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"In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi"

Georg Cantor

ADV Perspectives in Group Theory

– an open space –

ADV – 2A

Among the most important group properties are *periodicity* (each element has finite order), *simplicity* (no proper nontrivial normal subgroups), and *finite presentability* by generators and relators. A popular property of modern Geometric Group Theory is *intermediate growth* (between polynomial and exponential). Infinite finitely presented simple groups exist; for instance, the famous Higman-Thompson groups have these properties. A more complicated situation occurs when finite presentability is related with the other two properties.

The question about existence of finitely generated infinite periodic groups is the core of one of the most important problems of algebra, the Burnside Problem. Three branches of it were solved by E. Golod (1964), P. Novikov and S. Adian (1967) and E. Zelmanov (1991). A short survey on the Burnside Problem is [R. Grigorchuk and I. Lysenok: "Burnside Problem" in The Concise Handbook of Algebra, *Kluwer Academic Press* (2002), 111–115]. No example of a finitely presented group answering the Burnside question is known.

Question 1 *Is there a finitely presented infinite periodic group?*

Let G be a finitely generated group with a system of generators $S = \{s_1, \ldots, s_n\}$ and let $\gamma(n) = |\{g \in G : |g| = n\}|$ be the corresponding growth function, where $|g|$ is the length of g with respect to S and $|E|$ denotes the cardinality of the set E. The rate of growth of this function does not depend on the choice of generating system. In 1968, J. Milnor raised the question: *"Is it true that a growth of finitely generated group is either polynomial or exponential?"* (see ["Problem 5603", *Amer. Math. Monthly* 75 (1968), 685–686]). The answer was given in [R. Grigorchuk: "Degrees of growth of finitely generated groups and the theory of invariant means", *Izv. Akad. Nauk SSSR Ser. Mat.* 48 (1984), 939–985], where the first examples of groups of intermediate growth were constructed. All of them turned out to be infinitely presented and at the moment no example of a finitely presented group of intermediate growth is known.

Question 2 *Is the growth of every finitely presented group either polynomial or exponential?*

All known examples of finitely presented groups (including Higman-Thompson groups) satisfy the following impudent conjecture (at least, it seems, there are no groups that look like potential counterexamples to it).

Conjecture 1 *Every finitely presented group either contains a free subsemigroup on two generators or it is virtually nilpotent.*

Is this conjecture true? If it is, then the answer to the above questions is negative.

The groups of intermediate growth constructed by me are residually p-finite groups (i.e. groups approximated by finite p-groups). I proved that if the growth of a residually p-finite group G is less than the growth of the function $e^{\sqrt{n}}$, then it is polynomial [R. Grigorchuk: "On the Hilbert-Poincare series of the graded algebras associated to groups", *Mat. Sb.* 180 (1989), 207–225], and hence, by the famous result of M. Gromov, G is virtually nilpotent. Later, in a conversation with A. Lubotzky and A. Mann, it was remarked that the same holds for residually nilpotent groups.

Conjecture 2 *If the growth of a finitely generated group* G *is bounded from above by the function* $e^{\sqrt{n}}$*, then* G *has polynomial growth and hence it is virtually nilpotent.*

Variations of this conjecture (called *GAP Conjecture*), and its connection with the other GAP type conjectures in theory of random walks and spectral theory of discrete Laplacian on groups, can be found in [R. Grigorchuk: "Milnor's problem on the growth of groups and its consequences" in Frontiers in Complex Dynamics, *Princeton Univ. Press*, Princeton (2014), 705–773]. It would be nice to prove the above conjecture for the class of residually solvable groups (in particular for residually polycyclic groups), and for residually finite groups.

Rostislav Grigorchuk

ADV – 2B

A group class $\mathfrak{X}$ is said to be *countably recognizable* if a group G is an $\mathfrak{X}$-group whenever all its countable subgroups belong to $\mathfrak{X}$. Countably recognizable classes of groups were introduced and studied by R. Baer in 1962, but already in the fifties the property of being hyperabelian and that of being hypercentral were proved to be detectable from the behaviour of countable subgroups, respectively, by Baer and S.N. Černikov.

