Advances in Group Theory and Applications

1 (2016), pp. 77–96 ISSN: 2499-1287
DOI: 10.4399/97888548908176

On the Structure of Groups whose Non-Abelian Subgroups are Serial

M.R. Dixon — L.A. Kurdachenko — N.N. Semko

(*Received Jan. 7, 2016; Accepted Feb. 12, 2016 — Communicated by F. de Giovanni*)

To Hermann Heineken, for his 80th birthday

Abstract

Necessary and sufficient conditions are given for a locally finite group to have all non-abelian subgroups serial. We also obtain results for groups whose non-abelian subgroups are permutable.

Mathematics Subject Classification (2010): 20E15, 20E25, 20F50

Keywords: serial subgroup; locally finite group; permutable subgroup; metahamiltonian group; metaquasihamiltonian group.

1 Introduction

Substantial efforts have been made to classify groups all of whose proper subgroups have given properties. This type of research was started by R. Dedekind [5], where he classified the finite groups with all subgroups normal. More generally, other authors have considered groups with all subgroups subnormal, one highlight being the theorem of Möhres [23] that such a group is soluble. Further details concerning groups with all subgroups subnormal can be found in [20] and [19].

In a different direction, G.A. Miller and H. Moreno [22] obtained the structure of finite groups all of whose proper subgroups are abelian, a work generalized by O.J. Schmidt [35] who showed that finite groups with all proper subgroups nilpotent are soluble. It is not possible to extend this theorem to infinite groups since, as is well-known, there are 2-generator infinite simple groups with all proper subgroups abelian (see, for example, the book of A.Yu. Ol'shanskii [26]). On the other hand, one consequence of the amazing paper [1] shows that an infinite locally graded group with all proper subgroups nilpotent-by-Chernikov is necessarily soluble.

Many authors now are interested in groups in which only certain subgroups have some given property. Among the first results of this type are those of Romalis and Sesekin [32, 33, 34] who discussed metahamiltonian groups–those groups with all non-abelian subgroups normal. V.T. Nagrebetskij [24] and A.A. Mahnev [21] studied finite metahamitonian groups and indeed Nagrabetskij discussed finite groups whose non-nilpotent subgroups are normal. However the full description of metahamiltonian groups was obtained in a series of papers by N.F. Kuzenny and N.N. Semko (see [12, 13, 14, 15, 16, 17, 18]).

In order to broaden the discussion, Phillips and Wilson [27] studied a certain class of groups, containing the class of locally finite groups, satisfying the minimal condition on non-serial, non-locally nilpotent subgroups. This very interesting paper also included relevant results (Theorem C(i) in particular) concerning groups with all subgroups serial or abelian. Because of the existence of Tarski monsters they restricted attention to groups G such that every finitely generated non-nilpotent subgroup of G has a finite non-nilpotent image. In a follow-up work, Bruno and Phillips [3] continued this theme by considering groups whose non-normal subgroups are locally nilpotent. To conclude this short relevant history of the subject we mention the papers of H. Smith [37, 36] who studied groups with all non-subnormal subgroups nilpotent.

A more detailed idea of the structure of locally finite groups with all subgroups abelian or subnormal was obtained in the paper [10], where necessary conditions were obtained. In our current paper our main results give necessary and sufficient conditions for a locally finite group to have all non-abelian subgroups serial. Of course, every subgroup of a locally nilpotent group is serial (see [30, 12.4.4], for example) and for locally finite groups the converse also holds. Thus we may suppose, as is usually the case in [27], that our groups are not locally nilpotent. It is then a consequence of [27, Theorem C(i)] that such a locally finite group with all non-abelian subgroups serial is either centre-by-finite or Chernikov. It is easy to see that in the first case the serial subgroups are subnormal and in the second case that they are ascendant.

The proofs of the results are dependent upon whether or not the Sylow p-subgroups are abelian for all primes p. It is therefore convenient to list the results with this dichotomy in mind.

We shall use much of the terminology and notation of [29]. We shall let $\mathfrak{A}$ denote the class of abelian groups and we let $\mathfrak{A}^*$ denote the class of locally finite groups with all non-abelian subgroups serial. In Section 2 we give some preliminary results which are useful for the rest of the paper. In Section 3 we consider $\mathfrak{A}^*$-groups which have a non-abelian Sylow p-subgroup. The main result of this section is then the following theorem whose proof depends heavily, as in much of this work, on [10].

Theorem A *Let $G \in \mathfrak{A}^*$ and suppose that G has a non-abelian Sylow p-subgroup P, for some prime p. Suppose that G is not locally nilpotent. Then the following hold:*

(i) $G = P \rtimes Q$, *where Q is an abelian Sylow p'-subgroup of G;*

(ii) $C = C_P(Q)$ *is a G-invariant abelian subgroup of P such that P/C is a finite G-chief factor;*

(iii) $G/C_G(P/C)$ *is a cyclic p'-group;*

(iv) P/C *is a $\langle g\rangle$-chief factor, for every element $g \notin C_G(P/C)$;*

(v) $P = CD$, *where $D = [P, Q]$ is a non-abelian finite special p-subgroup and $|D/D'| \geqslant p^2$;*

(vi) $C \cap D = D' = \zeta(D) \leqslant \zeta(G)$ *and P is nilpotent of class at most 2;*

(vii) $D = [P, \langle g\rangle] = [D, \langle g\rangle]$, *for every element $g \notin C_G(P/C)$;*

(viii) $C_G(D) = C \times C_Q(D/D')$ *is abelian.*

Conversely, if a group satisfies conditions (i)-(viii), then every non-abelian subgroup of G is subnormal.

The case when all the Sylow subgroups are abelian is covered in Section 4 and here we obtain the following result.

Theorem B *Let* $G \in \mathfrak{A}^*$*, suppose that the Sylow* s*-subgroups of* G *are abelian for all primes* s *and that* G *is not locally nilpotent. Then the following hold:*

(i) *There is a prime* p *such that* G *contains a normal* p*-subgroup* P *and an abelian Sylow* p'*-subgroup* Q *such that* $G = P \rtimes Q$*;*

(ii) $[P, Q]$ *is a minimal* G*-invariant subgroup and* $P = C_P(Q) \times [P, Q]$*;*

(iii) $G/C_G([P, Q])$ *is a cyclic* p'*-group;*

(iv) $[P, Q]$ *is a minimal* $\langle g \rangle$*-invariant subgroup of* P *for all* $g \notin C_G([P, Q])$.

