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On Groups with Extreme Centralizers and Normalizers [1]

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Abstract

An FCI-group is a group in which every non-normal cyclic subgroup has finite index in its centralizer and an FNI-group is one in which every non-normal subgroup has finite index in its normalizer. FCI-groups and FNI-groups are characterized in the case where an infinite abelian normal subgroup is present and all periodic factors are locally finite. This applies in particular to locally soluble-by-finite groups.

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1 Introduction

A group G is called an *FCI-group* if for each $g \in G$ either $\langle g \rangle \triangleleft G$ or $|C_G(g) : \langle g \rangle|$ is finite (the author is grateful to Dr. A. Tortora and Dr. M. Tota for bringing this group theoretical property to his attention). Some obvious examples of FCI-groups are abelian groups,

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finite groups, free groups and Tarski groups. The class of FCI-groups is clearly subgroup closed, but it is not quotient closed.

Equivalently, we could say that a group G is an FCI-group if either $\langle g \rangle \lhd G$ or $|N_G(\langle g \rangle) : \langle g \rangle|$ is finite for all $g \in G$. Thus a stronger property than FCI arises when the condition on normalizers is applied to arbitrary subgroups, not just the cyclic ones. A group G is said to be an *FNI-group* if $|G : N_G(H)|$ is finite whenever H is a non-normal subgroup of G. The class of FNI-groups is quotient and subgroup closed. Obviously an FNI-group is an FCI-group, but the converse is false since non-cyclic free groups are not FNI-groups. The two properties can be thought of as forcing centralizers and normalizers to extreme positions – they are either very small or very large.

The classes of FCI-groups and FNI-groups were introduced by Fernández-Alcober *et al.* in [2, 3, 4]. If G is a periodic group, the FCI condition amounts to requiring that the centralizer of a non-normal cyclic subgroup be finite: a related group theoretic property has been considered for profinite groups by Shalev [9].

Notation

(i) $r_p(A), r_0(A)$: the p-rank and torsion-free rank of an abelian group A.

(ii) $\pi(G)$: the set of primes dividing the orders of elements of finite order in a group G.

(iii) $A[n]$: the subgroup of elements in an abelian group A with order dividing n.

(iv) A subgroup H is said to be d-*embedded* in a group G if $\langle x \rangle \lhd G$ for all $x \in H$. Thus H is a dedekind group and elements of G induce power automorphisms in H.

Despite the complexity of the classes of FCI-groups and FNI-groups, it is possible to describe large classes of these groups in a precise fashion, which is the objective of this work. There are five types of groups which appear in our classification of FCI-groups and FNI-groups.

(i) Dedekind groups;

(ii) a non-abelian group G with a finite d-embedded subgroup F such that G/F is infinite cyclic or infinite dihedral;

(iii) $G = \langle x, A \rangle$ where A is a non-periodic abelian group, $a^x = a^{-1}$ for all $a \in A$, $x^2 \in A[2]$ and $A[2]$ is finite;

(iv) $G = \langle x, A \rangle$ where A is an infinite periodic abelian subgroup which is d-embedded in G and $C_A(y)$ is finite for all $y \in G \setminus A$;

(v) $G = H \curlyvee \overline{G}$, a central product, where H is a finite hamiltonian 2-group which is d-embedded in G, and $\overline{G} = \langle x, B \rangle$ is a group of type (iv), with $H \cap \overline{G} = B_2 \leqslant H[2] \cap \langle x \rangle$ and $|B_2| = 1$ or 2.

Notice that if x has odd or infinite order in (v), then $G = H \times \overline{G}$ and B has no elements of order 2.

Our principal result is as follows.

Theorem 1 *Let G be an FCI-group with an infinite abelian normal subgroup and assume that every periodic factor of G is locally finite. Then G is a group of types $(i) - (v)$. In addition, if G is an FNI-group which is of type (iii), then it has finite torsion-free rank.*

The hypothesis on periodic factors in Theorem 1 cannot be omitted, even if an infinite abelian normal subgroup is present. Indeed Adian ([1], VII) has constructed a torsion-free group G which is a central extension of an infinite cyclic group by a free Burnside group of large prime exponent. The Burnside group is periodic, but not locally finite, and each element of prime order generates its centralizer. From this it follow readily that G is an FCI-group, but it is not of types (i)-(v).

As a converse we prove:

Theorem 2 *Groups of types $(i), (ii), (iv)$ and (v) are FNI-groups. A group of type (iii) is an FCI-group and if it has finite torsion-free rank, then it is an FNI-group.*

Theorem 1 can be applied to classify the FCI-groups and FNI-groups belonging to a large class of infinite groups. Let

$$\mathfrak{X}$$

denote the smallest class of groups containing all finite groups and all abelian groups which is locally closed and closed with respect to forming ascending series with factors in the class. For example, $\mathfrak{X}$ contains all locally soluble-by-finite groups.

Theorem 3 *Let* G *be an infinite group belonging to the class* $\mathfrak{X}$. *If* G *is an FCI-group, then it is of types* $(\mathrm{i}) - (\mathrm{v})$. *If* G *is an FNI-group which is of type* (iii), *then it has finite torsion-free rank.*

This theorem is a generalization of results of Fernández-Alcober *et al.* [2], who classify locally finite FCI-groups, albeit in a different form. Notice, however, that in the above classification non-periodic groups can occur in all five types. In [3] the locally nilpotent FCI-groups are found; also FCI-groups and FNI-groups in which there are bounds for the indices of centralizers or normalizers are studied in [4]. The groups of these kinds can be identified among the types (i)-(v) on our list.

