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# ADV Perspectives in Group Theory

**– an open space –**

## ADV – 1A

Let H be a *quasinormal subgroup* of a group G, i.e. $\langle H, K\rangle = HK$ for all subgroups K of G. Then it is known that H is an *ascendant subgroup* of G, i.e. there is an ascending series

$$H = H_0 \triangleleft H_1 \triangleleft \dots H_\sigma \triangleleft H_{\sigma+1} \triangleleft \dots H_\alpha = G$$

for a suitable ordinal $\alpha$ (see[S.E. Stonehewer: "Permutable subgroups of infinite groups", *Math. Z.* (1972), 1–16]). In fact it has been shown by Stonehewer and Napolitani (independently and unpublished) that $\alpha$ can always be chosen not exceeding $\omega + 1$. Examples with the smallest $\alpha$ equal to $\omega$ are easy to construct. Thus in the split extension G of the quasicyclic p-group P (for any prime p) by the infinite cyclic group X generated by x acting on P according to $x^{-1}gx = g^{p+1}$ for all g in P, we have X ascendant in G with $\omega$ as the smallest value of $\alpha$. But are there any examples for which the smallest value of $\alpha$ is $\omega + 1$?

*Stewart E. Stonehewer*

## ADV – 1B

A subgroup $H$ of a group $G$ is called *abnormal* in $G$ if $g \in \langle H, H^g \rangle$ for each element $g \in G$. It is well known that the absence of proper abnormal subgroups is a criterion of nilpotency of a finite group. One of the classes of generalized nilpotent groups consists of so called $\widetilde{N}$-*groups*. Recall that a group $G$ is called an $\widetilde{N}$-group if $G$ satisfies the following condition: for all subgroups $M < L \leqslant G$ the fact that $M$ is maximal in $L$ implies that $M$ is normal in $L$. Corollary 2.20 of the article [V.V. Kirichenko, L.A. Kurdachenko and I.Y. Subbotin: "Some related to pronormality subgroup families and the properties of a group", *Algebra Discrete Math.* 11 (2011), 75–108] yields that an $\widetilde{N}$-group has no proper abnormal subgroups. In this connection, the following two questions seem interesting.

**Question 1** *Does there exist a non* $\widetilde{N}$*-group with no proper abnormal subgroups?*

**Question 2** *Does there exist a group* $G$ *with no proper abnormal subgroups which contains a subgroup* $H$ *having a proper abnormal (in* $H$*) subgroup?*

*Igor Y. Subbotin*

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## ADV – 1C

A group $G$ is *locally graded* if every finitely generated non-trivial subgroup of $G$ has a finite non-trivial image. Denote by $\mathfrak{X}$ the class of groups, obtained from the class of periodic locally graded groups by using of the formation of local systems, subcartesian products and both ascending and descending normal series. We say that a group $G$ is of *finite special rank* $r(G) = r$ if every finitely generated subgroup can be generated by at most $r$ elements, and $r$ is the least integer with this property. This concept was introduced for arbitrary groups in [A.I. Maltsev: "On groups of finite rank", *Mat. Sb.* 22 (1948), 351–352]. If $G \in \mathfrak{X}$ and $G$ has finite special rank, then $G$ is almost locally soluble [N.S. Chernikov: "A theorem on groups of finite special rank", *Ukrainian Math. J.* 42 (1990), 855–861]. Thus the following problem is interesting.

**Question** *Let* G *be a finitely generated locally graded group, and suppose that* G *has a finite special rank. Is* G *almost soluble?*

*Leonid A. Kurdachenko*

## ADV – 1D

If G is an infinite locally graded group and all proper subgroups of G are soluble of derived length at most d, then G is also soluble of derived length at most d (see [D.I. Zaitsev: "Stably solvable groups", *Izv. Akad. Nauk SSSR Ser. Mat.* 33 (1969), 765–780] and [M.R. Dixon and M.J. Evans: "Groups with the minimum condition on insoluble subgroups", *Arch. Math. (Basel)* 72 (1999), 241–251]). If the bound on the derived lengths is removed then it has been shown that if G is a Fitting p-group with all proper subgroups hypercentral and soluble then G is soluble (see [A.O. Asar: "Locally nilpotent p-groups whose proper subgroups are hypercentral or nilpotent-by-Chernikov", *J. London Math. Soc.* 61 (2000), 412–422]). The following problem is therefore quite natural.