It is well known that soluble groups, as well as nilpotent groups, form a class of countable character, and many other relevant group classes have been proved to be countably recognizable. We refer to the introduction of [F. de Giovanni and M. Trombetti: "Countable recognizability and nilpotency properties of groups", *Rend. Circ. Mat. Palermo*, to appear; DOI 10.1007/s12215-016-0261-y] for a recent and complete account of this subject. In particular, it turns out that all classes of generalized nilpotent groups exhibited in the diagrams on pages 3 and 13 of [D.J.S. Robinson: "Finiteness Conditions and Generalized Soluble Groups" (Part 2), *Springer*, Berlin (1972)] have countable character, the only exception being the class of Gruenberg groups. Similarly, most of the classes of generalized soluble groups pictured in the diagram on page 80 of the same book have been proved to be countably recognizable; in this case, the only exceptions are the classes SN^* and SJ. In fact, it is known that SN^*, the class of groups admitting an ascending series with abelian factors, does not have countable character, while for the class SJ the problem seems still to be unsettled.

Question *Is the class* SJ *countably recognizable?*

Recall here that a group is SJ if it has a subnormal series (of arbitrary order type) all of whose factors are abelian.

Francesco de Giovanni
Marco Trombetti

ADV – 2C

Question *Let* G *be a finite group acting on a vector space* V *and* p *a prime divisor of* $|G|$ *such that the centralizer* $C_G(v)$ *contains a Sylow* p*-subgroup of* G *for any* $v \in V$. *What can be said about the structure of the group* G?

This is a problem in the investigation on character degrees in group theory. There are already a lot of significant developments made on this question. For example, when G is p-solvable, O. Manz and T.R. Wolf have developed a relatively complete theory (see their monograph ["Representations of Solvable Groups", *Cambridge University Press*, Cambridge (1993)] for an account on the subject); while, when p divides $|V|$ and G is an arbitrary finite group, readers are referred to the paper [M. Giudici, M. Liebeck, C. Praeger, J. Saxl and P. Tiep: "Arithmetic results on orbits of linear groups", *Trans. Amer. Math. Soc.* 368 (2016), 2415–2467].

Jiping Zhang

ADV – 2D

Let p be a prime. We say that a group G has *finite section* p*–rank* $sr_p(G) = k$ if every elementary abelian p-section of G has order at most p^k and there exists an elementary abelian p–section A/B such that $|A/B| = p^k$. Moreover, a group G is said to have *finite section rank* if $sr_p(G)$ is finite for each prime p. There are some examples of finitely generated residually finite p-groups which are infinite. On the other hand, A.Yu. Ol'shanskiĭ has constructed an infinite finitely generated periodic group, whose Sylow subgroups are cyclic ["An infinite group with subgroups of prime orders", *Izv. Akad. Nauk SSSR*

Ser. Mat. 44 (1980), 309–321]. This group is not residually finite, and thus the following problem is interesting.

Question 1 *Let* G *be a finitely generated periodic group of finite section rank, and suppose that* G *is residually finite. Is* G *finite?*

A slightly weakened variant of the above question is the following one.

Question 2 *Let* G *be a finitely generated periodic group, whose Sylow* p*-subgroups are finite for every prime* p*, and suppose that* G *is residually finite. Is* G *finite?*

Our question can be even weakened in the following form.

Question 3 *Let* G *be a finitely generated periodic group, whose Sylow* p*-subgroups are finite and have bounded orders for every prime* p*, and suppose that* G *is residually finite. Is* G *finite?*