Conversely, if the group G *also satisfies the conditions (i)-(iv) then every non-abelian subgroup of* G *is normal.*

In Section 5 we consider locally finite groups whose non-abelian subgroups are permutable, a topic that has been discussed for locally graded groups in [6], where the authors use the term *metaquasihamiltonian*. Here we give some more details concerning such groups. We recall that a subgroup H of a group G is permutable in G if $HK = KH$ for all subgroups K of G. Every normal subgroup is permutable and a result of Stonehewer [38] shows that permutable subgroups are always ascendant subgroups. We use Theorems A and B to obtain the following descriptions of non-locally nilpotent, locally finite groups whose non-abelian subgroups are permutable.

Theorem C *Let* G *be a locally finite group whose non-abelian subgroups are permutable. Suppose that* G *is not locally nilpotent and has a non-abelian Sylow* p*-subgroup* P *for some prime* p*. Then the following hold:*

(i) $G = P \rtimes Q$*, where* Q *is an abelian Sylow* p'*-subgroup of* G*;*

(ii) $P = [P, Q]$ *is a finite non-abelian* p*-subgroup of order* p^3*; indeed,* P *is either a quaternion group or* P *is a group of exponent* p*;*

(iii) $P' = \zeta(P) \leqslant \zeta(G)$*;*

(iv) P/P' *is a* G*-chief factor of order* p^2*;*

(v) P/P' *is a* $\langle g \rangle$*-chief factor for all* $g \notin C_G(P/P')$*;*

(vi) $G/C_G(P/P')$ *is a cyclic* p'*-group;*

(vii) $C_G(P) = P' \times C_G(P/P')$ *is abelian;*

(viii) $P = [P, \langle g \rangle]$ *for every element* $g \notin C_G(P/P')$.

Conversely, if the group G *also satisfies conditions (i)-(viii) then every non-abelian subgroup of* G *is normal in* G.

Our version of Theorem B with permutability in place of serial is as follows.

Theorem D *Let* G *be a locally finite group whose non-abelian subgroups are permutable. Suppose that* G *is not locally nilpotent and that the Sylow* s*-subgroups of* G *are abelian for all primes* s*. Then the following hold:*

(i) *There is a prime* p *such that* G *contains a normal* p*-subgroup* P *and an abelian Sylow* p'*-subgroup* Q *such that* $G = P \rtimes Q$*;*

(ii) $[P, Q]$ *is a minimal* G*-invariant subgroup and* $P = C_P(Q) \times [P, Q]$*;*

(iii) $G/C_G([P, Q])$ *is a cyclic* p'*-group;*

(iv) $[P, Q]$ *is a minimal* $\langle g\rangle$*-invariant subgroup of* P *for all* $g \notin C_G([P, Q])$.

Conversely, if the group G *also satisfies the conditions (i)-(iv) then every non-abelian subgroup of* G *is normal.*

Finally, in Section 6, we use these results to obtain the following detailed description of non-locally nilpotent, locally graded groups of 0-rank at least 2 whose non-abelian subgroups are permutable.

Theorem E *Let* G *be a locally graded group whose non-abelian subgroups are permutable. Suppose that* G *is not locally nilpotent and that the* 0*-rank of* G *is at least* 2*. Then* G *is a group of one of the following two types:*

(I) G *has the following properties*

- *(Ia)* G' *is a finite minimal normal subgroup of* G*;*
- *(Ib)* G' *is an elementary abelian* p*-group for some prime* p*;*
- *(Ic)* $C_G(G')$ *is abelian and* $G/C_G(G')$ *is a cyclic* p'*-group;*
- *(Id)* G' *is a minimal* $\langle g\rangle$*-invariant subgroup of* G *for all* $g \notin C_G(G')$.

(II) G *has the following properties*

- *(IIa)* $P = G'$ *has order* p^3 *and is either a quaternion group or a group of exponent* p*;*
- *(IIb)* P/P' *is a* $\langle g\rangle$*-chief factor for each* $g \notin C_G(P/P')$*;*
- *(IIc)* $P' \leqslant \zeta(G)$*;*
- *(IId)* $G = P \rtimes Q$*, for some abelian subgroup* Q*;*
- *(IIe)* $G/C_G(P/P')$ *is a cyclic* p'*-group;*

(IIf) $C_G(P) = P' \times C_Q(P/P')$ *is abelian.*

Conversely, if G *is a group of type (I) or (II) then every non-abelian subgroup of* G *is normal.*

In a future paper we hope to address the remaining case of groups of 0-rank precisely 1. We note that in [6, Lemma 2.8] the authors prove that a locally graded group of 0-rank at least 2 whose non-abelian subgroups are permutable is finite-by-abelian.

2 Preliminary Results

In this preliminary section we shall obtain certain results which will often be used in the sequel. First we note the following fact and give its easy proof.

Lemma 2.1 *Let* G *be a group whose non-abelian subgroups are serial. If* L *is a normal non-abelian subgroup of* G *then every subgroup of* G/L *is serial. In particular, if* G/L *is locally finite, then* G/L *is locally nilpotent.*

Proof — Let H/L be a subgroup of G/L. Then H is non-abelian and hence is serial in G, so there is a series $\mathcal{S}$ from H to G. If $S \in \mathcal{S}$, then the subgroups S/L comprise a series from H/L to G/L and the result follows. □

Our next result is also easy to prove.

Lemma 2.2 *Let* G *be a group whose non-abelian subgroups are serial and let* $V \triangleleft U \leqslant G$ *be subgroups such that* U/V *is non-abelian. Then every non-abelian subgroup of* U/V *is serial.*

We shall often require information concerning Sylow subgroups and here give a slight extension of [31, Lemma 1].