2 Preliminary results

We begin establishing some properties of groups of types (i)-(v).

Lemma 1 *Groups of types (i)-(v) are metabelian.*

Proof — Consider a group G of type (ii): the other types are obviously metabelian. There is a finite d-embedded subgroup F such that G/F is infinite cyclic or infinite dihedral. In the first case write $G = \langle x \rangle F$. If F is abelian, G is certainly metabelian. Assume that F is hamiltonian. Notice that a power automorphism of a quaternion group of order 8 is inner. Hence $G' = F'[F, x] \leqslant Z(F)$, which is abelian. Next let G/F be infinite dihedral and write $G = \langle x, a, F \rangle$ where $a^x \equiv a^{-1} \bmod F$. Then $G' = F'[F, x][F, a]\langle [a, x] \rangle^F \leqslant Z(F)\langle [a, x] \rangle$ since $[a, x]$ centralizes F. It follows that G' is abelian. □

For a complete understanding of groups of types (iv) and (v) it is necessary to know which power automorphisms of an abelian group have finitely many fixed points. The simplest situation is when the group is non-periodic and the answer is well known.

Lemma 2 *Let* α *be a non-trivial power automorphism of a non-periodic abelian group* A. *Then* α *is inversion in* A *and* $C_A(\alpha)$ *is finite if and only if* $A[2]$ *is finite, i.e.,* $r_2(A)$ *is finite.*

It is a less trivial task to determine when the fixed point subgroup is finite if the group is periodic. Let A be a periodic abelian group with a power automorphism α and let α_p denote the automorphism induced by α in the p-component A_p. Then α_p can be represented by a p-adic integer, which is uniquely determined modulo the exponent of A_p if this is finite. For an account of the power automorphisms of abelian groups see [7].

Lemma 3 *Let* A *be a periodic abelian group with a power automorphism* α *of infinite order. Denote the order of* $(\alpha_p)|_{A[p]}$ *by* e_p*. Then* $C_A(\alpha^k)$ *is finite for all* $k > 0$ *if and only if the following conditions hold.*

(i) A_p *has finite rank for all primes* p*, and if* $|\alpha_p|$ *is finite, then* A_p *is finite.*

(ii) There do not exist infinitely many primes $p_1, p_2, \dots$ *in* $\pi(A)$ *such that* $e_{p_1} = e_{p_2} = \dots$

Proof — Assume first that $C_A(\alpha^k)$ is finite for all $k > 0$. First of all α^{p-1} centralizes $A[p]$, so $A[p]$ is finite and $r_p(A)$ is finite. Next, if α_p has finite order m, then α^m centralizes A_p and the latter is finite. Thus (i) is valid.

Next suppose there exist infinitely many primes p_i, $i = 1, 2, \dots$, in $\pi(A)$ such that the e_{p_i} are all equal to k. Then $\alpha_{p_i}^k$ centralizes $A[p_i]$ for all i and $C_A(\alpha^k)$ is infinite. This contradiction establishes (ii).

To prove the converse assume that conditions (i) and (ii) are valid, but $C_A(\alpha^k)$ is infinite for some $k > 0$. Suppose that $C_{A_p}(\alpha^k)$ is infinite. Then A_p is infinite and, as $r(A_p)$ is finite, $C_{A_p}(\alpha^k)$ has infinite exponent and therefore α^k centralizes A_p. But (i) gives the contradiction that A_p is finite. Therefore $C_{A_p}(\alpha^k)$ is finite for all $p \in \pi(A)$, which implies that $C_{A_p}(\alpha^k) \neq 1$ for infinitely many primes p. For any such prime $(\alpha_p^k)|_{A[p]} = 1$, so that e_p divides k. Therefore infinitely many of the e_p's are equal, which contradicts (ii). □

When the power automorphism has finite order, the fixed point problem is solved in [2]. A short proof of this result follows.

Lemma 4 *Let* A *be a periodic abelian group and* α *a non-trivial power automorphism of* A *with finite order* m*. Let* m_p *denote the order of* α_p*. Then* $C_A(\alpha^k)$ *is finite for all* $1 \leqslant k < m$ *if and only if the following conditions hold.*

(i) $r_2(A)$ *is finite.*

(ii) $\langle A_p \mid m_p < m \rangle$ *is finite.*

(iii) $\langle A_p \mid m_p = m,\ p > 2,\ p \not\equiv 1 \mod m \rangle$ *is finite.*

Proof — Assume that $C_A(\alpha^k)$ is finite for $1 \leqslant k < m$. Certainly $r(A_2)$ is finite since $A[2]$ is centralized by α. If $m_p < m$, then $\alpha^{m_p} \neq 1$ centralizes A_p, so A_p is finite. Suppose there are infinitely many such primes p in $\pi(A)$. Then infinitely many of the m_p are equal – to k say – and therefore $C_A(\alpha^k)$ is infinite. This contradiction establishes (ii).