Leonid A. Kurdachenko

ADV – 2E

Wreath products (including their variations such as verbal wreath products, etc.) are the most popular operations used to study products of group varieties. Here is one of typical schemes of their application. The product $\mathfrak{U}\mathfrak{V}$ of varieties $\mathfrak{U}$ and $\mathfrak{V}$ consists of all extensions of groups $A \in \mathfrak{U}$ by groups $B \in \mathfrak{V}$. In general, a product variety is very hard to handle using this direct definition only. By G. Birkhoff's theorem, if $\mathfrak{W} = \operatorname{var}(G)$, that is, if the variety $\mathfrak{W}$ is generated by the group G, then $\mathfrak{W} = \mathsf{QSC}(G)$, i.e., all the groups of $\mathfrak{W}$ can be obtained as some homomorphic images of certain subgroups of some Cartesian powers of G. Therefore, if we find such specific groups A and B that the Cartesian wreath product $A \operatorname{Wr} B$ generates $\mathfrak{U}\mathfrak{V}$, then by G. Birkhoff's theorem $\mathfrak{U}\mathfrak{V} = \mathsf{QSC}(A \operatorname{Wr} B)$. B.H. Neumann, H. Neumann, P.M. Neumann in ["Wreath products and varieties of groups", *Math. Z.* 80 (1962), 44–62] ask: *"If the groups* A*,* B *belong to the varieties* $\mathfrak{U}$*,* $\mathfrak{V}$*, respectively, then* $A \operatorname{Wr} B$ *belongs to the product variety* $\mathfrak{U}\mathfrak{V}$*. If* A *generates* $\mathfrak{U}$ *and* B *generates* $\mathfrak{V}$*, then one might hope that* $A \operatorname{Wr} B$ *generates* $\mathfrak{U}\mathfrak{V}$*; but this is in general not the case".* Then they bring examples where $A \operatorname{Wr} B$ may or may not generate $\mathfrak{U}\mathfrak{V} = \operatorname{var}(A)\operatorname{var}(B)$. We tried to cover as wide as possible classes of groups A and B for which it is possible to detect, if $A \operatorname{Wr} B$ generates $\operatorname{var}(A)\operatorname{var}(B)$. And we have already classified the cases, when A and B are arbitrary abelian groups [V.H. Mikaelian: "Metabelian varieties of groups and wreath products of abelian groups", *J. Algebra* 313 (2007), 455–485], arbitrary finite groups [V.H. Mikaelian: "The criterion of Shmel'kin and varieties generated by wreath products of finite groups", *Algebra i Logika*, to appear; see arXiv:1503.08474], when A is any nilpotent group of restricted exponent and B is any abelian group [V.H. Mikaelian: "A classification theorem for varieties generated by wreath products of groups"; see arXiv:1607.02464].

Question *For as wide as possible classes of groups* A *and* B *is it possible to classify the cases when the equality* $\operatorname{var}(A \operatorname{Wr} B) = \operatorname{var}(A)\operatorname{var}(B)$ *holds or does not hold?*

Clearly, we assume new cases that were not covered in the literature earlier. By what we so far know, this is going to be a complicated task even when both A and B are soluble.

Vahagn H. Mikaelian

ADV – 2F

A subgroup A is called a *supplemented subgroup* in a group G, if there is a proper subgroup B of G such that $G = AB$. If, in addition, $A \cap B = \langle e \rangle$, then A is called a *complemented subgroup* of G. The subgroup B is called a *supplement* (correspondingly, a *complement*) of A in G. Complementation is one of the key concept in group theory. It is enough to mention the famous theorem of complementability of Philip Hall subgroups in finite soluble groups. Finite groups in which every subgroup is complemented were introduced in [P. Hall: "Complemented groups", *J. London Math. Soc.* 12 (1937), 201–204], and the structure of such (finite and infinite) groups was completely described by N.V. Černikova (see, for instance, ["On the fundamental theorem about completely factorizable groups" in Groups with Systems of Complemented Subgroups, *Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR*, Kiev (1971), 49–58]). Several papers written by many authors were dedicated to study of complementability and groups with different families of complemented subgroups. However, the following natural topic is still open.

Problem *Investigate the groups in which for any subgroups* A, B *and* C, *the fact that* A *is complemented (supplemented) in* B *and* B *is complemented (supplemented) in* C *implies that* A *is complemented (supplemented) in* C.