Lemma 2.3 *Let* G *be a locally finite group, let* π *be a set of primes and suppose that* P *is a serial* π*-subgroup of* G*. Then* P^G *is a* π*-subgroup of* G*. In particular, if* P *is a Sylow* π*-subgroup of* G*, then* $P = O_\pi(G)$ *and is precisely the set of* π*-elements of* G*.*

Proof — Let $h \in P$ and let H be a finite subgroup of G containing h. By [8], $P \cap H$ is subnormal in H, so $(P \cap H)^H$ is a π-group. Hence $\langle h \rangle^H$ is a π-group so $\langle h \rangle^G$ is also a π-group. Therefore P^G is likewise a π-group. In particular, if P is a Sylow π-subgroup then $P = P^G = O_\pi(G)$, as required. □

Lemma 2.4 *Let* $G \in \mathfrak{A}^*$. *If* K, L *are normal non-abelian subgroups of* G *such that* $K \cap L = 1$ *then* G *is locally nilpotent.*

Proof — By Lemma 2.1, G/K and G/L are both locally nilpotent. Since G embeds in $G/K \times G/L$ it follows that G is locally nilpotent. □

Our next result shows that the Sylow subgroups are actually quite restricted in structure and gives us a fundamental dichotomy that exists in $\mathfrak{A}^*$-groups.

Corollary 2.5 *Let* $G \in \mathfrak{A}^*$. *If there are distinct primes* p, q *such that* G *has a non-abelian Sylow* p*-subgroup and a non-abelian Sylow* q*-subgroup then* G *is locally nilpotent.*

Proof — Let P be a non-abelian Sylow p-subgroup of G and let Q be a non-abelian Sylow q-subgroup of G. By Lemma 2.3, P, Q are normal in G and $G/P, G/Q$ are locally nilpotent, by Lemma 2.1. Since $P \cap Q = 1$, Lemma 2.4 implies that G is locally nilpotent, as required. □

Using the same argument as in Lemma 2.9 of [10] (where we note that a serial subgroup of a finite group is subnormal) we also have

Lemma 2.6 *Let* G *be an* $\mathfrak{A}^*$*-group and suppose that* G *is not locally nilpotent. Let* p *be a prime and suppose that* A *is a finite minimal normal abelian* p*-subgroup such that* G/A *is locally nilpotent. Then* A *is a* $\langle g \rangle$*-chief factor for each* $g \notin C_G(A)$.

Finally in this section we give a couple of structural results concerning $\mathfrak{A}^*$-groups which are not locally nilpotent. The first is immediate from [27, Theorem C(i)].

Lemma 2.7 *Let* $G \in \mathfrak{A}^*$ *and suppose that* G *is not locally nilpotent. Then* G *is abelian-by-finite.*

Lemma 2.8 *If* $G \in \mathfrak{A}^*$ *and* G *is not locally nilpotent, then* G *is soluble.*

Proof — By Lemma 2.7, G contains a normal abelian subgroup A such that G/A is finite. Suppose, for a contradiction, that S is a non-abelian simple factor of G/A. Then the proper subgroups of the finite group S are abelian and, by the theorem of Miller and Moreno [22] mentioned in the introduction, S is soluble. This contradiction implies that G is soluble. □

3 The structure of $\mathfrak{A}^*$-groups with a non-abelian Sylow p-subgroup

In this section we obtain necessary and sufficient conditions for a non-locally nilpotent group G to be an $\mathfrak{A}^*$-group when G contains a non-abelian Sylow p-subgroup for some prime p.

Lemma 3.1 *Let $G \in \mathfrak{A}^*$, suppose that G is not locally nilpotent and that G contains a non-abelian Sylow p-subgroup P for some prime p. Then P' is abelian.*

Proof — We note that P is a normal subgroup of G by Lemma 2.3 and that P contains all p-subgroups of G. Let $x, y \in P'$. Since G is locally finite, there is a finite non-abelian subgroup P_1 of P such that $x, y \in P_1$. Similarly, since G is not locally nilpotent, there is a finite subgroup H which is not nilpotent. Hence the finite group $K = \langle H, P_1 \rangle$ is also finite non-nilpotent and has the normal non-abelian Sylow p-subgroup $K \cap P$. The non-abelian subgroups of K are serial and hence subnormal in K. By [10, Lemma 3.1] we deduce that $(K \cap P)'$ is abelian. However $P_1' \leqslant (K \cap P)'$ so that $[x, y] = 1$. It follows that P' is abelian. □

Lemma 3.2 *Let $G \in \mathfrak{A}^*$ and suppose that G is not locally nilpotent. Suppose that P is a non-abelian Sylow p-subgroup of G. Then P is normal in G and $G = P \rtimes Q$ where Q is an abelian Sylow p'-subgroup of G.*

Proof — By Lemma 2.3, P is a normal subgroup of G. If G is not Chernikov, then every non-abelian subgroup of G is subnormal, by [27, Theorem C(i)]. In this case the result follows from [10, Lemma 3.3]. If G is Chernikov then $G = P \rtimes Q$, for some Sylow p'-subgroup of G, by [7, Theorem 2.4.5]. Furthermore, by Lemma 2.1, G/P is locally nilpotent so Q is also locally nilpotent. However the Sylow q-subgroups of G are abelian for $q \neq p$ and it follows that Q is abelian. □

Lemma 3.3 *Let $G \in \mathfrak{A}^*$ and suppose that the Sylow p-subgroup P of G is non-abelian. Suppose that G is not locally nilpotent and $G = P \rtimes Q$, for the abelian Sylow p'-subgroup Q. Then $[P', Q] = 1$.*

Proof — We note that P' is abelian, by Lemma 3.1. If G is not Chernikov, then the result follows by [10, Lemma 3.5], so we may suppose that G is Chernikov. Since P is Chernikov, $\Omega_n(P) = \langle x | x^{p^n} = 1 \rangle$ is finite for all n and P is the union of such subgroups. Since P is non-abelian, there is a natural number k such that $\Omega_n(P)$ is non-abelian, for all $n \geqslant k$. Let Q_1 be any finite subgroup of Q with the property that $F = \Omega_n(P) \rtimes Q_1$ is non-nilpotent, for some $n \geqslant k$. Since F is finite and all the non-abelian subgroups of F are subnormal, [10, Lemma 3.5] implies that $[\Omega_n(P)', Q_1] = 1$. Since this is true for every such subgroup Q_1 we deduce that $[\Omega_n(P)', Q] = 1$, and since this is true for every $m \geqslant n$ we deduce that $[P', Q] = 1$. This completes the proof. □

Lemma 3.4 *Let $G \in \mathfrak{A}^*$ and suppose that G has a non-abelian Sylow p-subgroup for some prime p. Suppose that G is not locally nilpotent. Then P/P' contains a finite minimal G-invariant subgroup V/P' such that G/V is abelian. In particular, $G/C_G(V/P')$ is a cyclic p'-group.*