Next assume that $p > 2$, $m_p = m$ and $p \not\equiv 1 \mod m$. Note that m divides $p^{e-1}(p-1)$ where A_p has exponent p^e. Then p must divide m, for otherwise m divides $p-1$, and there can be only finitely many such primes p. Also $\alpha^{\frac{m}{p}}$ is a power automorphism of order p, so it centralizes $A[p]$ and thus $r_p(A)$ is finite. Moreover, since $p > 2$, this can only happen if A_p has finite exponent, so it is actually finite. Hence (iii) is valid.

Conversely, assume that the three conditions hold, but $C_A(\alpha^k)$ is infinite for some k satisfying $1 \leqslant k < m$. Conditions (ii) and (iii) show that $C_{A_p}(\alpha^k)$ is infinite for some prime p satisfying $m_p = m$ and either $p = 2$ or else $p > 2$ and $p \equiv 1 \mod m$. In the latter case α_p has order dividing $p-1$, so $C_{A_p}(\alpha^k) = 1$. Thus we are left with the sole possibility that $p = 2$ and $2 = m_2 = m$. But then α_2 is inversion in A_2, so $C_{A_2}(\alpha) = A[2]$, which is finite by (i). □

The next lemma records a frequently used property of FCI-groups and appears in [2].

Lemma 5 *Let* $N \lhd G$ *where* N *is finite. If* G *is an FCI-group, then* G/N *is an FCI-group.*

Proof — Let $g \in G$. If $\langle g \rangle \lhd G$, then $\langle g \rangle N/N \lhd G/N$, so assume this is not the case. Then $|C_G(g) : \langle g \rangle|$ is finite. Set $L = C_G(gN)$. If $l \in L$, then $g^l = gx$ where $x \in N$. Hence g has finitely many conjugates in L and $|L : C_L(g)|$ is finite, so $|L : \langle g \rangle|$ is finite. Therefore $|L/N : \langle g \rangle N/N|$ is finite and G/N is an FCI-group. □

Note that FCI is not a quotient closed property, as is shown by non-cyclic free groups. A key role in the theory of FCI-groups is played by the FC-centre.

Lemma 6 *Let* G *be an FCI-group which is not cyclic-by-finite. Then* G *has a unique maximal* d*-embedded subgroup, namely its FC-centre.*

Proof — By Zorn's Lemma there is a maximal d-embedded subgroup N of G. Suppose that $\langle g \rangle \lhd G$ and $g \notin N$. Then $N < \langle g \rangle N$, so there exists $a \in N$ and $i > 0$ such that $\langle g^i a \rangle$ is not normal in G. Hence $|C_G(g^i a) : \langle g^i a \rangle|$ is finite. Now $(g^i a)^G \leqslant \langle g \rangle \langle a \rangle = M$, say, and

$|G : C_G(M)|$ is finite since $\langle g \rangle$ and $\langle a \rangle$ are normal in G. It follows that $|G : C_G(g^i a)|$ is finite and hence $|G : \langle g^i a \rangle|$ is finite, which gives the contradiction that G is cyclic-by-finite. Hence N is the unique largest d-embedded subgroup of G.

Next let F denote the FC-centre of G; then clearly $N \leqslant F$. If $x \in F \setminus N$, then $|G : C_G(x)|$ is finite and also $|C_G(x) : \langle x \rangle|$ is finite, since $\langle x \rangle$ is not normal in G. Hence $|G : \langle x \rangle|$ is finite and again G is cyclic-by-finite, a final contradiction. □

Notice that $\mathrm{Dih}(\infty) \times \mathbb{Z}_3$ is an FCI-group of type (ii) but it does not have a unique maximum d-embedded subgroup. Thus Lemma 6 does not hold for cyclic-by-finite FCI-groups.

The cyclic-by-finite case

It is straightforward to identify those FCI-groups that are cyclic-by-finite.

Lemma 7 *Let G be an infinite cyclic-by-finite FCI-group. Then G is either abelian or a group of type (ii).*

Proof — Assume that G in non-abelian. Let $A \triangleleft G$ where A is infinite cyclic and G/A is finite. Set $C = C_G(A)$; then $|G/C| = 1$ or 2. Also $|C : A|$ is finite and $A \leqslant Z(C)$, so C' is finite. Hence the elements of finite order in C form a finite subgroup F and C/F is infinite cyclic. If $f \in F$, then $\langle f \rangle \triangleleft G$; for otherwise $C_G(f)$ is finite, which is impossible because $A \leqslant C_G(f)$. Hence F is d-embedded in G. If $C = G$, then G/F is infinite cyclic. Otherwise $|G : C| = 2$ and $G = \langle x, C \rangle$ where x inverts elements of A and hence of C/F. Since $x^2 \in C$ and x^2F is fixed by x, it follows that $x^2 \in F$ and G/F is an infinite dihedral group. Hence G is a group of type (ii). □

3 Proof of Theorem 1

We begin by analyzing the centralizer of a maximal abelian normal subgroup in an FCI-group.

Lemma 8 *Let G be an FCI-group with an infinite maximal abelian normal subgroup A and which is such that periodic factors are locally finite. Then $A = C_G(A)$ and, if G is not cyclic-by-finite, A is d-embedded in G and G/A is abelian.*

Proof — Assume $A \neq C = C_G(A)$ and let $c \in C \setminus A$. Then $\langle c \rangle$ cannot be normal in G: for otherwise $\langle c, A \rangle$ is abelian and normal in G, contradicting the maximality of A. Therefore $|C_G(c) : \langle c \rangle|$ is finite and $C_G(c)$ is cyclic-by-finite. Since $A \leqslant C_G(c)$ and A is infinite, it follows that A is non-periodic.