Igor Y. Subbotin

AdV – 2G

If G is a free group, then of course G is locally graded, so certainly a homomorphic image of a locally graded need not be locally graded: we just need to take a free group F of free rank 2 and find a normal subgroup R such that F/R is isomorphic to an infinite simple 2-generator p-group all of whose proper subgroups are of prime order, for a large enough prime p. On the other hand, H. Smith has shown in ["On homomorphic images of locally graded groups", *Rend. Sem. Mat. Padova* 91 (1994), 53–60] that if G is locally graded and H is a G-invariant subgroup of the hypercentre of G, then G/H is also locally graded. More generally, it has been shown in [P. Longobardi, M. Maj and H. Smith: "A note on locally graded groups" *Rend. Sem. Mat. Padova* 94 (1995), 275–277] that if G is locally graded and H is a G-invariant subgroup of the Hirsch-Plotkin radical of G, then G/H is locally graded, and indeed, if H is a normal radical subgroup of G, then G/H is also locally graded. It is therefore of some interest to find other cases of this phenomenon. The following problems are therefore quite natural.

Question 1 *Let* G *be an infinite locally graded group. If* H *is a normal locally soluble subgroup of* G, *then is* G/H *locally graded? If* H *is a normal locally finite subgroup of* G, *then is* G/H *locally graded?*

Question 2 *If* G *is an infinite locally graded group, what types of normal subgroups can be factored to still obtain a locally graded group?*

Martyn Dixon

ADV – 2H

A group G with a lattice order is called a *right ℓ-group* if

$$a \leqslant b \implies ac \leqslant bc$$

holds for $a, b, c \in G$. If the left-hand version of this implication also holds, G is said to be an *ℓ-group*. Totally ordered right ℓ-groups are also called *right ordered groups* (Paul Conrad, 1959). The Cayley-Holland Theorem implies that every ℓ-group is right orderable.

Question 1 *Is every right ℓ-group right orderable?*

A forthcoming paper will show that the next question is a special case of Question 1. Let X be an orthomodular lattice (e.g., the lattice of closed subspaces of a Hilbert space or the projection lattice of a von Neumann algebra), and let G be a group. A G*-valued measure* on X is a function $\mu \colon X \to G$ which satisfies

$$\mu(x \vee y) = \mu(x) + \mu(y)$$

for $x, y \in X$ with $x \perp y$.

Question 2 *Does any orthomodular lattice* X *admit an injective* G*-valued measure* $\mu \colon X \to G$ *into a right-ordered group* G*?*

Wolfgang Rump

ADV – 2I

The notion of growth for finitely generated groups has been thoroughly investigated since Milnor posed his famous problem in [J. Milnor: "Problem 5603", *Amer. Math. Monthly* 75 (1968), 685–686], answered by Grigorchuk (there exist finitely generated groups of intermediate growth), and by Gromov (a finitely generated group has polynomial growth if and only if it is virtually nilpotent). This classical notion was recently extended to endomorphisms $\phi : G \longrightarrow G$ of arbitrary groups G in a natural way, by using the language of algebraic entropy (see the work of D. Dikranjan and A. Giordano Bruno in ["Discrete dynamical systems in group theory", *Note Mat.* 33

(2013), 1–48] and ["Entropy on abelian groups", *Adv. Math.* 298 (2016), 612–653]).

Let $\mathcal{F}(G)$ be the family of all non-empty finite subsets of G; the *growth function* of ϕ with respect to $F \in \mathcal{F}(G)$ is $\gamma_{\phi,F} : \mathbb{N}_+ \longrightarrow \mathbb{N}_+$ defined by $n \mapsto |F \cdot \phi(F) \cdot \phi^2(F) \cdot \ldots \cdot \phi^{n-1}(F)|$. Then:

- ϕ has *polynomial growth* if $\gamma_{\phi,F}$ is polynomial for every $F \in \mathcal{F}(G)$;
- ϕ has *exponential growth* if there exists $F \in \mathcal{F}(G)$ such that $\gamma_{\phi,F}$ is exponential;
- ϕ has *intermediate growth* if $\gamma_{\phi,F}$ is not exponential for every $F \in \mathcal{F}(G)$ and there exists $F \in \mathcal{F}(G)$ such that $\gamma_{\phi,F}$ is intermediate.

If G is a finitely generated group and $\phi = id_G$ is the identity map, then G has polynomial (respectively, exponential, intermediate) growth in the classical terminology if and only if id_G has polynomial (respectively, exponential, intermediate) growth according to the above definition.

