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Groups in which all Subgroups are Subnormal-by-Finite

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To professor Hermann Heineken on his eightieth birthday

Abstract

We prove that a locally finite group G in which every subgroup is a finite extension of a subnormal subgroup of G is nilpotent-by-Černikov.

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1 Introduction

A subgroup H of the group G is said to be *subnormal-by-finite* if there exists a subnormal subgroup S of G such that $S \leqslant H$ and $|H : S|$ is finite. The aim of this paper is to carry on the investigation started by Hermann Heineken in [4] abount the class of groups in which every subgroup is subnormal-by-finite. Based on Heineken's results, we take advantage of the many informations now available on groups in which every subgroup is subnormal ($\mathfrak{N}_1$-groups) and of some of the techniques developed in treating them (notably by W. Möhres, a PhD student of Heineken's). In [2] it is proved that a locally finite $\mathfrak{N}_1$-group is nilpotent-by-Černikov; our main result states that this is true for groups in which every subgroup is subnormal-by-finite.

Theorem *Let* G *be a locally finite group in which every subgroup is subnormal-by-finite. Then*

(A) G *is nilpotent-by-Černikov;*

(B) there exists an integer $d \geqslant 1$ *such that every subgroup of* G *admits a subgroup of finite index which is subnormal of defect at most* d *in* G*.*

We recall that the Hirsch-Plotkin radical of a group G is the largest normal locally nilpotent subgroup of G, while the Baer radical of G is the subgroup generated by all cyclic subnormal subgroups of G. Clearly, the Baer radical of G is a characteristic subgroup of G, and it is always contained in the Hirsch-Plotkin radical. A Baer group is a group which coincides with its Baer radical, that is a group in which every cyclic (and thus every finitely generated) subgroup is subnormal.

Let us state a Theorem gathering those results in Heineken's paper [4] which will be the starting point of our investigation.

Theorem 1.1 (Heineken [4]) *Let* G *be a group in which every subgroup is subnormal-by-finite.*

1. *If* G *is a Baer group then* $G \in \mathfrak{N}_1$*;*
2. *if* G *is locally finite then the Hirsch-Plotkin radical* H *of* G *has finite index in* G*, and there exists an integer* $c \geqslant 1$ *such that all but a finite number of primary components of* H *are nilpotent of class at most* c*.*

Since, by a Theorem of Möhres, $\mathfrak{N}_1$-groups of finite exponent are nilpotent, we may easily deduce the following Corollary, which will be relevant in our arguments.

Corollary 1.2 *Let* G *be a locally finite group in which every subgroup is subnormal-by-finite; if* G *has finite exponent then* G *is nilpotent-by-finite.*

Proof — By point 2 in Heineken's Theorem, we may assume that the group G is locally nilpotent. Let B be the Baer radical of G; then $B \in \mathfrak{N}_1$ and so, by the aforementioned result of Möhres, B is nilpotent. Suppose, by contradiction, that G/B is infinite; then, being locally nilpotent, it admits an infinite abelian subgroup. By the subnormal-by-finite property it follows from that there exists a subnormal subgroup H of G containing B, such that H/B is an infinite abelian group. Then, H is a soluble locally nilpotent group of finite exponent, hence, by a well known fact, H is a Baer group. Since H is subnormal in G, we reach the contradiction $H \leqslant B$. □

Let us conclude this introduction with a remark concerning the concept of commensurable pairs of subgroups. Two subgroups A, B of the group G are said to be *commensurable* if $A \cap B$ has finite index both in A and in B. Suppose, in this case, that A is subnormal in G. Then the normal core $S = (A \cap B)_A$ is a normal subgroup of A and $|A/S|$ is finite; hence S is subnormal in G and $|B : S|$ is finite. It follows that, for a subgroup B of a group G, being subnormal-by-finite is equivalent to be commensurable with a subnormal subgroup. Thus, groups with all subgroups subnormal-by-finite are precisely those groups in which every subgroup is commensurable with a subnormal subgroup.

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2 Preliminaries

The arguments leading to the proof of Theorem 1 often reduce to the case of a metabelian group. We thus collect in this section the relevant results (many of those are certainly known) concerning actions of abelian groups on abelian groups. Let us start by recalling that if A is an abelian group and $x \in \mathrm{Aut}(A)$, then $[A, \langle x\rangle] = \{[a, x] \mid a \in A\}$.