Next $A\langle c \rangle / \langle c \rangle$ is finite. Consequently $A \cap \langle c \rangle \neq 1$, so C/A is periodic and hence locally finite (this is the only point in the proof of Theorem 1 where the hypothesis on periodic factors is used). Since $A \leqslant Z(C)$, it follows that C' is locally finite. This shows that the elements of finite order in C form a subgroup T containing C'. If $t \in T$, then $C_G(t)$ contains A, so it is infinite and thus $\langle t \rangle \triangleleft G$. Therefore $\langle t, A \rangle$ is abelian and normal in G, so $t \in A$ and hence $T \leqslant A$.

Notice that C/T is a torsion-free abelian group of rank 1 since A is cyclic-by-finite and C/A is periodic. Let $u, v \in C$ and observe that $\langle u, v \rangle T/T$ is cyclic, equal to $\langle w \rangle T/T$ let us say. Write $u = w^m s$ and $v = w^n t$ where $s, t \in T$. Then $[u, v] = 1$ since $T \leqslant A \leqslant Z(C)$. Therefore C is abelian and $C = A$ by maximality.

Finally, assume that G is not cyclic-by-finite. Let $a \in A$. If $|C_G(a) : \langle a \rangle|$ is finite, A is cyclic-by-finite, and, as Aut(A) is clearly finite, G/A is finite and G is cyclic-by-finite. It follows that $|C_G(a) : \langle a \rangle|$ is infinite, so that $\langle a \rangle \triangleleft G$ and A is d-embedded in G. Therefore G/A is abelian, being a group of power automorphisms of A. □

At this point we begin the proof of Theorem 1 in earnest. Let G be an FCI-group satisfying the hypotheses of Theorem 1; thus G has an infinite maxi- mal abelian normal subgroup A and all periodic factors of G are locally finite. We can assume that G is neither dedekind nor cyclic-by-finite by Lemma 7, assumptions that will be maintained throughout the proof. The next lemma identifies the groups of type (iii).

Lemma 9 *If A is non-periodic, then G is a group of type (iii). If G is an FNI-group, then $r_0(A)$ is finite.*

Proof — We know from Lemma 8 that $C_G(A) = A$ and A is d-embedded in G. Since A is non-periodic and $A \neq G$, we have $|G : A| = 2$ and $G = \langle x, A \rangle$ where $a^x = a^{-1}$ for all $a \in A$. Also $x^2 \in A$, so $x^4 = 1$ and $x^2 \in A[2]$. Now $\langle x \rangle$ cannot be normal in G, since otherwise $A^2 = [A, x] \leqslant \langle x \rangle$ and $A^4 = 1$. Therefore $C_G(x)$ is finite and hence $C_A(x) = A[2]$ is finite. Thus G is a group of type (iii).

Now assume that G is an FNI-group and yet $r_0(A)$ is infinite. Then there exists in A an infinite, linearly independent subset $\{a_1, a_2, \dots\}$

with each a_i of infinite order. Put $H = \langle x, a_1^4, a_2^4, \dots \rangle$; then H is not normal in G since $a_1^2 = [x, a_1] \notin H$. On the other hand, $N_G(H) \geqslant \langle x, a_1^2, a_2^2, \dots \rangle$, which implies that $|N_G(H) : H|$ is infinite and G is not an FNI-group. □

4 Proof of Theorem 1 (continued)

Let G be a group satisfying the conditions of Theorem 1 and assume that it is neither dedekind nor cyclic-by-finite. We use the previously established notation, but with the stipulation that the maximal abelian normal subgroup A is periodic.

Lemma 10 *There is a subgroup X such that $G = XA$ and $X \cap A$ is finite.*

Proof — Since G is not a dedekind group, it has a non-normal subgroup $\langle g \rangle$. Hence $|C_G(g) : \langle g \rangle|$ is finite and $C_G(g)$ is cyclic-by-finite, so $C_A(g) = C_G(g) \cap A$ is finite. Let p be a prime. By Lemma 8 the element g induces a power automorphism in A_p, say $a \mapsto a^{\alpha_p}$, $(a \in A)$, where α_p is a p-adic integer. Now $C_{A_p}(g) \neq 1$ if and only if $p \in \pi(A)$ and $\alpha_p \equiv 1 \bmod p$, and if this happens $A[p]$ is finite and $r_p(A_p)$ is finite. Define

$$\pi = \{p \in \pi(A) | \alpha_p \equiv 1 \bmod p\}.$$

Since $C_A(g)$ is finite, π is a finite set and hence A_π has finite rank. It follows that there is a finite subgroup F of A_π such that A_π/F is divisible and $A_\pi = [A_\pi, g]F$. If on the other hand q is a prime and $q \notin \pi$, then $[A_q, g] = A_q$. Therefore $[A/F, gF] = A/F$ and it is easy to see that $C_{A/F}(gF)$ is finite.