In the spirit of Milnor Problem, we propose the following open problems.

Problem *Characterize the groups admitting no endomorphism of intermediate growth.*

The polynomial - exponential dichotomy holds for abelian groups [D. Dikranjan and A. Giordano Bruno: "The Pinsker subgroup of an algebraic flow", *J. Pure Appl. Algebra* 216 (2012), 364–376], for locally finite groups [A. Giordano Bruno and P. Spiga: "Growth of group endomorphisms", *J. Group Theory*, in press] and for locally virtually solvable groups [A. Giordano Bruno and P. Spiga: "Milnor-Wolf Theorem for the growth of endomorphisms of locally virtually soluble groups", in preparation].

Question *Does there exist a finitely generated group of polynomial or exponential growth admitting an endomorphism of intermediate growth?*

Dikran Dikranjan
Anna Giordano Bruno
Pablo Spiga

ADV – 2J

Let F_n denote the free group of rank n. We say that $x \in F_n$ is a *primitive* element of F_n if it is contained in a set of free generators of F_n.

It is known that if $n \geqslant 3$, there exist normal subgroups N of F_n such that F_n/N can be generated by fewer than n elements, even though N contains no primitive elements of F_n. The first examples of such N were obtained by G.A. Noskov ["Primitive elements in a free group", *Mat. Zametki* 30 (1981), 497–500]. Many more examples are now known (see [M.J. Evans: "Nielsen equivalence classes and stability graphs of finitely generated groups" in Ischia Group Theory 2006, *World Sci. Publ.* (2007), 103–119] for a survey of these results). However, there is a related question of considerable importance that remains unanswered. First note that $\mathrm{Aut}(F_n)$ acts transitively on the (set of) primitive elements of F_n, and so proper characteristic subgroups of F_n contain no primitive elements of F_n.

Question *Let N be a proper characteristic subgroup of F_n. Can F_n/N be generated by fewer than n elements?*

The answer is unknown even if we impose the additional hypothesis that N has finite index in F_n.

It is interesting to observe that if N is a *fully invariant* proper subgroup of F_n, then it is verbal, and so F_n/N is the free group of rank n in a variety of groups. Consequently F_n/N has an abelian image that requires n generator and so F_n/N itself certainly requires n generators.

Martin Evans

ADV – 2K

An old question of J.E. Roseblade ["On groups with all subgroups subnormal", *J. Algebra* 2 (1965), 402–412] asks whether it is true that a group in which every n-generated subgroup is n-subnormal (i.e. subnormal of defect at most n) is nilpotent (of class bounded by a funtion of n).

For $n \geqslant 1$ we say that a group G is CE_n if

$$[g, x_1, x_2, \dots, x_n] \in \langle x_1, x_2, \dots, x_n \rangle$$

for every $g, x_1, x_2, \dots, x_n \in G$ (not necessarily distinct). Clearly, every CE_n-group is $(n+1)$-Engel, but not the converse: the wreath product $C_p \wr A$, where C_p is a cyclic group of order the prime p and A an infinite elementary abelian p-group, is $(p+1)$-Engel but it is not C_n for any $n \geqslant 1$.

Question *Is every* CE_n*-group locally nilpotent?*

The question is thus a weak form of that about local nilpotency of bounded Engel groups. O the other hand, if we ask whether every CE_n-group is nilpotent (I do not know of any counterexamples) we have a stronger version of Roseblade's question.

Carlo Casolo

ADV – 2L

Let R be a commutative ring with a unity, G a group and RG the group ring of G over R. It is well known that one can associate a normal subgroup of G with a two-sided associative ideal I of RG, namely the set

$$\{x \in G \,|\, x - 1 \in I\}.$$

But the problem of identifying such a subgroup is in general quite hard.

