P}_1$-group. □

Proof of Theorem B — Let D be the largest divisible subgroup of A. Clearly, D is normal in G and the index $|A : D|$ is finite. Assume first that $D = \{1\}$, so that A is finite and it follows from Lemma 2.6 that G contains a $\mathcal{P}_1$-projector S. Then $G = SA$, and S has finite index in G, so that in particular S is finitely generated. In this case, in order to obtain the second part of the statement it is enough to take $B = A$.

Suppose that $D \neq \{1\}$, so that D is a direct product of $n \geqslant 1$ Prüfer subgroups, and we shall prove the statement by induction on n. Let D^* be a proper divisible non-trivial G-invariant subgroup of D. As the hypotheses are obviously inherited by the factor group G/D^*, by induction there exists a finitely generated $\mathcal{P}_1$-subgroup E^*/D^* of G/D^* such that $G = E^*A$ and the index $|E^* \cap A : D^*|$ is finite. Of course, D^* has no infinite E^*-invariant sections of rank 1, and so again by induction E^* contains a finitely generated $\mathcal{P}_1$-subgroup E such that $E^* = ED^*$ and $E \cap D^*$ is finite. Then

$$G = E^*A = ED^*A = EA$$

and $E \cap A$ is finite, because D^* has finite index in $E^* \cap A$. Assume furthermore that G splits over A, and $\mathfrak{L}$ be the set of all complements of A in G. Then $\overline{G} = G/D^*$ splits over $\overline{A} = A/D^*$ and

$$\overline{\mathfrak{L}} = \{\overline{X} \mid X \in \mathfrak{L}\}$$

is a set of complements of $\overline{A}$ in $\overline{G}$, where $\overline{X} = XD^*/D^*$ for each $X \in \mathfrak{L}$, so that the induction assumption yields that $\overline{A}$ contains a finite $\overline{G}$-invariant subgroup $\overline{B} = B^*/D^*$ such that

$$\{\overline{XB} \mid \overline{X} \in \overline{\mathfrak{L}}\}$$

is contained in a class of conjugate subgroups of $\overline{G}$. Choose an element X of $\mathfrak{L}$. As X is a complement of B^* in XB^*, again by induction there exists a finite subgroup B of B^*, which is X-invariant and so also G-invariant, such that UB is conjugate to XB for each complement U of B^* in XB^*. Let Y be any other element of $\mathfrak{L}$. Then $XB^* = Y^gB^*$ for some element g of G, so that Y^g is a complement of B^* in XB^* and hence there exists $h \in G$ such that $Y^gB = X^hB$. It follows that the set $\{XB \mid X \in \mathfrak{L}\}$ is a class of conjugate subgroups of G.

Therefore it can be assumed that D has no infinite proper G-invariant subgroups, so that in particular it is a p-group for some prime number p. As $A = WD$ for a suitable finite characteristic subgroup W, we may further suppose without loss of generality that $A = D$ is divisible. Since G/A is a $\mathcal{P}_1$-group, it follows that its commutator subgroup K/A is nilpotent.

Suppose first that the normal subgroup $[A, K]$ of G is properly contained in A. Since $[A, K]$ is a divisible and A has no infinite proper G-invariant subgroups, it follows that $[A, K] = \{1\}$, so that in particular K is nilpotent and $[A, G'] = \{1\}$. If A is contained in K', the factor group G/K' has the $\mathcal{P}_1$-property, so that G is a $\mathcal{P}_1$-group by Lemma 3.5, and so A has rank 1. This contradiction shows that $A \cap K'$ is a proper subgroup of A, and hence it is finite. In particular, the normal subgroup $A \cap G''$ of G is finite, and so it is enough to prove that the statement holds for the factor group $G/A \cap G''$. Thus without loss of generality it can be assumed that $A \cap G'' = \{1\}$. In this case the statement follows from Lemma 3.4 applied to the metabelian group G/G''.

Assume now that $[A, K] = A$, so that K cannot be nilpotent. As K/A is nilpotent, there exists a nilpotent subgroup S of K such that $AS = K$ and the index $|N_K(S) : S|$ is finite (see [3], Theorem 4). Assume for a contradiction that $A \cap S$ is infinite, so that it contains an infinite S-invariant subgroup C whose proper S-invariant subgroups are finite. Clearly, C is contained in $Z(S)$, since S is nilpotent, so that C also lies in the centralizer $C_A(K)$. Then $C_A(K)$ is an infinite normal subgroup of G, and hence $C_A(K) = A$ and $[A, K] = \{1\}$. This contradiction shows that $A \cap S$ must be finite, so that also its normal closure $N = (A \cap S)^G$ is finite. Thus we may replace G by G/N, and hence that $A \cap S = \{1\}$. Then

$$\{S^x \mid x \in G\}$$

is a set of complements of A in K, and hence there exists a finite K-invariant subgroup E of A such that

$$\{S^x E \mid x \in G\}$$

is contained in a class of conjugate subgroups of K (see [3], Theorem 4). Moreover, by replacing E by its normal closure E^G, we may also suppose that E is normal in G. Therefore

$$G = KN_G(SE) = AN_G(SE).$$

If $A \leqslant N_G(SE)$, the subgroup SE is normal in G and

$$A = [A, K] \leqslant K' \leqslant SE,$$

since $K = A(SE)$, and this is a contradiction because in this case $A = E(A \cap S)$ would be finite. It follows that the normal subgroup $A \cap N_G(SE)$ of G is properly contained in A, so that it is finite and by Lemma 2.6 there exists a finitely generated $\mathcal{P}_1$-subgroup U of $N_G(SE)$ such that

$$N_G(SE) = \big(A \cap N_G(SE)\big)U.$$

Then

$$G = AN_G(SE) = AU,$$

and the first part of the proof is established also in this case.

Again under the assumption $[A, K] = A$, suppose finally that G splits over A, and let $\mathfrak{L}$ be the set of all complements of A in G. For each $X \in \mathfrak{L}$, put $X^* = X \cap K$, so that

$$\mathfrak{L}^* = \{X^* \mid X \in \mathfrak{L}\}$$

is a set of complements of A in K. As K/A is nilpotent but K is not nilpotent, we have $N_G(X^*) \neq G$ for all $X \in \mathfrak{L}$. Moreover, X is obviously contained in $N_G(X^*)$, so that $G = AN_G(X^*)$ and hence $A \cap N_G(X^*)$ is a proper G-invariant subgroup of A. Then $A \cap N_G(X^*)$ is finite, and so the index $|N_K(X^*) : X^*|$ is finite for every $X \in \mathfrak{L}$.

Application of [3], Theorem 4, to the abelian-by-nilpotent group K yields that the set
$$\{X^*B^* \mid X^* \in \mathfrak{L}^*\}$$
lies in a class of conjugate subgroups of K for a suitable finite K-invariant subgroup B^* of A, which can be chosen to be characteristic in A. Then
$$\{N_G(X^*B^*) \mid X^* \in \mathfrak{L}^*\}$$
is a collection of conjugate proper subgroups of G. On the other hand, the intersection $B = A \cap N_G(X^*B^*)$ is a normal subgroup of G, so that it is finite and independent on the choice of X^*. Moreover, we have
$$N_G(X^*B^*) = AX \cap N_G(X^*B^*) = X\big(A \cap N_G(X^*B^*)\big) = XB,$$
and hence $\{XB \mid X \in \mathfrak{L}\}$ is a class of conjugate subgroups of G. The proof is complete. □

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