Lemma 2.1 *Let A be an abelian divisible group and x an automorphism of A of finite order. Then $[A, x] = [A, x, x]$ and $A = [A, x]C_A(x)$.*

PROOF — Let $m = |x|$ and write $[A, x, x] = N$. Notice that A/N is divisible; given $a \in A$, there exists $b \in A$ such that $Na = Nb^m$. Hence, modulo N,

$$[a, x] = [b^m, x] = [b, x]^m = [b, x^m] = 1,$$

thus proving $N = [A, x]$. Then $[a, x] = [c, x, x]$ for some $c \in A$ and therefore $a[c, x]^{-1} \in C_A(x)$. □

Given a group G and an element $x \in G$, we write

$$D_G(x) = \{g \in G \mid [g, x, x] = 1\}.$$

This is not in general a subgroup; however such is the case in some groups which will be relevant in our proofs.

Lemma 2.2 *Let* G *be a metabelian group. Then* $D_G(x)$ *is a subgroup of* G *for every* $x \in G$.

Proof — Let G be metabelian, $x \in G$, and $a, b \in D_G(x)$. Then

$$[a^{-1}, x, x] = [x, a, x^a]^{a^{-1}} = [x, a, x[x, a]]^{a^{-1}} = [x, a, x]^{a^{-x}} = 1,$$

whence $a^{-1} \in D_G(x)$. Similarly, as G' is abelian, we have

$$[[a, x]^b, x] = [[a, x], x^{b^{-1}}]^b = [[a, x], x[x, b^{-1}]]^b = [a, x, x]^{[x,b^{-1}]b} = 1,$$

and therefore

$$[ab, x, x] = [[a, x]^b[b, x], x] = [[a, x]^b, x]^{[b,x]}[b, x, x] = 1.$$

Thus, $ab \in D_G(x)$. □

Lemma 2.3 *Let* p *be a prime, and* A *a normal abelian divisible subgroup of the* p*-group* G*, such that* G/A *is abelian. Then for every* $x \in G \setminus A$,

(i) $G = [A, x]D_G(x)$;

(ii) $[A, x] \cap D_G(x)$ *has finite exponent and is normal in* G;

(iii) $(A \cap D_G(x))/([A, x] \cap D_G(x))$ *is divisible.*

Proof — Let $x \in G \setminus A$; then $D_G(x) \leqslant G$ by Lemma 2.2. Let $g \in G$, then $[g, x] \in A$, hence, by Lemma 2.1, there exist $a \in A$, $c \in C_A(x)$ with $[g, x] = [a, x]c$. Therefore,

$$[ga^{-1}, x] = [g, x]^{a^{-1}}[a^{-1}, x] = [g, x][a, x]^{-1} = c.$$

This shows that $ga^{-1} \in D_G(x)$, and so $g \in AD_G(x) = [A, x]D_G(x)$.

Observe that $[A, x]$ is normal in G because G/A is abelian; hence

$$[A, x] \cap D_G(x) \trianglelefteq [A, x]D_G(x) = G.$$

Let $p^m = |x|$, and let $u \in [A, x] \cap D_G(x)$. Then $[u, x, x] = 1$ and, by Lemma 2.1, $u = [a, x]$ for some $a \in A$; then

$$[a^{p^m}, x, x] = [[a, x]^{p^m}, x] = [u^{p^m}, x] = [u, x]^{p^m} = [u, x^{p^m}] = 1.$$

Hence $a^{p^m} \in D_G(x)$, and

$$u^{p^{2m}} = [a, x]^{p^{2m}} = [a^{p^m}, x]^{p^m} = [a^{p^m}, x^{p^m}] = 1,$$

which yields (ii).