Define $X/F = C_{G/F}(gF)$. For any $u \in G$ we have $[u, g] \in A$, since G/A is abelian by Lemma 8. Hence $[u, g] = [a, g]f$ where $a \in A, f \in F$. Therefore $[ua^{-1}, g] = f$, so that $ua^{-1} \in X$ and $u \in XA$, which shows that $G = XA$. Also

$$(X \cap A)/F = C_{G/F}(gF) \cap (A/F) = C_{A/F}(gF),$$

which is finite. Therefore $X \cap A$ is finite. □

With the notation of Lemma 10 we define

$$U = X \cap A, \quad E = C_X(A/U) \quad \text{and} \quad N = EA.$$

Regarding these subgroups, we prove:

Lemma 11 *The subgroup* E *is finite and* N *is the unique maximum* d*-embedded subgroup of* G.

Proof — Let $e \in E$. Then $[A, e] \leqslant U$ and U is finite by Lemma 10, so $[A, e^k, e^k] = 1$ where $k = |\mathrm{Aut}(U)|$. Hence $[A, e^{kl}] = [A, e^k]^l = 1$ where $l = |U|$, so $E/C_E(A)$ is periodic. Also $[A, E] \leqslant U$, so $[A_p, E] = 1$ if $p \notin \pi = \pi(U)$, and thus $C_E(A) = C_E(A_\pi)$. Now $E/C_E(A_p)$ is finite, being a periodic group of power automorphisms of A_p. Since $\pi(U)$ is finite, it follows that $E/C_E(A)$ is finite. But $C_E(A) = U$, because $C_G(A) = A$, and we conclude that E is finite.

Let D denote the FC-centre of G, which by Lemma 6 is the maximum d-embedded subgroup of G. Let $u \in N$ and write $u = ea$ where $e \in E$, $a \in A$. Then

$$u^G = (ea)^G \leqslant E^A\langle a\rangle \leqslant EU\langle a\rangle = E\langle a\rangle,$$

which is finite. Hence $|G : C_G(u)|$ is finite and $u \in D$, which shows that $N \leqslant D$. Next $D = D \cap (XA) = (D \cap X)A$. If $t \in D \cap X$, then $[A, t] \leqslant \langle t\rangle \cap A \leqslant U$, so $t \in E$. Therefore $D \leqslant EA = N$. □

Next comes a major step in the proof.

Lemma 12 *The group* X/E *is cyclic.*

Proof — Write $\overline{X} = X/E$, noting that $\overline{X}$ is abelian and X is finite-by-abelian, since A is d-embedded in G. We show first that $\overline{X}$ is locally cyclic. Suppose $\overline{X}$ has a subgroup $\langle\overline{x}\rangle \times \langle\overline{y}\rangle$ where $\overline{x} = xE$ has infinite order and $\overline{y} = yE \neq 1$. Put $H = \langle x, y\rangle$, which is finitely generated, finite-by-abelian, and hence is centre-by-finite. Therefore some positive power of x centralizes y. Now $\langle y\rangle$ is not normal in G, since otherwise $y \in X \cap N = E$ by Lemma 11. Hence $|C_G(y) : \langle y\rangle|$ is finite and some positive power of x belongs to $\langle y\rangle$, which is a contradiction.

Next suppose that $\overline{X}$ has a subgroup $\langle\overline{x}\rangle \times \langle\overline{y}\rangle$ where $\overline{x} = xE$ and $\overline{y} = yE$ both have prime order q. Assume first that $\pi(A)$ is infinite and let $p \in \pi(A)$ be odd. Since the power automorphism group of $(A/U)[p]$ is cyclic, some element $x^{r_p}y^{s_p}$ centralizes $(A/U)[p]$ where $0 \leqslant r_p, s_p < q$ and $(r_p, s_p) \neq (0, 0)$. Since there are only finitely many such pairs (r_p, s_p), there exists $(r, s) \neq (0, 0)$ such that $0 \leqslant r, s < q$ and $z = x^r y^s$ centralizes $(A/U)[p]$ for infinitely many primes p in $\pi(A) \setminus \pi(U)$. For such primes p we have $[A[p], z] \leqslant U \cap A_p = 1$. Therefore $|C_G(z) : \langle z\rangle|$ is infinite, which implies that $\langle z\rangle \lhd G$ and $z \in X \cap N = E$, a contradiction.

It follows that $\pi(A)$ is finite, which means that some A_p is infinite. First assume that p is odd. Then there is an element $z = x^r y^s$, with $x, y \in X$, that centralizes $(A/U)[p]$, where $0 \leqslant r, s < q$ and $(r, s) \neq (0, 0)$. Note that $r_p(A)$ is finite, since otherwise $C_{A_p}(z)$ is infinite, $\langle z \rangle \triangleleft G$ and $z \in X \cap N = E$. Hence A_p has infinite exponent. Next z induces a power automorphism in $(A/U)_p$ of the form $a \mapsto a^\alpha$ where the p-adic integer α satisfies $\alpha \equiv 1 \bmod p$. But α has finite order and $p > 2$, which shows that $\alpha = 1$ and $[A_p, z] \leqslant U$. Therefore $C_{A_p}(z)$ is infinite and $\langle z \rangle \triangleleft G$, so that, as before, $z \in E$, a contradiction.