Proof — By Lemma 3.1, P' is abelian and if G is not Chernikov then the result follows from [10, Lemma 3.6]. Hence we may suppose that G is Chernikov. Let Q be an abelian Sylow p'-subgroup such that $G = P \rtimes Q$, guaranteed by Lemma 3.2. If L is the Hirsch-Plotkin radical of G then $P \leqslant L$ and hence $L = P \times (L \cap Q)$. It follows that $L \cap Q \leqslant C_G(P)$ and that $G/L = QL/L \simeq Q/Q \cap L$ is a finite p'-group. Let $A/P' = \Omega_1(P/P')$. Since G is Chernikov, A/P' is finite and since $L \cap Q \leqslant C_G(P)$ it follows that $L/P' \leqslant C_{G/P'}(P/P')$. Consequently, $G/C_G(P/P')$ is a finite p'-group, as is $G/C_G(A/P')$. It follows from [11, Corollary 5.15] that

$$A/P' = A_1/P' \times A_2/P' \times \cdots \times A_t/P',$$

for certain minimal G-invariant subgroups A_i/P' of A/P'.

Suppose, for a contradiction, that A_j/P' is G-central for all j. Then $[A, G] \leqslant P'$ and [10, Lemma 3.4] implies that P/P' lies in the upper hypercentre of G/P'. It follows that G/P' is hypercentral and hence abelian so Q stabilizes the series $1 \leqslant P' \leqslant P$. Since $\pi(P) \cap \pi(Q) = \emptyset$ we have $[P, Q] = 1$ and hence $G = P \times Q$ is locally nilpotent, contrary to our hypothesis.

Consequently there exists a factor A_m/P' which is not G-central. Then A_mQ/P', and hence A_mQ, is non-abelian. Thus A_mQ is serial in G and the Sylow p'-subgroup QA_m/A_m is serial in G/A_m. By Lemma 2.3, QA_m is normal in G and hence $[P, Q] \leqslant A_m$. If also A_n/P' is not G-central, for some $n \neq m$, then, as above, $[P, Q] \leqslant$

A_n. In this case we have $[P, Q] \leqslant A_m \cap A_n = P'$, which leads to a contradiction, as above. Hence, for $j \neq m$, A_j/P' is G-central and we have $[A/P', G] = A_m/P'$, since A_m/P' is a minimal G-invariant subgroup.

Let $V/P' = A_m/P'$. Then $[P, Q] \leqslant V$ implies that $G/V = P/V \times QV/V$, an abelian group since $P' \leqslant V$. Finally, $G/C_G(V/P')$ is an abelian p'-group so is cyclic, by [9, Theorem 3.1]. This completes the proof. □

Proof of Theorem A — (i) follows from Lemma 3.2.

To prove (ii) and (iii) note that, by Lemma 3.4, P/P' contains a finite minimal G-invariant subgroup V/P' such that G/V is abelian and $G/C_G(V/P')$ is a cyclic p'-group. As in the proof of [10, Proposition 3.7] we deduce that $P/P' = C/P' \times [P/P', G]$, where $C = C_P(Q)$ is abelian, $C/P' = C_{P/P'}(G)$ and $[P/P', G] = V/P'$. Since $P = CV$ and $C \cap V = P'$ it follows that P/C and V/P' are G-isomorphic. Then $G/C_G(P/C) \simeq G/C_G(V/P')$ is a cyclic p'-group.

(iv) follows from Lemma 2.6.

Next let $D = [P, Q]$ so that $V = DP'$. As in the proof of [10, Proposition 3.7] we deduce that $D \cap P' \leqslant \zeta(D)$ and $P = CD$. Furthermore, $P/P' = C/P' \times DP'/P'$. Clearly $D' \leqslant D \cap P' \leqslant \zeta(D)$. Since $V/P' = DP'/P' \simeq D/(D \cap P')$ it follows that $D/(D \cap P')$ is a finite G-chief factor so either $D = \zeta(D)$ or $D \cap P' = \zeta(D)$. In the former case D is abelian and, as in the proof of [10, Proposition 3.7], a contradiction is obtained. Hence $D \cap P' = \zeta(D)$ and $D/\zeta(D)$ is a finite G-chief factor. Since D is non-abelian it follows that $D/\zeta(D)$ has order at least p^2.

Suppose, for a contradiction, that $D' \neq \zeta(D)$. Since $P' \leqslant C_P(Q)$ and $D \cap P' = \zeta(D)$ we have $C_{D/D'}(Q) \neq 1$. By [2, Proposition 2.12] we have $D/D' = E/D' \times B/D'$, where $E/D' = C_{D/D'}(Q)$ and

$$B/D' = [D/D', QD'/D'] = [D, Q]D'/D'.$$

Since $E/D' \neq 1$, $B \neq D$. Since the p'-group Q stabilizes the series $1 \leqslant D' \leqslant E$ we have $[E, Q] = 1$ so $E \leqslant C$. Then $P = CD = CEB = CB$ and

$$D = [P, Q] = [CB, Q] = [B, Q] \leqslant B,$$

which yields our contradiction. Hence $D' = \zeta(D)$.

Since $D/\zeta(D)$ is an elementary abelian p-group we have $D' = \zeta(D) = \mathrm{Frat}(D)$, the Frattini subgroup of D. If y is an arbitrary element of D, then $y^p \in D' = \zeta(D)$ and so, for arbitrary $x \in D$, we have $[x, y]^p = [x, y^p] = 1$. Hence D' is elementary abelian so that $D = [P, Q]$ is special. Finally, since $D/\zeta(D)$ is finite, D' is finite, by [25] and it follows that D is likewise finite.

Since $D/\zeta(D)$ is a G-chief factor and P is a normal locally nilpotent subgroup we have $D/\zeta(D) \leqslant \zeta(P/\zeta(D))$ so $[P, D] \leqslant \zeta(D) = D'$. Hence $[D, P, D] = [P, D, D] = 1$ and, by the three subgroup lemma, we deduce that $[D', P] = 1$ so that $D' = \zeta(D) \leqslant \zeta(P)$. However, $D' \leqslant C_P(Q) = C$ and D/D' is a G-chief factor so $D' = D \cap C$. This completes the proof of (v) and (vi).