To prove (iii) just observe that

$$\frac{A \cap D_G(x)}{[A, x] \cap D_G(x)} \simeq \frac{[A, x](D_G(x) \cap A)}{[A, x]} = \frac{[A, x]D_G(x) \cap A}{[A, x]} = \frac{A}{[A, x]}$$

is a factor of the divisible group A. □

Lemma 2.4 *Let* A *be a normal abelian divisible subgroup of the* p*-group* G*, and suppose that* G/A *a countable abelian group. Then there exists* $H \leqslant G$ *such that* H *is a Baer group and* $AH = G$*.*

Proof — If G/A is finite there exists a finitely generated (hence nilpotent) subgroup H of G such that $AH = G$. Thus, suppose G/A be infinite and let $X = \{x_n \mid n \in \mathbb{N}\} \subseteq G$ be a set of representatives of G modulo A, with $x_0 = 1$.

We let $y_0 = x_0 = 1$, $H_0 = D_G(x_0) = G$ and $N_0 = \{1\}$. Suppose that for $n \in \mathbb{N}$ we have found elements $y_0, y_1, \dots y_n$ in G and subgroups $H_0, H_1, \dots, H_n$ and $N_0, N_1, \dots, N_n$, such that, for every $i = 0, \dots, n$:

(i) $y_i \in H_i$ and $Ay_i = Ax_i$;

(ii) $AH_i = G$ and $H_i \leqslant H_{i-1}$ (for $i \geqslant 1$);

(iii) N_i is a normal subgroup of H_i of finite exponent, $N_i \leqslant A$ and $(A \cap H_i)/N_i$ is divisible;

(iv) $\langle y_i \rangle^{H_i}$ has finite exponent.

We now prove the existence of $N_{n+1} \trianglelefteq H_{n+1} \leqslant G$ and $y_{n+1} \in H_{n+1}$ such that these same properties hold up to $n + 1$.

Since $AH_n = G$ there exists $y_{n+1} \in H_n$ such that $Ay_{n+1} = Ax_{n+1}$. Now, $(A \cap H_n)/N_n$ is a normal divisible subgroup of H_n/N_n and $H_n/(A \cap H_n) \simeq AH_n/A$ is abelian.

Let $H_{n+1}/N_n = D_{H_n/N_n}(N_n y_{n+1})$, then $H_n = (A \cap H_n)H_{n+1}$ by Lemma 2.3. Hence, $y_{n+1} \in H_{n+1} \leqslant H_n$ and

$$AH_{n+1} = A(A \cap H_n)H_{n+1} = AH_n = G.$$

Also, writing $\overline{A} = (A \cap H_n)/N_n$, $\overline{H}_{n+1} = H_{n+1}/N_n$, Lemma 2.3 yields that

$$[\overline{A}, y_{n+1}] \cap \overline{H}_{n+1}$$

is a normal subgroup of finite exponent of $\overline{H}_{n+1}$. Letting N_{n+1} be its inverse image modulo N_n, we have that $N_{n+1} \trianglelefteq H_{n+1}$; moreover, as N_n has finite exponent, N_{n+1} has finite exponent. Also,

$$\frac{A \cap H_{n+1}}{N_{n+1}} \simeq \frac{\overline{A} \cap \overline{H}_{n+1}}{N_{n+1}/N_n}$$

is divisible by Lemma 2.3. Finally, from $H_{n+1}/N_n = D_{H_n/N_n}(N_n y_{n+1})$ it follows, by definition, that the normal closure

$$\langle N_n y_{n+1} \rangle^{H_{n+1}/N_n} = \langle y_{n+1} \rangle^{H_{n+1}} N_n/N_n$$

is abelian and thus of finite exponent. Since N_n has finite exponent we conclude that $\langle y_{n+1} \rangle^{H_{n+1}}$ is metabelian of finite exponent.

Thus, we recursively determine an infinite descending chain of subgroups of G,

$$H_0 \geqslant H_1 \geqslant H_2 \geqslant \dots$$

and find elements $y_i \in H_i$ $(i \in \mathbb{N})$ such that properties (i)–(iv) above are satisfied for every $i \geqslant 1$. Now, observe that, for every $i \leqslant j$, $\langle y_i \rangle^{H_i}$ is normalized by H_j; therefore, for every $n \in \mathbb{N}$, $L_n = \langle y_0 \rangle^{H_0} \langle y_1 \rangle^{H_1} \cdots \langle y_n \rangle^{H_n}$ is a soluble p-group of finite exponent, hence a Baer group. We let

$$H = \bigcup_{n \in \mathbb{N}} L_n.$$

Then $L_n \trianglelefteq H$, for every $n \in \mathbb{N}$, and it thus follows immediately that H is a Baer group. To finish the proof, we have just to observe that