**Question** *Let* G *be an infinite locally graded group and suppose that every proper subgroup of* G *is soluble. Is* G *itself soluble?*

*Martyn R. Dixon*

## ADV – 1E

Let p be a prime number, and let $\mathcal{K}$ be the algebraic closure of the field with p elements. It is well-known that the multiplicative group $\mathcal{K}^*$ of $\mathcal{K}$ is a direct product of groups of type $q^\infty$, one for each prime $q \neq p$. Consider a non-empty set $\pi$ of prime numbers other than p, and let Q be the $\pi$-component of $\mathcal{K}^*$. If F is the subfield of $\mathcal{K}$ generated by Q, and A is the additive group of F, the semidirect product $G(p,\pi) = Q \ltimes A$, where the action of an element x of Q on A is the multiplication by x (in F), is called the *Čarin group* of type $(p,\pi)$ (see [V.S. Čarin: "A remark on the minimal condition for subgroups", *Dokl. Akad. Nauk SSSR* 66 (1949), 575–576]). If the set $\pi$ consists of a single prime $q \neq p$, the group $G(p, \pi)$ is denoted by $G(p, q)$. Notice that $A$ is an infinite abelian group of exponent $p$, and it is the unique minimal normal subgroup of $G(p, \pi)$. Thus $G(p, q)$ satisfies the minimal condition on normal subgroups but it does not satisfy the minimal condition on subgroups. This example also shows that the minimal condition on normal subgroups in not inherited by normal subgroups. The consideration of non-split central extensions of an abelian group C by $G(p, q)$ (for suitable different prime numbers $p$ and $q$) would allow the construction of examples with relevant properties. Thus the following problem is of interest.

**Question** *If $p$ and $q$ are different prime numbers, determine the structure of the Schur multiplicator of the Čarin group $G(p, q)$.*

*Francesco de Giovanni*

## ADV – 1F

Let G be a finite group. In 1976 Chouinard proved that a $\mathbb{Z}G$-module $M$ is projective if (and only if) its restriction to $\mathbb{Z}E$ is projective for any elementary abelian subgroup $E$ of G (see [L. G. Chouinard: "Projectivity and relative projectivity over group rings", *J. Pure Appl. Algebra* 7 (1976), 287–302]). It is not difficult to show that projectivity of $M$ over $\mathbb{Z}G$ is equivalent to the norm map $N_G : R \to R^G$ being surjective, where $R = \mathrm{End}(M)$ is the endomorphism ring endowed with the diagonal action $g(\mu) = g \circ \mu \circ g^{-1}$, $\mu \in R$ and $R^G = \{z \in R : g(z) = z, g \in G\}$. The latter condition is equivalent to the existence of $x_G \in R$ with $N_G(x_G) = \mathrm{id}$. With this terminology, Chouinard's theorem says that (notation as above) if for any elementary abelian subgroup $E$ of G there is $x_E \in R$ with $N_E(x_E) = 1$ then there exists $x_G$ with $N_G(x_G) = 1$. Aljadeff and Ginosar extended Chouinard's theorem for any unital ring $R$ (see [E. Aljadeff and Y. Ginosar: "Induction from elementary abelian subgroups", *J. Algebra* 179 (1996), 599–606]). Based on a compactness argument Shelah showed there is a "general" polynomial formula (depending on G but independent of R) of the form $x_G = p(\{g(x_E)\}_{E, g \in G})$. The following problem may be viewed as an effective version of Chouinard's theorem.

**Question** *Given a finite group* $G$*, find a polynomial* $x_G = p(\{g(x_E)\})$.

Peter Palfy showed that the polynomial $y = xg(x)x + xg(x) - x^2g(x)$ is a formula for the cyclic group $G$ of order 4 generated by $g$ where $y = x_G$ and $x = x_E$, $E = \{e, g^2\}$. The problem is easily reduced to p-groups and with more efforts to extraspecial groups (see [E. Aljadeff and C. Kassel: "Norm formulas for finite groups and induction from elementary abelian subgroups", *J. Algebra* 303 (2006), 677–706]). Formulas where found for cyclic p-groups [E. Aljadeff and C. Kassel: "Explicit norm one elements for ring actions of finite abelian groups", *Israel J. Math.* 129 (2002), 99–108] (and hence for arbitrary abelian groups), quaternion and dihedral groups (see the first mentioned paper by Aljadeff and Kassel).