As a consequence we must have $p = 2$: a slight modification of the argument is called for in this case. First note that $A[2]$ is finite and A_2 has infinite exponent. The power automorphism group of $(A/U)[4]$ has order 2, so there is an element $z = x^r y^s$, with $x, y \in X$, that centralizes $(A/U)[4]$, where $0 \leqslant r, s < q$ and $(r, s) \neq (0, 0)$. Let α represent the power automorphism of $(A/U)_2$ induced by z. Then $\alpha \equiv 1 \bmod 4$ and α has finite order. Since A_2 has infinite exponent, it follows that $\alpha = 1$ and $[A_2, z] \leqslant U$, which implies that $z \in E$. This contradiction completes the proof that $\overline{X}$ is locally cyclic.

Finally, $X \neq E$ since otherwise $G = XA = EA = N$, a dedekind group. Let $y \in X \setminus E$. Then $\langle yU \rangle$ is not normal in G/U, because the contrary leads to $[A, y] \leqslant \langle y \rangle U \cap A = U$ and $y \in E$. From Lemma 5 we see that G/U is an FCI-group, so $C_{G/U}(yU)$ is cyclic-by-finite; but $C_{G/U}(yU) \geqslant X/U$ since X/U is abelian. Therefore X/U is finitely generated and $\overline{X} = X/E$ is cyclic. □

From now on we will write

$$X = \langle x \rangle E, \text{ so that } G = \langle x \rangle N.$$

It is now possible to identify the groups of type (iv).

Corollary 1 *If* N *is abelian, then* G *is a group of type (iv).*

Proof — With the above notation we have $N = A$ by maximality of A, so that $G = \langle x \rangle A$. Let $y \in G \setminus A$. Now $\langle y \rangle$ cannot be normal in G since otherwise $y \in N = A$. Hence $|C_G(y) : \langle y \rangle|$ is finite and thus $C_A(y)$ is finite. It follows that G is a group of type (iv). □

The final step in the proof of Theorem 1 identifies G as a group of the type (v) if N is non-abelian.

Lemma 13 *If* N *is non-abelian, then* G *is a group of type (v).*

Proof — Since N is a dedekind group, it has the form $N = H \times N_{2'}$ where H is a hamiltonian 2-group and $N_{2'}$ is abelian. The element x induces a power automorphism in H which is either trivial or conjugation by an element k of order 4 in H: in the latter event we can replace x by xk^{-1}. Thus we may assume that $[H, x] = 1$ and

$$G = \langle x\rangle N = \langle x\rangle HN_{2'} = H\overline{G}$$

where $\overline{G} = \langle x\rangle N_{2'}$. Since $[H, \overline{G}] = 1$, it follows that $H \cap \overline{G} \leqslant Z(G)$ and $G = H \curlyvee \overline{G}$ is a central product. Notice that $\langle x\rangle$ is not normal in G; for otherwise $G = N$ is a dedekind group. Hence $|C_G(x) : \langle x\rangle|$ is finite and thus $C_N(x)$ is finite. Since $[H, x] = 1$, it follows that H is finite.

Let $B = C_{\overline{G}}(N_{2'})$; then B centralizes $N_{2'}$ and H, and hence N. Therefore $[A, B] = 1$ and $B \leqslant C_G(A) = A$. Thus B is infinite, periodic abelian, and is d-embedded in G. Since $A \leqslant N$, we have $B \leqslant N$ and hence $B_2 \leqslant H \cap \overline{G}$. In addition $H \cap \overline{G} \leqslant C_{\overline{G}}(N_{2'}) = B$ and thus $H \cap \overline{G} \leqslant B_2$. It follows that $H \cap \overline{G} = B_2$. Also $H \cap \overline{G} \leqslant Z(H) = H[2]$ and $\overline{G}/N_{2'}$ is cyclic, so B_2 is cyclic and hence $|B_2| = 1$ or 2. In addition $B = (B \cap \langle x\rangle)N_{2'}$ so that $B_2 \leqslant \langle x\rangle$.

To complete the proof we need to show that $\overline{G} = \langle x\rangle B$ is a group of type (iv). Recall that B is d-embedded in G. Let $y \in \overline{G} \setminus B$ and assume that $\langle y\rangle \lhd \overline{G}$. Then

$$y \in N \cap \overline{G} = (H \cap \overline{G}) \times N_{2'} = B_2 \times N_{2'} \leqslant B.$$

By this contradiction $|C_G(y) : \langle y\rangle|$ is finite and hence $C_B(y)$ is finite, showing that $\overline{G}$ is of type (iv). □

5 Proof of Theorem 2

It must be proven that groups of types (i), (ii), (iv), (v), as well as groups of type (iii) which have finite torsion-free rank, are FNI-groups, and also that all groups of type (iii) are FCI-groups.

(i) Obviously dedekind groups are FNI.

(ii) Let G be a group of type (ii) and K a non-normal subgroup of G; then $K \not\leqslant F$ since F is d-embedded in G. If K is infinite, then $|G : K|$ is finite and so $|N_G(K) : K|$ is finite. Thus we may assume that K is finite. Now G/F cannot be infinite cyclic, since otherwise $K \leqslant F$. Thus G/F is infinite dihedral and there is an infinite cyclic group A/F with $|G : A| = 2$. Clearly $K \nleq A$ and if $k \in K \setminus A$, then k induces inversion in A/F. If $a \in N_A(K)$, then KF contains $[a, k]F = a^{-2}F$. Thus a has finite order and it follows that $N_A(K)$ is periodic and hence finite. Finally, $N_G(K)$ is finite since $|G : A|$ is finite, and G is an FNI-group.