$$H \supseteq \{y_0, y_1, \dots\},$$

whence $AH/A = A\langle x_0, x_1, \dots \rangle/A = G/A$, and so $AH = G$. □

Lemma 2.5 *Let A be a reduced abelian p-group, and* $A_\omega = \bigcap_{n \geqslant 1} A^{p^n}$. *Let* α *be an automorphism of A of order a power of p. If* $[A,_n \alpha] \leqslant A_\omega$ *for some* $n \geqslant 1$*, then* $[A,_n \alpha]$ *has finite exponent and* $[A,_{n+1} \alpha] = 1$.

For the proof, let us recall the following elementary fact (which may be easily proved by induction on n).

Lemma 2.6 *Let α be an automorphism of finite order q of the abelian group A, such that $[A,_n \alpha] = 1$ for some $n \geqslant 1$. Then $[A^{q^{n-1}}, \alpha] = 1$.*

Proof of Lemma 2.5 — Let $B = [A,_n \alpha] \leqslant A_\omega$. Let $m \geqslant 1$; then there exists $k \geqslant 1$ such that $[A,_{n+k} \alpha] \leqslant [B,_k \alpha] \leqslant B^{p^m}$. It follows from Lemma 2.6 that $[A^{p^t}, \alpha] \leqslant B^{p^m}$ for some $t \geqslant 1$, and in particular one has $[B, \alpha] \leqslant B^{p^m}$. This holds for every $m \geqslant 1$, and so $[B,\alpha] \leqslant \bigcap_{m\geqslant 1} B^{p^m} = B_\omega$. Therefore, $[A,_{n+1} \alpha] = [[A,_n \alpha], \alpha] \leqslant [B, \alpha] \leqslant B_\omega$, whence, by Lemma 2.6 again, there exists $s \geqslant 1$ such that

$$B^{p^s} \leqslant [A, \alpha]^{p^s} = [A^{p^s}, \alpha] \leqslant B_\omega.$$

Since B is reduced, this implies that B has finite exponent, and $[A,_{n+1} \alpha] \leqslant B_\omega = 1$, as wanted.

We end this section by stating, for the convenience of the reader, a fundamental, and much less elementary, result of W. Möhres.

Lemma 2.7 (Möhres [6]) *Let G be a nilpotent p-group, and N a normal subgroup such that G/N is an infinite elementary abelian group. Then, for every finite subgroup H of G and any finite subset $\mathfrak{U}$ of $G \setminus H$, there exists a subgroup K of G with $H \leqslant K$, $\mathfrak{U} \cap K = \emptyset$ and NK/N infinite.*

3 Proof of the main result

As mentioned in the Introduction, it is proved in [2] that a locally finite $\mathfrak{N}_1$-group G has a nilpotent normal subgroup N such that G/N is an abelian group of finite rank. A Theorem of Khukhro and Makarenko [5] allows to improve this result in a convenient way, by saying that such N can be taken to be characteristic in G.

Proposition 3.1 *Let G be a periodic $\mathfrak{N}_1$-group. Then there exists a characteristic nilpotent subgroup C of G such that G/C is abelian of finite rank.*

Proof — Let G be a periodic $\mathfrak{N}_1$-group; then there exists an integer $k \geqslant 1$ such that all but a finite number of primary components of G are nilpotent of class at most k. We may thus assume that G is a p-group for some prime p. Let N a normal nilpotent subgroup such that G/N is abelian of finite rank. If c is the nilpotency class of N, then, by Theorem 1.2 of Khukhro and Makarenko [5], G admits a characteristic subgroup K, which is nilpotent of class at most c and such that G/K has finite rank. Since G/K is a Baer group it is not difficult to show that there exists a finite characteristic subgroup C/K of G/K, such that G/C is abelian. Now C is nilpotent because it is Baer and a finite extension of the nilpotent group K. □

Let us also recall from the Introduction another result of Möhres that we are going to use repeatedly.

Proposition 3.2 (Möhres [6]) *Let the periodic $\mathfrak{N}_1$-group G be the extension of a nilpotent group by an abelian group of finite exponent; then G is nilpotent.*

In this rest of section, to shorten our statements, we say that a group G is sbyf if every subgroup of G is subnormal-by-finite.