The equation $P = C_P(Q)D$ implies that $[P, \langle g \rangle] \leqslant D$. By Lemma 2.6, D/D' is a $\langle g \rangle$-chief factor so that $[D, \langle g \rangle]D'/D' = D/D'$. However, D is special, so $D' = \mathrm{Frat}(D)$ and we obtain $D = [D, \langle g \rangle]$. Thus (vii) follows and (viii) follows easily.

To prove the converse, assume that G is a group satisfying conditions (i)-(viii). Let H be a non-abelian subgroup of G. Then $H = (H \cap P) \rtimes U$, where U is a Sylow p'-subgroup of H. Since H is non-abelian, $H \cap P \neq 1$. Also $|G : N_G(Q)|$ is finite, so the Sylow p'-subgroups of G are conjugate in G by [7, Lemma 2.3.2]. Hence without loss of generality we may suppose that $U \leqslant Q$. Note that if $U \leqslant C_G(P)$ then $H = (H \cap P)U$ is subnormal in G since $PC_Q(P)$ is nilpotent. Hence we may suppose that $U \nleqq C_Q(P)$. Since H is non-abelian, (ii) implies that $H \cap P \nleqq C = C_P(Q)$. By (iv), P/C is a U-chief factor and hence $P = (H \cap P)C$ and $W = [H \cap P, U] \nleqq C$. Thus $[H \cap P, U] \leqslant [P, U] = D$ and $C \cap D = \mathrm{Frat}(D)$ does not contain W. Hence $W\mathrm{Frat}(D)/\mathrm{Frat}(D) \neq 1$. The subgroup W is U-invariant so

$$W\mathrm{Frat}(D)/\mathrm{Frat}(D) = W^U\mathrm{Frat}(D)/\mathrm{Frat}(D) = D/\mathrm{Frat}(D).$$

Hence $D = W\mathrm{Frat}(D)$ so that $W = [H \cap P, U] = D$. It follows that $D \leqslant H$ and since G/D is abelian, H is normal in G. This completes the proof. □

4 Structure of $\mathfrak{A}^*$-groups with abelian Sylow p-subgroups for all primes p

In this short section we assume that the Sylow subgroups are always abelian. The next result follows in precisely the same manner as the corresponding result, Lemma 4.3, of [10].

Lemma 4.1 *Let* $G \in \mathfrak{A}^*$, *suppose that* G *is not locally nilpotent and that the Sylow* q*-subgroups of* G *are abelian, for all primes* q. *Then* G' *is a* p*-group, for some prime* p.

Proof of Theorem B — If $G \in \mathfrak{A}^*$ then conditions (i)-(iv) hold using a similar proof to [10, Theorem 4.4], so we merely prove the converse and assume that G is a group satisfying the conditions. Let H be a non-abelian subgroup of G. Then $H = (P \cap H) \rtimes U$, where $P \cap H$ is the normal (abelian) Sylow p-subgroup of H and U is an (abelian) Sylow p'-subgroup of H. As in the proof of Theorem A we may suppose that $U \leqslant Q$. Since H is non-abelian, $P \cap H \neq 1$ and also $[P \cap H, U](\neq 1)$ is H-invariant. Clearly $[P \cap H, U] \leqslant [P, Q]$. Since H is not nilpotent, condition (iv) implies that $[P, Q] = [P \cap H, U] \leqslant H$. Since $G/[P, Q]$ is abelian it follows that H is normal in G, as required. □

The following corollary is essentially observed in [27, Theorem C(i)].

Corollary 4.2 *Let* G *be a locally finite group that is not locally nilpotent. The following are equivalent:*

(i) Every non-abelian subgroup of G *is serial in* G;

(ii) Every non-abelian subgroup of G *is ascendant in* G;

(iii) Every non-abelian subgroup of G *is subnormal in* G.

For locally nilpotent groups this result is clearly not true since, as we mentioned in the Introduction, every subgroup of a locally nilpotent group is serial and for locally finite groups the converse also holds. We note also that J. Wilson [40] has constructed an example of a group G in which every subgroup is serial, but which is not locally nilpotent. However groups whose subgroups are ascendant form a proper subclass of the class of locally nilpotent groups (see the comment after Theorem 13 of [4]). On the other hand groups whose subgroups are all subnormal are soluble [23], but there exist insoluble, periodic hypercentral groups, it being clear that in a hypercentral group every subgroup is ascendant.

5 Structure of locally finite groups whose non-abelian subgroups are permutable

In this section we discuss locally finite groups whose non-abelian subgroups are permutable.

Lemma 5.1 *Let* G *be a group and let* L *be a normal subgroup of* G*. Let* K *be a subgroup of* L *and let* $g \in G$ *be such that* $L \cap \langle g \rangle = 1$*. If* $K\langle g \rangle = \langle g \rangle K$ *then* $K^{\langle g \rangle} = K$.

Proof — Let $H = K\langle g \rangle = K^{\langle g \rangle}\langle g \rangle$. Then

$$K^{\langle g \rangle} = K^{\langle g \rangle} \cap K\langle g \rangle = K(K^{\langle g \rangle} \cap \langle g \rangle) = K,$$

by hypothesis. □

We shall need the following old result due to L. Redei [28].

Lemma 5.2 *Let* G *be a finite non-abelian* p*-group whose proper subgroups are abelian. Then* G *is one of the following types:*

(i) G *is quaternion of order* 8;

(ii) $G = \langle a, b \rangle$ *and* $|a| = p^k, |b| = p^t, a^b = a^m$*, where* $m = 1 + p^{k+1}$, $k \geqslant 2$, $t \geqslant 1$ *and* $|G| = p^{k+t}$;

(iii) $G = \langle a, b \rangle$ *and* $|a| = p^k, |b| = p^t, [a, b] = c$, $|c| = p, |G| = p^{k+t+1}$.

Proof of Theorem C — Every permutable subgroup of a group is ascendant, by [38] and hence we can apply Theorem A. Then $G = P \rtimes Q$, for some abelian Sylow p'-subgroup Q. Let $C = C_P(Q)$, a G-invariant abelian group. Also $D = [P, Q]$ is a finite non-abelian special p-group and $D/D' \simeq P/C$ is a G-chief factor. Hence D/D' is P-central.