*Eli Aljadeff*

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## ADV – 1G

In 1938, O.Yu. Šmidt asked wheter every group all of whose proper subgroups are finite is of type $p^\infty$. Generalizing this problem, A.G. Kuroš and S.N. Černikov asked wheter there exist uncountable groups whose proper subgroups have strictly smaller cardinality, while B. Jónsson extended this problem to algebras. Nowadays, uncountable (countably infinite) groups with no proper equipotent subgroup are called *Jónsson groups* (*Šmidt groups*).

In 1979–80, A.Yu. Ol'šanskiĭ provided an example of a simple Šmidt group, while S. Shelah ["On a problem of Kurosh, Jónsson groups, and applications", in Word Problems II - the Oxford Book, *North-Holland*, Amsterdam (1980), 373–394] constructed a Jónsson group of cardinality $\aleph_1$.

One of the recent trends in group theory is the investigation of the behaviour of *large* proper subgroups of a group. However, those examples constitute an obstacle to this study, so one usually needs some generalized soluble condition to exclude them: A. Macintyre remarked that Jónsson groups are simple over the centre. Even so, to obtain nice results in this last kind of investigation one must require also some condition on the cardinality of the group (see [F. de Giovanni and M. Trombetti: "Uncountable groups with restritions on subgroups of large cardinality", *J. Algebra* 447 (2016), 383–396]). In fact, the following question seems to be open.

**Question** *Is it possible to construct an uncountable group* $G$ *(with no simple homomorphic images) all of whose proper normal subgroups have cardinality strictly smaller than* $G$*?*

*Marco Trombetti*

## ADV – 1H

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set $\mathbb{P}$ of all primes, that is, $\mathbb{P} = \cup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. A finite group G is said to be: *σ-primary* if G is a $\sigma_i$-group for some $i \in I$; *σ-full* if G possesses a Hall $\sigma_i$-subgroup for each $i \in I$.

A subgroup $H$ of a group $G$ is said to be *σ-subnormal* in $G$ [A.N. Skiba: "On σ-subnormal and σ-permutable subgroups of finite groups", *J. Algebra*, 436 (2015), 1–16] if there is a subgroup chain $A = A_0 < A_1 < \cdots < A_n = G$ such that either $A_{i-1}$ is normal in $A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$.

**Question** *Let* $A$ *be a subgroup of a finite σ-full group* $G$*. Is it true then that* $A$ *is σ-subnormal in* $G$ *if and only if* $H \cap A$ *is a Hall* $\sigma_i$*-subgroup of* $A$ *for every* $i \in I$ *and every Hall* $\sigma_i$*-subgroup* $H$ *of* $G$*?*

Note that in the case when $\sigma = \{\{2\},\{3\},\{5\},\ldots\}$, the answer to this question is positive [P.B. Kleidman: "A proof of the Kegel-Wielandt conjecture on subnormal subgroups", *Ann. Math.* 133 (1977), 369–428].

*Alexander N. Skiba*

## ADV – 1I

A result of R. Gilmer and T. Parker ["Divisibility properties in semigroup rings", *Michigan Math. J.* 21 (1974), 65–86] determines when a group ring RG is a UFD. Here R is an integral domain and G is an abelian group. This setup yields immediately that G is torsion-free. If we replace the integral domain R by the semiring of nonnegative integers $\mathbb{N}$, then the group-semiring $\mathbb{N}G$ admits no zero divisors even if G is not torsion free. In this case we still cannot expect unique factorization, since any element $g \in G$ of prime order $p$ satisfies

$$(1+g)(1+g+g^2+\cdots+g^{p-1}) = 2(1+g+g^2+\cdots+g^{p-1}),$$

both are factorization to irreducible elements in $\mathbb{N}G$. We demand a weaker condition on factorization.

**Question** *Give sufficient and necessary conditions on a group* $G$ *(not necessarily abelian) such that factorization* $x = x_1 \cdot x_2 \cdot \ldots \cdot x_r$ *of any element* $x \in \mathbb{N}G$ *to irreducible elements admits factorization of the corresponding augmentation values* $\epsilon(x) = \epsilon(x_1) \cdot \epsilon(x_2) \cdot \cdots \cdot \epsilon(x_r)$ *in a unique way (up to order).*

*Yuval Ginosar*

## ADV – 1J

A subgroup of a group $G$ is called *modular* if it is a modular element of the lattice of all subgroups of $G$. Clearly, every normal subgroup of a group is modular, but arbitrary modular subgroups need not be normal. Abelian groups and the so-called Tarski groups (i.e. infinite groups whose proper non-trivial subgroups have prime order) are obvious example of groups in which every subgroup is modular (the so-called $M$-*groups*). The structure of $M$-groups has been described by Iwasawa and Schmidt.