Lemma 3.3 *Let G be a p-group, and A a normal abelian subgroup of G with G/A abelian. If A is divisible and G is a sbyf-group, then G is $\mathfrak{N}_1$-by-finite. If, further, $G/C_G(A)$ has finite exponent, G is nilpotent-by-finite.*

Proof — If G/A is finite there is nothing to prove. Otherwise, we may well assume that G/A is countable. Then, by Lemma 2.4, there exists $H \leqslant G$ such that H is a Baer group and $AH = G$. Since G is a sbyf-group there exits $S \lhd\lhd G$ with $S \leqslant H$ and $|H : S| < \infty$. Then S is contained in the Baer radical B of G, as well as A. Hence $AS \leqslant B$ and so $|G : B| \leqslant |H : B| < \infty$. Finally, B is a $\mathfrak{N}_1$-group by Theorem 1.1.

If $G/C_G(A)$ has finite exponent, then AH is a periodic nilpotent-by-(finite exponent) $\mathfrak{N}_1$-group and so it is nilpotent by Proposition 3.2. □

Lemma 3.4 *Let G be a p-group, and A a normal abelian subgroup of G with G/A elementary abelian. If A is reduced and G is a sbyf-group, then G is nilpotent-by-finite.*

Proof — Let B be the Baer radical of G. Then $A \leqslant B$ and B is a $\mathfrak{N}_1$-group by Theorem 1.1. Since B is abelian-by-(finite exponent), Proposition 3.2 ensures that B is nilpotent. If it has finite index in G we are done. Otherwise, let C/A be a complement of B/A in G/A, then the Baer radical of C is A, and we may well consider C instead of G, thus assuming that the Baer radical of G is A. We then observe the following fact:

(1) *For every finite subgroup F of G, any $n \geqslant 1$ and $x \in F \setminus A$, there exists $a \in A$ such that $[a,_n x] \notin F$.*

In fact, if $x \notin A$ then $\langle x\rangle$ is not subnormal in G; this implies that all terms $[G,_n \langle x\rangle]^{\langle x\rangle}$ of the normal closure series of $\langle x\rangle$ in G are infinite; hence $[A,_n x]$ is infinite for every $n \geqslant 1$, and claim (1) follows.

Now, for every $m \geqslant 1$, let $A_m = \{a^{p^m} \mid a \in A\}$, and $A_\omega = \bigcap_{m\geqslant 1} A_m$.

We first suppose $A_\omega = 1$. We start by setting $H_0 = 1$ and $\mathcal{U}_0 = \emptyset$. Let $n \geqslant 0$, and suppose that for every $0 \leqslant i \leqslant n$ we have defined a finite subgroup H_i and a finite subset $\mathcal{U}_i \subseteq A$ such that for every $1 \leqslant i \leqslant n$,

(i) $|AH_i/A| = p^i$;

(ii) $H_{i-1} \leqslant H_i$ and $\mathcal{U}_{i-1} \subseteq \mathcal{U}_i$;

(iii) $H_i \cap \mathcal{U}_i = \emptyset$;

(iv) for each $x \in H_i \setminus AH_{i-1}$ there exists an element $u(x) \in \mathcal{U}_i$ such that $u(x) \in [A,_i x] \setminus H_{i-1}$.

As $\mathcal{U}_n$ is finite, there exists $m \geqslant 1$ such that $\mathcal{U}_n \cap A_m = \emptyset$. Now, G/A_m is a soluble p-group of finite exponent, hence it is a Baer group and so it is nilpotent by Proposition 3.2. Also, G/A_m admits a quotient isomorphic to the infinite elementary abelian group G/A. We apply Möhres Lemma 2.7 to the subgroup A_mH_n/A_m and the subset $\overline{\mathcal{U}} = \{uA_m \mid u \in \mathcal{U}_n\}$ (recall that we are assuming that G/A is infinite), obtaining a subgroup S/A_m of G/A_m such that $S \geqslant A_mH_n$, $|AS/A| = p^{n+1}$ and $S/A_m \cap \overline{\mathcal{U}} = \emptyset$. Then, we find a finitely generated, hence finite, subgroup H_{n+1} of S such that $H_n \leqslant H_{n+1}$ and $AH_{n+1} = AS$. Clearly, $H_{n+1} \cap \mathcal{U}_n = \emptyset$. Next, for every $x \in H_{n+1} \setminus AH_n$ we may select, by (1), an element $u(x) \in [A,_{n+1} x] \setminus H_{n+1}$; finally we let

$$\mathcal{U}_{n+1} = \mathcal{U}_n \cup \{u(x) \mid x \in H_{n+1} \setminus AH_n\}.$$