There has been a lot of work done on the structure and properties of the sequence of *dimension subgroups* of RG whose n-th term is defined as

$$D_n(G) := G \cap (1 + \Delta(G)^n),$$

where $\Delta(G)$ denotes the augmentation ideal of RG. In particular, the so called *dimension subgroup problem* has been extensively investigated over the years by several authors. For a comprehensive account we refer the reader to the monograph [I.B.S. Passi: "Groups Rings and their Augmentation Ideals", *Springer*, Berlin (1979)], Chapter III of [S.K. Sehgal: "Topics in Group Rings", *Marcel Dekker*, New York (1978)], Section 1.8 of [S.K. Sehgal: "Units in Integral Group Rings", *John Wiley & Sons*, New York (1993)] and Section 9 of [S.K. Sehgal: "Group Rings" in Handbook of Algebra 3, *North-Holland*, Amsterdam (2003), 455-541].

Motivated by questions connected with the above mentioned problem two other sequences of subgroups of G related to the Lie structure of RG have been introduced: those of *Lie dimension subgroups* and *restricted Lie dimension subgroups* of RG. In more details, one sets $RG^{[1]} := RG$ and, for $n > 1$, $RG^{[n]}$ as the two-sided ideal of RG generated by all the left-normed Lie commutators $[x_1, x_2, \ldots, x_n]$, where $x_i \in RG$ and $[y, z] : yz - zy$. On the other hand, put $RG^{(1)} := RG$ and inductively, for $n > 1$, $RG^{(n)} := [RG^{(n-1)}, RG]RG$, the two-sided ideal of RG generated by all the Lie commutators $[x, y]$ with $x \in RG^{(n-1)}$ and $y \in RG$. The n-*th Lie dimension subgroup* of RG is defined as

$$D_{(n)}(G) := G \cap (1 + RG^{(n)}),$$

whereas the corresponding n-*th restricted Lie dimension subgroup* is the subgroup of G

$$D_{[n]}(G) := G \cap (1 + RG^{[n]}).$$

We observe that since any element $\alpha \in RG$ can be written as $\epsilon + \beta$, where $\epsilon \in R$ and $\beta \in \Delta(G)$, one can replace the Lie powers of RG with the Lie powers of $\Delta(G)$. The investigation of these series becomes relevant in the study of *Lie nilpotent* (*strongly Lie nilpotent*, respectively) group rings, namely those RG for which there exists an integer n such that $RG^{[n]} = \{0\}$ ($RG^{(n)} = \{0\}$, respectively). In particular, their identification (which is known when R is a field) brings out the impact of Lie structure of RG on the structure of G.

Along this line we propose to study two series naturally arising in the area of *Lie solvable group rings* which appeared in the Problem section of Oberwolfach meeting in 2007. Recall that in any ring A, we let $[x_1, x_2]^\circ := [x_1, x_2]$ and, recursively,

$$[x_1, \ldots, x_{2^n}]^\circ := [[x_1, \ldots, x_{2^{n-1}}]^\circ, [x_{2^{n-1}+1}, \ldots, x_{2^n}]^\circ].$$

The ring A is called *Lie solvable* if there exists an integer n such that $[x_1, \ldots, x_{2^n}]^\circ = 0$ for any $x_i \in A$. We define

$$\delta^{[0]}(RG) := \delta^{(0)}(RG) := RG$$

and, for any $n \geqslant 1$, $\delta^{[n]}(RG)$ as the associative two-sided ideal generated by all the Lie commutators of RG of the form $[x_1, \dots, x_{2^n}]^\circ$ and, inductively

$$\delta^{(n)}(RG) := [\delta^{(n-1)}(RG), \delta^{(n-1)}(RG)]RG$$

(if $\delta^{(n)}(RG) = \{0\}$ for some n, the group ring is said to be *strongly Lie solvable*). Let us consider the normal subgroups of G

$$S^{[n]}(G) := G \cap (1 + \delta^{[n]}(RG))$$

and

$$S^{(n)}(G) := G \cap (1 + \delta^{(n)}(RG)).$$

Problem *Compute the subgroups $S^{[n]}(G)$ and $S^{(n)}(G)$ when $R = \mathbb{Z}$ or a field.*

The problem should be easier when G is a free group or R is a field.

A.K. Bhandari and Passi ["Residually Lie nilpotent group rings", *Arch. Math. (Basel)* 58 (1992), 1–6] and, independently, D.M. Riley ["Restricted Lie dimension subgroup", *Comm. Algebra* 19 (1991), 1493–1499] proved that if R is a field of characteristic different from 2 and 3, for any group G and integer n one has that $D_{(n)}(G) = D_{[n]}(G)$.