Let K be a minimal non-abelian subgroup of D. Then $K \nleq D'$ since D' is abelian and, as K is permutable in G, Lemma 5.1 implies that K is Q-invariant. Hence KD'/D' is G-invariant so $KD' = D$. Then $D/D' \simeq K/(K \cap D')$ is elementary abelian and Lemma 5.2 shows that it has order p^2. Thus $D/D' = \langle aD', bD' \rangle$, for some $a, b \in G$ and since $D' = \mathrm{Frat}(D)$ it follows that $D = \langle a, b \rangle$. Moreover, $c = [a, b] \in \zeta(D)$ and has order p. Hence $|D| = p^3$. Lemma 5.2 shows that D is a quaternion group, or D is a non-abelian group of exponent p, or $D =$

$\langle a\rangle \rtimes \langle b\rangle$, where $|a| = p^2, |b| = p$ and $a^b = a^{1+p}$. However, in the latter case $\langle a\rangle$ is characteristic in D so $\langle aD'\rangle$ is a G-invariant subgroup of the G-chief factor D/D'. Then $D/D' = \langle aD'\rangle$ has order p, which is impossible by Theorem A. Hence D is quaternion or non-abelian of exponent p.

The factor group $G/C_G(D/D')$ is a cyclic p'-group, so let $g \in Q$ be such that $G = \langle g\rangle C_G(D/D')$. Since D/D' is G-chief, there exists an element $y \in \mathbb{F}_2\langle g\rangle$ such that $(aD')^y = bD'$ and hence $a^y = b$ or $a^y = bc^k$, for some k such that $1 \leqslant k \leqslant p$, where we note that if $p = 2$, then $c^2 = b$ and if $p > 2$ then $c^p = 1$. Suppose that $d \in C_C(a, b)$ and $d \notin \langle a, b\rangle$. Since $d \in C = C_P(Q)$ and $y \in Q$ we have $(ad)^y = bd$ or $(ad)^y = bc^k d$. However, $bd, bc^k d$ are not in $\langle ad, b\rangle$ since $c \in \langle ad, b\rangle$, but $a \notin \langle ad, b\rangle$. Hence $(ad)^y \notin \langle ad, b\rangle$, contrary to Lemma 5.1, as $\langle ad, b\rangle \leqslant P$, but$P \cap \langle y\rangle = 1$. Consequently, $C_C(a, b) = \langle c\rangle$.

Suppose that $C \neq C_C(a)$. Then $C/C_C(a)$ has order p and there is an element u such that $C = \langle u, C_C(a)\rangle$. Then $\langle u, a\rangle$ is non-abelian. Again we have $a^y = b$ or $a^y = bc^k$, for $1 \leqslant k \leqslant p$ so that $ay \notin \langle u, a\rangle$ and we obtain a contradiction to Lemma 5.1. A similar contradiction is reached if we suppose that $C \neq C_C(b)$. Hence $C = \langle c\rangle$ and $P = D$. The conditions (i)-(viii) given in the the statement of the Theorem now hold.

Conversely, suppose that G satisfies the conditions (i)-(viii), let H be a non-abelian subgroup of G and let $P = \langle a, b\rangle$. Then $H = (H \cap P) \rtimes U$, where U is a Sylow p'-subgroup of H. Since H is non-abelian we have $H \cap P \neq 1$ and, as in the proof of Theorem A, we may suppose that $U \leqslant Q$. If $H \cap P$ is non-abelian then $H \cap P = P$, by (ii), which is normal in G so H is normal in G, since G/P is abelian.

If $H \cap P$ is abelian, then either $|H \cap P| = p$ or $|H \cap P| = p^2$. By (iii) we have $c = [a, b] \in \zeta(G)$, so $H \cap P \neq \langle c\rangle$. Hence $H \cap P \neq P'$. Since H is non-abelian we have $U \nleq C_Q(P)$ so, by (v), P/P' is a U-chief factor. Since $H \cap P$ is U-invariant it follows that $P = (H \cap P)P'$. We deduce that $P = H \cap P$, which is a contradiction since P is non-abelian. This completes the proof. □

Theorem D is an immediate consequence of Theorem B.

6 The structure of some non-periodic groups whose non-abelian subgroups are permutable

Finally, in this section we turn to the proof of Theorem E. We require the following preliminary result.

Lemma 6.1 *Let G be a locally graded group whose non-abelian subgroups are permutable. Suppose that G is not locally nilpotent and that the 0-rank of G is at least 2. If G contains a non-abelian p-subgroup, for some prime p, then the Sylow p-subgroup P of G satisfies the following:*

(i) $P \triangleleft G$ *and* $P = G'$;

(ii) P *has order* p^3 *and is either a quaternion group or a group of exponent* p;

(iii) P/P' *is a* $\langle g \rangle$*-chief factor for each* $g \notin C_G(P/P')$;

(iv) $P' \leqslant \zeta(G)$;

(v)) $G = P \rtimes Q$, *for some abelian subgroup* Q;

(vi) $G/C_G(P/P')$ *is a cyclic* p'*-group;*

(vii) $C_G(P) = P' \times C_Q(P/P')$ *is abelian.*

Proof — By [6, Lemma 2.8], G' is finite and hence G is an FC-group. It follows from [39, Theorem 1.4] that $G/\zeta(G)$ is periodic. Let F be a free abelian subgroup of $\zeta(G)$ such that $\zeta(G)/F$ is periodic and note that G/F is then locally finite, since it is a periodic FC-group. We note that every non-abelian subgroup of G/F is permutable. Let T be the torsion subgroup of G so that $T \cap F = 1$. Remak's theorem implies that there is an embedding

$$G \longrightarrow G/T \times G/F.$$

However G/T is abelian, by [39, Theorem 1.6], so G/F is not locally nilpotent, since G is not locally nilpotent.

Let R be a non-abelian p-subgroup of G and let P denote a Sylow p-subgroup of G containing R. Then P is non-abelian also and hence $|P| \geqslant p^3$. Furthermore G/F contains a non-abelian Sylow p-subgroup P_1/F which is normal in G/F and of order p^3, by Theorem C. It follows that $P_1/F = PF/F \simeq P$ and since P_1/F is a quaternion group or non-abelian of exponent p, the same is true for P.

By Theorem C we have $(G/F)' = G'F/F = PF/F$ so that $G' \leqslant PF$. Since P is the torsion subgroup of PF and G' is finite we have $G' \leqslant P$ so $G' = P$. Hence P is normal in G. Again by Theorem C, we have $P'F/F \leqslant \zeta(G/F)$ so $[P', G] \leqslant F \cap P = 1$ and hence $P' \leqslant \zeta(G)$.