A celebrated theorem of B.H. Neumann states that in a group $G$ every subgroup is *almost normal* (i.e. its normalizer has finite index) if and only if $G$ is central-by-finite, that is the centre of $G$ has finite index. If $\phi$ is an isomorphism from the subgroup lattice of a group $G$ onto the subgroup lattice of a group $\overline{G}$, then the image of any almost normal subgroup of $G$ is modular in a subgroup of finite index of $\overline{G}$. A subgroup $H$ of a group $G$ is called almost modular if there exists a subgroup $K$ of finite index of $G$ such that $H$ is modular in $K$, and the structure of groups in which every subgroup is almost modular has been completely described, at least in the periodic case. In particular, it turns out that a group $G$ with such a property contains a subgroup of finite index whose subgroups are modular in $G$ (see [F. de Giovanni, C. Musella and Y.P. Sysak: "Groups with almost modular subgroup lattice", *J. Algebra* 243 (2001), 738–764]).

It has been proved by Baer and Sato that large classes of M-groups admit lattice-isomorphism onto abelian group. From this remarkable fact, the following problem arises.

**Question** *Is it true that (most) groups in which all subgroups are almost modular admit lattice-isomorphisms onto central-by-finite groups?*

*Carmen Musella*

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## ADV – 1K

A group G is said to have *finite (Prüfer) rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property; if such an r does not exist, G is said to have *infinite rank*. A group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Let now $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\overline{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ under the operators **$\acute{\mathbf{P}}$, $\grave{\mathbf{P}}$, R, L** (see the first chapter of [D.J.S. Robinson: "Finiteness Conditions and Generalized Soluble Groups", *Springer*, Berlin (1972)] as a general reference for definitions and properties of closure operations on group classes). The class $\overline{\mathfrak{D}}$ has been introduced in [N.S. Černikov: "A theorem on groups of finite special rank", *Ukranian Math. J.* 42 (1990), 855–861]. Finally, a group G is said to be *strongly locally graded* if every section of G is a $\mathfrak{D}$-group. Using a result contained in [V.N. Obraztsov: "An embedding theorem for groups and its corollaries", *Mat. Sb.* 180 (1989), 529–541], it is possible to construct a group of infinite rank whose proper subgroups have finite rank. On the other hand, if all proper subgroups of a strongly locally graded group G have finite rank, then G itself has finite rank (see for instance [M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi "Groups with restrictions on subgroups of infinite rank", *Rev. Mat. Iberoamericana* 30 (2014), 537–550]). It would be interesting to know if the property of being rich in subgroups of infinite rank can be extended to locally graded groups of infinite rank.

**Question** *Let* G *be a locally graded group whose proper subgroups have finite rank. Does* G *itself have finite rank?*

*Maria De Falco*

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# ADV − 1L

The Tits' alternative asserts that a finitely generated linear group is either virtually solvable or contains two elements $g, h$ which generate a non-abelian free group. It is an open question if for $n \geqslant 3$ every non-torison element $g \in SL(n, \mathbb{Z})$ has a partner $h$ such that $\langle g, h \rangle$ is a non-abelian free group. The question below is closely related to the question whether a complex element of $SL(3, \mathbb{Z})$ has such a partner. A second motivation comes from studying permutation actions of linear groups. Call a group $\Gamma$ *invariably generated* if for every transitive action of $\Gamma$ there exits $g \in \Gamma$ without fixed points. It was asked by Kantor, Lubotzky and Shalev if $SL(n, \mathbb{Z})$ is invariably generated when $n \geqslant 3$. A negative answer to the question below would imply that $SL(3, \mathbb{Z})$ is invariably generated. Finally, a positive answer to the question below would provide the first example (as far has we know) of a discrete subgroup $\Gamma \leqslant SL(3, \mathbb{R})$ of infinite covolume which acts minimally on the 2-dimensional real projective space.

**Question** *An element* $g \in SL(3, \mathbb{Z})$ *is called complex if for every* $m \geqslant 1$ *the matrix* $g^m$ *has a non-real eigenvalue. Does there exist an infinite-index Zariski-dense subgroup of* $SL(3, \mathbb{Z})$ *which contain a complex element?*

*Chen Meiri & Tsachik Gelander*