It is then immediate to check that we may extend our sequences $(H_i)_{i\leqslant n}$, $(\mathcal{U}_i)_{i\leqslant n}$ with $H_{n+1}, \mathcal{U}_{n+1}$ in such a way properties $(i) - (iv)$ continue to hold.

We then set

$$H = \bigcup_{n\in\mathbb{N}} H_n$$

and observe that $H \cap \mathcal{U}_n = \emptyset$ for every $n \geqslant 1$. Let S be a subnormal subgroup of G such that $S \leqslant H$ and $|H : S| < \infty$; clearly, we may assume $S \trianglelefteq H$; then $H = SH_n$ for some $n \geqslant 1$. By property (i), $H/(A \cap H) \simeq AH/A$ is infinite, hence $S/(S \cap A)$ is infinite.

Thus, $S \cap (H_j \setminus AH_{j-1})$ is not empty for every $j \geqslant n+1$. Let d be the defect of subnormality of S in G, and $t \geqslant \max\{d+1, n+1\}$. Then there exists $x \in S \cap (H_t \setminus AH_{t-1})$ and we get the contradiction

$$u(x) \in [A,_t x] \cap \mathfrak{U}_t \subseteq S \setminus H,$$

proving our claim in case $A_\omega = 1$.

For the general case, let N/A_ω a normal nilpotent subgroup of G/A_ω such that G/N is finite. It then follows from Lemma 2.5 that N is a Baer group. Hence, N is nilpotent and we are finished. □

We are now ready for the proof of the Theorem stated in the Introduction. We begin with the crucial case.

Lemma 3.5 *Let G be a p-group with a normal abelian subgroup A such that G/A is elementary abelian. If G is a sbyf-group then it is nilpotent-by-finite.*

Proof — Let B be the Baer radical of G. Then $B \geqslant A$ and B is nilpotent by Möhres Lemma (Proposition 3.2). If $|G/B|$ is finite we are done. Otherwise, let L/A be a complement of B/A in G/A; then L/A is infinite and the Baer radical of L is A. So, possibly replacing G by L, we may assume that A is the Baer radical of G.

Let D be the divisible radical of A; then D is normal in G and G/D is nilpotent-by-finite by Lemma 3.4. We may thus assume that G/D is nilpotent. As A/D is abelian and G/A elementary abelian one easily shows that $G'D/D$ has finite exponent. Since $G'D \leqslant A$ is abelian and D divisible, we thus have $G'D = DM$ where $M = \{u \in G'D \mid u^{p^m} = 1\}$ for some $m \geqslant 0$. Now, DM/M is divisible and G/DM abelian; we may then apply Lemma 2.4 to the group G/M, obtaining that there exists $M \leqslant H \leqslant G$ such that H/M is a Baer group and $DH = G$. Then H/M is a $\mathfrak{N}_1$-group, hence it is nilpotent by Proposition 3.2. Let S be a subnormal subgroup of G with $M \leqslant S \leqslant H$ and $|H : S| < \infty$. Then DS has finite index in G, since $G = DH$; moreover, $[D, DS] \leqslant M$, since abelian normal divisible subgroups of a periodic nilpotent group are central. Therefore $[D, DS] = 1$, because $[D, DS]$ is divisible while M has finite exponent. Thus $D \leqslant Z(DS)$ and it follows that DS is a nilpotent subgroup of G of finite index in G. □

Proof of main Theorem — (A) Let G be a locally finite sbyf-group. By Theorem 1.1, we may assume that G is locally nilpotent and indeed a p-group for some prime p.

Let B be the Baer radical of G. Then B is a $\mathfrak{N}_1$-group, hence, by Proposition 3.1, there exists a characteristic nilpotent subgroup N of B such that B/N is abelian of finite rank. Now $N \trianglelefteq G$; we prove by induction on the derived length d of N that G/N is a Černikov group.