(iii) Let $G = \langle x \rangle A$ be a group of type (iii); we show first that G is an FCI-group. Let $g \in G$ where $\langle g \rangle$ is not normal in G; thus $g \notin A$. Since $x^2 \in A$, we can write $g = xa$ where $a \in A$. Then $C_A(g) = A[2]$ since x inverts in A, and $A[2]$ is finite by hypothesis. Also $G = \langle g \rangle A$ and $C_G(g) = C_G(g) \cap (\langle g \rangle A) = \langle g \rangle C_A(g) = \langle g \rangle A[2]$. Hence $|C_G(g) : \langle g \rangle|$ is finite and G is an FCI-group. Now let us assume that $r_0(G)$ is finite. Let K be a non-normal subgroup of G. Then $K = \langle xa, K \cap A \rangle$ where $a \in A$. Next $|N_G(K) : K|$ is finite if and only if $|N_A(K) : K \cap A|$ is finite, since $G = KA$. Observe that $N_A(K)$ consists of all $c \in A$ such that $[xa, c] = c^2 \in K \cap A$. Hence $N_A(K)/K \cap A = (A/K \cap A)[2]$, which is finite because $r_2(A)$ and $r_0(A)$ are finite.

(iv) Let $G = \langle x \rangle A$ be a group of type (iv). To show that G is an FNI-group, let K be a non-normal subgroup; thus $K \nleqslant A$. Write $K = \langle ya, K \cap A \rangle$ where $y \in \langle x \rangle \setminus A$ and $a \in A$. Then $N_A(K)$ is the set of all $b \in A$ such that $[ya, b] = [y, b] \in K \cap A$, which shows that $N_A(K)/K \cap A = C_{A/K \cap A}(y)$. Now $C_A(y)$ is finite and, keeping in mind that y induces a power automorphism in A, one can easily prove that $C_{A/K\cap A}(y)$ is also finite; hence $|N_A(K) : K \cap A|$ is finite. Finally,

$$|N_G(K)A : KA| \leqslant |\langle x \rangle A : \langle y \rangle A| \leqslant |\langle x \rangle : \langle y \rangle|,$$

which is finite since $y \neq 1$. Consequently $|N_G(K) : K|$ is finite and G is an FNI-group.

(v) Let $G = H \curlyvee \overline{G}$ be a group of type (v) with $\overline{G} = \langle x \rangle B$. We show that G is an FNI-group. Let $L = B_{2'}$ and note that $C_{\overline{G}}(L) = C_{\overline{G}}(B)$ since $B_2 \leqslant H[2] \leqslant Z(G)$. Suppose there exists $y \in C_{\overline{G}}(L) \setminus B$. Since $\overline{G}$ is a group of type (iv), $C_{\overline{G}}(y)$ is finite and B is finite. By this contradiction $C_{\overline{G}}(L) = B$ and $C_G(L) = HB = H \times L$. Hence $C_G(L)$ is d-embedded in G. Now let $K \leqslant G$; if $[L, K] = 1$, then $K \leqslant C_G(L)$ and $K \lhd G$. Thus we may assume that $[L, K] \neq 1$ and prove that $|N_G(K) : K|$ is finite. By hypothesis K contains an element yc where $y \in \langle x \rangle$, $c \in HB$ and $[L, y] \neq 1$. Suppose that $a \in N_L(K)$; then we have $[yc, a] = [y, a] \in K \cap L$, and $a(K \cap L) \in C_{L/K\cap L}(y)$. Therefore $N_L(K)/K\cap L = C_{L/K\cap L}(y)$. Now $y \notin B$, so $C_L(y)$ is finite and, as noted in the proof of (iv), we can deduce that $C_{L/K\cap L}(y)$ is also finite. Hence $N_L(K)/K \cap L$, i.e., $N_G(K) \cap L/K \cap L$ is finite. It suffices to prove that $|N_G(K)L : KL|$ is finite. Observe that $|N_G(K)L : KL| \leqslant |G : \langle yc\rangle L|$ and $\langle yc\rangle BH = \langle y\rangle BH$. Since B_2 and H are finite, $|\langle yc\rangle BH : \langle yc\rangle L|$ is finite. Therefore

$$\begin{aligned} |G : \langle yc\rangle L| &= |\langle x\rangle BH : \langle y\rangle BH| \cdot |\langle y\rangle BH : \langle yc\rangle L| \\ &\leqslant |\langle x\rangle : \langle y\rangle| \cdot |\langle y\rangle BH : \langle yc\rangle L|, \end{aligned}$$

which is finite since $y \neq 1$. Therefore $|N_G(K)L : KL|$ is finite and G is an FNI-group.

6 Proof of Theorem 3

Let G be an infinite $\mathfrak{X}$-group. It is easy to see that periodic factors of G are locally finite. Therefore in order to establish Theorem 3 it is enough to prove that G has an infinite abelian normal subgroup. The first step in the proof is to establish the following result.

Lemma 14 *Let G be an FCI-group in which every abelian normal subgroup is finite. Then:*

(i) locally finite subgroups of G are finite;

(ii) abelian subgroups of G are cyclic-by-finite.