If $g \notin C_G(P/P')$ but $[gF, PF/F] \leqslant P'F/F$, then $[g, P] \leqslant P'F \cap P = P'$, which is a contradiction. Hence $gF \notin C_{G/F}((PF/F)/(P'F/F))$ and, by

Theorem C, $(PF/F)/(P'F/F)$ is a $\langle gF\rangle$-chief factor. From this, it follows that P/P' is a $\langle g\rangle$-chief factor. Since $C_G(P/P')/F = C_{G/F}((PF/F)/(P'F/F))$, Theorem C implies that $G/C_G(P/P')$ is a cyclic p'-group.

Theorem C also implies that $G/F = P_1/F \rtimes Q/F$ for some subgroup Q and since $P_1 = PF$ we deduce that $G = P \rtimes Q$. Of course Q is abelian, since $P = G'$. Finally since $p \notin \Pi(Q)$ and $P' \leqslant \zeta(G)$ we have $C_Q(P/P') = C_Q(P)$ and it follows that $C_G(P) = P' \times C_Q(P) = P' \times C_Q(P/P')$ is abelian. This proves the result. □

Proof of Theorem E — As in the proof of Lemma 6.1, G' is finite, G is an FC-group and there is a free abelian subgroup F of $\zeta(G)$ such that G/F is locally finite, but not locally nilpotent. We note that every non-abelian subgroup of G/F is permutable so we may apply Theorems C and D where appropriate.

Suppose first that the Sylow s-subgroups of G/F are abelian for all primes s. By Theorem D there is a prime p such that the Sylow p-subgroup P/F of G/F is normal and also there is an abelian p'-subgroup Q/F such that $G/F = P/F \rtimes Q/F$. Furthermore $P/F = C/F \times D/F$, where $C/F = C_{P/F}(Q/F)$ and $D/F = [P/F, Q/F] = G'F/F$ is a minimal G-invarant subgroup of G/F. Then $D = G' \times F$ since G' is finite. Since G' is G-isomorphic to D/F it follows that G' is a minimal normal subgroup of G, so is an elementary abelian p-group for some prime p. As in the proof of Lemma 6.1 we can prove that $C_G(G')/F = C_{G/F}(D/F)$ which implies that $G/C_G(G')$ is a cyclic p'-group. By Theorem D, $C_G(G')/F = C_{G/F}(D/F)$ is abelian and hence $C_G(G')' \leqslant F \cap G' = 1$, so $C_G(G')$ is abelian. Finally in this case, if $g \notin C_G(G')$ then $gF \notin C_{G/F}(D/F)$ and, by Theorem D, D/F is a $\langle gF\rangle$-chief factor. It then follows that G' is a $\langle g\rangle$-chief factor. Therefore we have (I).

Suppose now that G/F has a non-abelian Sylow p-subgroup R/F for some prime p. Then Theorem C shows that R/F is a finite p-subgroup of order p^3 and is either quaternion or of exponent p. Furthermore $R/F = G'F/F$. However, G' is finite so $R = G' \times F$ and $P = G' \simeq R/F$ is non-abelian of order p^3. In this situation Lemma 6.1 applies so that G is a group of type (II).

Conversely, suppose that G is a group of type (I) and let H be a non-abelian subgroup of G. Then $1 \neq H' \leqslant G'$. Since $C_G(G')$ is abelian $H \nleqq C_G(G')$. Let $1 \neq a \in H'$ and let $g \in H$ be such that $g \notin C_G(G')$. Since G' is a minimal $\langle g\rangle$-invariant subgroup we have $\langle a\rangle^{\langle g\rangle} = G'$. Hence $G' \leqslant \langle a, g\rangle \leqslant H$, so H is a normal subgroup of G.

Finally, suppose that G is a group of type (II) and let H be a non-abelian subgroup of G. Since $C_G(P)$ is abelian, $H \nleqslant C_G(P)$. If $p \notin \Pi(H)$ then $H \cap P = 1$ and property (IId) implies that H is abelian, a contradiction. Hence the Sylow p-subgroup $P \cap H$ of H is nontrivial. If $P \cap H$ has order p^3, then $P \cap H = P = G' \leqslant H$ so that H is normal in G. Suppose then that $P \neq P \cap H$. If $P \cap H = P'$ then $H \leqslant P' \times Q$ so that H is abelian, contrary to our assumption. Hence $H \cap P \neq P'$ and $H \nleqslant C_G(P)$, since H is non-abelian. Hence there is an element $a \in (H \cap P) \setminus P'$, and there is an element $g \in H$ such that $gC_G(P)$ is a nontrivial p'-element. It follows that $g \notin C_G(P/P')$ and our hypotheses imply that $\langle a \rangle^{\langle g \rangle} P'/P' = P/P'$. Since $P' = \mathrm{Frat}(P)$ we deduce that $\langle a \rangle^{\langle g \rangle} = P$. Hence $H \cap P = P$, contrary to our assumption.

The result follows. □

REFERENCES

[1] A. O. Asar: "Locally nilpotent p-groups whose proper subgroups are hypercentral or nilpotent-by-Chernikov", *J. London Math. Soc.* 61 (2000), 412–422.

[2] A. Ballester-Bolinches – L. A. Kurdachenko – J. Otal – T. Pedraza: "Infinite groups with many permutable subgroups", *Rev. Mat. Iberoamericana* 24 (2008), 745–764.

[3] B. Bruno – R. E. Phillips: "Groups with restricted nonnormal subgroups", *Math. Z.* 176 (1981), 199–221.

[4] S. N. Chernikov: "Infinite locally soluble groups", *Mat. Sb.* 7 (1940), 35–64 (Russian).

[5] R. Dedekind: "Über Gruppen deren sämtliche Teiler Normalteiler sind", *Math. Ann.* 48 (1897), 548–561.

[6] M. De Falco – F. de Giovanni – C. Musella – R. Schmidt: "Groups in which every non-abelian subgroup is permutable", *Rend. Circ. Mat. Palermo* (2) 52 (2003), 70–76.

[7] M. R. Dixon: "Sylow Theory, Formations and Fitting Classes in Locally Finite Groups", *World Scientific*, Singapore (1994).