Let N be abelian and suppose that G/N is not Černikov; then it admits an elementary abelian subgroup of infinite rank, and so, by property sbyf, there exists $N \leqslant H \lhd\lhd G$ such that H/N is an infinite elementary abelian group. By Lemma 3.5 there exists a nilpotent normal subgroup K of H such that H/K is finite; clearly, we may assume $N \leqslant K$. Then K/N is an infinite elementary abelian group. On the other hand $K \lhd\lhd G$, hence K is contained in the Baer radical B of G, and this yields a contradiction, because, as B/N has finite rank, $(B \cap H)/N$ is finite.

Let now $d \geqslant 2$. By the previous case there exists $N' \leqslant L \trianglelefteq G$ such that G/L has finite rank and L/N' is nilpotent. Since N is nilpotent, it follows that L is nilpotent by P. Hall criterion (see), and we are done.

(B) Let G be a locally finite sbyf-group, which we may assume to be a p-group for some prime p. By the previous point, G admits a normal nilpotent subgroup N such that G/N has a normal subgroup D/N of finite index which is the product of a finite number of groups of type C_{p^∞}. Now, every subgroup of N is subnormal in N with defect at most c, where c is the nilpotency class of N. Let $d = c + 2$.

In proving that any $H \leqslant G$ admits a finite index subgroup which is subnormal of defect at most d in G, we may well suppose $H \leqslant D$. Then $K = N \cap H$ is a subnormal subgroup of G of defect at most $c + 1 < d$ and it is normal in H. If H/K is finite we are done. Thus, suppose that H/K is infinite. Then, $H/K \simeq NH/N$ has a normal divisible subgroup A/K of finite index; observe that $AN/N \leqslant D/N$; in particular, AN is subnormal of defect at most 2 in G. Let S be a subnormal subgroup of G such that $S \leqslant H$ and $|H : S| < \infty$; clearly, we may suppose $S \geqslant K$. Then $S \geqslant A$, because A/K has no proper subgroups of finite index. It follows that A is subnormal in S and hence is subnormal in G, so we may replace S by A. Having observed that $A \cap N = K$, let $L = K^N$. Then L is normalized by A, thus $L \trianglelefteq NA$ and NA/L is nilpotent. Now, (abelian) divisible subgroups of a nilpotent p-group are central, hence in particular $AL/L \simeq A/K$ is normal in NA/L, yielding $LA = A^{NA}$. By an easy inductive argument we deduce that the defect of A in NA is equal to the defect of K in N, which in turn does not exceed c. Since NA has defect at most 2 in G, we conclude that A has defect at most $c + 2 = d$ in G, as wanted. □

4 Concluding remarks and questions

1. I do not know of any example of a locally finite sbyf-group which is not a finite extension of a $\mathfrak{N}_1$-group, nor was I able to show that this cannot happen. Thus, I have to leave unsettled the question as to whether every locally finite sbyf-group is $\mathfrak{N}_1$-by-finite.

2. Groups in which every subgroup is *normal-by-finite*, called CF-groups, are the subject of a number of interesting papers (see [1] and [3], to quote just a couple of them). In particular, Buckley, Lennox, B.H. Neumann, H. Smith and Wiegold proved in [1] that a locally finite CF-group is abelian by finite.

For $d \geqslant 1$, let us say that a subgroup is d-subnormal if it is subnormal of defect at most d, and that a group G is d-sbyf if every subgroup H of G contains a subgroup S of finite index which is d-subnormal in G. Point (B) of the main Theorem states that every locally finite sbyf-group is d-sbyf for some $d \geqslant 1$. It would be of some interest to know whether this is true in general (that is, not just for locally finite groups). By the mentioned result in [1], locally finite 1-sbyf groups are abelian-by-finite. On the other hand, the groups of Heineken-Mohamed are examples of locally finite 2-sbyf groups that are not nilpotent-by-finite. However, the question remains whether there exists a function $\gamma(d)$ of d such that every locally finite d-sbyf group G has a normal subgroup N such that G/N is Černikov and N is nilpotent of class at most $\gamma(d)$. By the results in this paper, the question clearly reduces to nilpotent groups.

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