Proof — (i) Suppose that L is an infinite, locally finite subgroup of G. Then L has an infinite abelian subgroup A by the Hall-Kulatilaka-Kargapolov Theorem (see [8], 3.43). Since A cannot be normal in G, there exists $a \in A$ such that $\langle a\rangle$ is not normal in G. Hence $|C_G(a):\langle a\rangle|$ is finite, which implies that A is finite, a contradiction.
(ii) Let B an infinite abelian subgroup of G. By (i) B cannot be periodic, so it contains an element b of infinite order. Then $\langle b\rangle$ is not normal in G, so $|C_G(b) : \langle b\rangle|$ is finite and thus B is cyclic-by-finite. □

Corollary 2 *Let G be an FCI-group which has an infinite ascendant abelian subgroup. Then G has an infinite abelian normal subgroup.*

Proof — Let H be an infinite ascendant abelian subgroup of G and assume that G has no infinite abelian normal subgroups. Then H^G

is locally nilpotent, being generated by ascendant abelian subgroups (see [8], 2.3). By Lemma 14 abelian subgroups of H^G are finitely generated and it follows from results of Mal'cev ([6], Theorem 5; see also [8], 6.3) that H^G is a finitely generated, infinite nilpotent group. Therefore $Z(H^G)$ is an infinite abelian normal subgroup of G. □

A consequence of Corollary 2 is a weakening of the hypotheses in Theorem 1: *it suffices to assume that the group has an infinite ascendant abelian subgroup.*

The final step in the proof of Theorem 3 is provided by next lemma.

Lemma 15 *Let* G *be an infinite* $\mathfrak{X}$*-group. If* G *is an FCI-group, then it has an infinite abelian normal subgroup.*

Proof — Using the notation of group classes and closure operations (see [8], 1.1), we have

$$\mathfrak{X} = \bigcup_{\alpha} (\mathbf{L}\acute{\mathbf{P}})^{\alpha} \mathfrak{X}_0$$

where the union is over all ordinals α and $\mathfrak{X}_0$ is the class of all groups which are finite or abelian. Assume the result is false, so that there is a least ordinal α such that the result fails for some group G in the class $(\mathbf{L}\acute{\mathbf{P}})^{\alpha}\mathfrak{X}_0$. Clearly α cannot be a limit ordinal and the result holds for all infinite FCI-groups in $\mathfrak{Y} = (\mathbf{L}\acute{\mathbf{P}})^{\alpha-1}\mathfrak{X}_0$.

Next $G \in \mathbf{L}\acute{\mathbf{P}}\mathfrak{Y}$ and by Lemma 14 the group G cannot be locally finite, so it has a finitely generated infinite subgroup X. Then $X \in \acute{\mathbf{P}}\mathfrak{Y}$ since $\mathfrak{Y}$ is subgroup closed, and hence X has an ascending series $\{X_\beta\}$ with factors in the class $\mathfrak{Y}$. There is a least ordinal β such that X_β is infinite. Now β cannot be a limit ordinal since otherwise X_β would be locally finite and hence finite. Hence $X_{\beta-1}$ is finite, while $X_\beta/X_{\beta-1}$ is an infinite $\mathfrak{Y}$-group. In addition Lemma 2 shows that $X_\beta/X_{\beta-1}$ is an FCI-group. Consequently $X_\beta/X_{\beta-1}$ has an infinite abelian normal subgroup $A/X_{\beta-1}$ by minimality of α. Now A is finite-by-abelian, so it has a nilpotent normal subgroup with finite index. Since its abelian subgroups are finitely generated, we see easily that A has an infinite subnormal abelian subgroup B. But B is subnormal in X, so X has an infinite abelian normal subgroup by Corollary 2.

We can now apply Theorem 1 and Lemma 1 to show that X is metabelian. Since this conclusion is valid for every finitely generated, infinite subgroup X, the group G itself is metabelian. This implies that G has an infinite abelian normal subgroup, a final contradiction. □

Theorem 3 is now an immediate consequence of Theorem 1 and Lemma 15.

REFERENCES

[1] S.I. Adian: "The Burnside Problem and Identities in Groups", *Springer*, Berlin (1979).

[2] G.A. Fernández-Alcober, L. Legarreta, A. Tortora and M. Tota: "A finiteness condition on centralizers in locally finite groups", *arXiv*: 1510.03733.

[3] G.A. Fernández-Alcober, L. Legarreta, A. Tortora and M. Tota: "A finiteness condition on centralizers in locally nilpotent groups", *Monatsh. Math.*, to appear, DOI: 10.1007/s00605-015-0854-0.

[4] G.A. Fernández-Alcober, L. Legarreta, A. Tortora and M. Tota: "Some finiteness conditions on normalizers or centralizers in groups", *arXiv*: 1412.7330v2.

[5] J.C. Lennox and D.J.S. Robinson: "The Theory of Infinite Soluble Groups", *Clarendon Press*, Oxford (2004).

[6] A.I. Mal'cev: "On certain classes of infinite soluble groups", *Mat. Sb.* 28 (1951), 567–588 = *Amer. Math. Soc. Translations* 2 (1956), 1–21.

[7] D.J.S. Robinson: "Groups in which normality is a transitive relation", *Proc. Cambridge Philos. Soc.* 60 (1964), 21–38.

[8] D.J.S. Robinson: "Finiteness Conditions and Generalized Soluble Groups", *Springer*, Berlin (1972).

[9] A. Shalev: "Profinite groups with restricted centralizers", *Proc. Amer. Math. Soc.* 122 (1994), 1279–1284.

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