[8] B. Hartley: “Serial subgroups of locally finite groups”, *Proc. Cambridge Philos. Soc.* 71 (1972), 199–201.

[9] L. Kurdachenko – J. Otal – I. Subbotin: “Groups with Prescribed Quotient Groups and Associated Module Theory”, *World Scientific*, Singapore (2002).

[10] L. A. Kurdachenko – S. Atlihan – N. N. Semko: “On the structure of groups whose non-abelian subgroups are subnormal”, *Central Eur. J. Math.* 12 (2014), 1762–1771.

[11] L. A. Kurdachenko – J. Otal – I. Ya. Subbotin: “Artinian Modules over Group Rings”, *Birkhäuser*, Basel (2007).

[12] N. F. Kuzennyĭ – N. N. Semko: “The structure of solvable non-nilpotent meta-Hamiltonian groups”, *Mat. Zametki* 34 (1983), 179–188.

[13] N. F. Kuzennyĭ – N. N. Semko: “The structure of solvable meta-Hamiltonian groups”, *Dokl. Akad. Nauk Ukrain. SSR Ser. A* (1985), 6–8, 87.

[14] N. F. Kuzennyĭ – N. N. Semko: “On the structure of nonperiodic meta-Hamiltonian groups”, *Izv. Vyssh. Uchebn. Zaved. Mat.* (1986), 32–40, 86.

[15] N. F. Kuzennyĭ – N. N. Semko: ”The structure of periodic metabelian and meta-Hamiltonian groups with a nonelementary commutator group”, *Ukrain. Mat. Zh.* 39 (1987), 180–185, 271.

[16] N. F. Kuzennyĭ – N. N. Semko: “The structure of periodic metabelian meta-Hamiltonian groups with an elementary commutator subgroup of rank two”, *Ukrain. Mat. Zh.* 40 (1988), 743–750, 814.

[17] N. F. Kuzennyĭ – N. N. Semko: “The structure of periodic nonabelian meta-Hamiltonian groups with an elementary commutator subgroup of rank three”, *Ukrain. Mat. Zh.* 41 (1989), 170–176, 286.

[18] N. F. Kuzennyĭ – N. N. Semko: “Meta-Hamiltonian groups with elementary commutator subgroup of rank two”, *Ukrain. Mat. Zh.* 42 (1990), 168–175.

[19] J. C. Lennox – D. J. S. Robinson: “The Theory of Infinite Soluble Groups”, *Clarendon Press*, Oxford (2004).

[20] J. C. Lennox – S. E. Stonehewer: "Subnormal Subgroups of Groups", *Oxford University Press*, Oxford (1987).

[21] A. A. Mahnev: "Finite meta-Hamiltonian groups", *Ural. Gos. Univ. Mat. Zap.* 10 (1976), 60–75, 151.

[22] G. A. Miller – H. Moreno: "Non-abelian groups in which every subgroup is abelian", *Trans. Amer. Math. Soc.* 4 (1903), 398–404.

[23] W. Möhres: "Auflösbarkeit von Gruppen, deren Untergruppen alle subnormal sind", *Arch. Math. (Basel)* 54 (1990), 232–235.

[24] V. T. Nagrebeckiĭ: "Finite nonnilpotent groups, any nonabelian subgroup of which is invariant", *Ural. Gos. Univ. Mat. Zap.* 6 (1967), 80–88.

[25] B. H. Neumann: "Groups with finite classes of conjugate elements", *Proc. London Math. Soc.* (3) 1 (1951), 178–187.

[26] A. Yu Ol'shanskii: "Geometry of Defining Relations in Groups", *Kluwer*, Dordrecht (1991).

[27] R. E. Phillips – J. S. Wilson: "On certain minimal conditions for infinite groups", *J. Algebra* 51 (1978), 41–68.

[28] L. Rédei: "Das "schiefe Produkt" in der Gruppentheorie mit Anwendung auf die endlichen nichtkommutativen Gruppen mit lauter kommutativen echten Untergruppen und die Ordnungszahlen, zu denen nur kommutative Gruppen gehören", *Comment. Math. Helv.* 20 (1947), 225–264.

[29] D. J. S. Robinson: "Finiteness Conditions and Generalized Soluble Groups", *Springer*, Berlin (1972).

[30] D. J. S. Robinson: "A Course in the Theory of Groups", *Springer*, Berlin (1996).

[31] D. J. S. Robinson: "Sylow permutability in locally finite groups", *Ricerche Mat.* 59 (2010), 313–318.

[32] G. M. Romalis – N. F. Sesekin: "Metahamiltonian groups", *Ural. Gos. Univ. Mat. Zap.* 5 (1966), 101–106.

[33] G. M. Romalis – N. F. Sesekin: "Metahamiltonian groups II", *Ural. Gos. Univ. Mat. Zap.* 6 (1968), 50–52.

[34] G. M. Romalis – N. F. Sesekin: "Metahamiltonian groups III", *Ural. Gos. Univ. Mat. Zap.* 7 (1969/1970), 195–199.

[35] O. J. Schmidt: "Groups all of whose subgroups are nilpotent", *Mat. Sb.* 31 (1924), 366–372.

[36] H. Smith: "Torsion-free groups with all non-nilpotent subgroups subnormal", ', *Topics in infinite groups - Quaderni Mat.* 8 (2001), 297–308.

[37] H. Smith: "Groups with all non-nilpotent subgroups subnormal", *Topics in infinite groups - Quaderni Mat.* 8 (2001), 309–326.

[38] S. E. Stonehewer: "Permutable subgroups of infinite groups", *Math. Z.* 125 (1972), 1–16.

[39] M. J. Tomkinson: "FC-Groups", *Pitman*, Boston (1984).

[40] J. S. Wilson: "On periodic generalized nilpotent groups", *Bull. London Math. Soc.* 9 (1977), 81–85.

Martyn R. Dixon
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350 (U.S.A.)
e-mail: mdixon@ua.edu

Leonid A. Kurdachenko
Department of Algebra
National University of Dnepropetrovsk
Gagarin prospect 72, Dnepropetrovsk, Ukraine 49010
e-mail: lkurdachenko@i.ua, lkurdachenko@gmail.com

Nikolaj N. Semko
Department of Mathematics
National State Tax Service Academy of Ukraine
09200 Irpin (Ukraine)
e-mail: n-semko@mail.ru