Question 1 *Is it true that $S^{(n)}(G) = S^{[n]}(G)$?*

Finally, a beautiful result of N. Gupta and F. Levin ["On the Lie ideals of a ring", *J. Algebra* 81 (1983), 225–231] establishes that for any ring with unity A and any integer $n \geqslant 1$, if $\mathcal{U}(A)$ is its group of units, then

$$\gamma_n(\mathcal{U}(A)) \subseteq 1 + A^{[n]},$$

where $\gamma_n(\mathcal{U}(A))$ is the n-th term of the lower central series of $\mathcal{U}(A)$.

Question 2 *Is it true that $\delta_n(\mathcal{U}(RG)) \subseteq 1 + \delta^{[n]}(RG)$?*

Here $\delta_n(\mathcal{U}(RG))$ denotes the n-th term of the derived series of the group $\mathcal{U}(RG)$, is true. This will provide an upper bound for the derived length of $\mathcal{U}(RG)$ in terms of the Lie derived length of RG as well.

Sudarshan K. Sehgal

ADV – 2M

Let L be a finite-dimensional restricted Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$ with p-map $x \mapsto x^{[p]}$ for $x \in L$. It is well known that every irreducible L-module M is finite-dimensional and admits a character $\chi \in \mathrm{Hom}_{\mathbb{F}}(L, \mathbb{F})$ such that the central element $x^p - x^{[p]}$ in the universal enveloping algebra of L acts as the scalar $\chi(x)^p$ on M for every $x \in L$. Under the assumption that L is solvable, we proved in [J. Feldvoss, S. Siciliano and Th. Weigel: "Restricted Lie algebras with maximal 0-PIM", *Transform. Groups* 21 (2016), 377–398] that the number of isomorphism classes of irreducible L-modules with a fixed p-character is at most $p^{MT(L)}$, where $M(T)$ denotes the maximal dimension of a torus in L. We ask in general the following question.

Question *Let* L *be a finite-dimensional restricted Lie algebra over an algebraically closed field* $\mathbb{F}$ *of characteristic* $p > 0$, *and let* χ *be a linear form on* L. *Is the number of isomorphism classes of irreducible* L*-modules with* p*-character* χ *bounded above by* $p^{MT(L)}$?

Note that previous question is also quoted in [G. Benkart and J. Feldvoss: "Some problems in the representation theory of simple modular Lie algebras" in Lie Algebras and Related Topics, *Contemp. Math.* 652 (2015), 207–228]; for further partial results on this problem we refer to Section 4 of that paper and references therein.

Salvatore Siciliano
Thomas Weigel

ADV – 2N

By a well known theorem of Roseblade (see [J. E. Roseblade: "On groups in which every subgroup is subnormal", *J. Algebra* 2 (1965), 402–412]), any group with all subgroups subnormal of a defect at most n is nilpotent of class at most $f(n)$, where f is some function depending only on the positive integer n. This function is not well understood in general. Restricting oneself to groups that are torsion-free one knows however that $f(n) = n$ for $\leqslant 4$ (for $n = 4$ this was proved in [H. Smith and G. Traustason: "Torsion-free groups with all subgroups 4-subnormal", *Comm. Algebra* 33 (2005), 4567–4585]).

Question *Are torsion-free groups with all subgroups* n*-subnormal always nilpotent of class at most* n*?*

Gunnar Traustason

ADV – 2P

Let $r(m)$ denote the residue class $r + m\mathbb{Z}$, where $0 \leqslant r < m$. Given disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$, let the *class transposition*

$$\tau_{r_1(m_1),r_2(m_2)}$$

be the permutation of $\mathbb{Z}$ which interchanges $r_1 + km_1$ and $r_2 + km_2$ for every $k \in \mathbb{Z}$ and which fixes everything else. The set of all class transpositions generates a countable simple group $\mathrm{CT}(\mathbb{Z}) < \mathrm{Sym}(\mathbb{Z})$ (see [S. Kohl: "A simple group generated by involutions interchanging residue classes of the integers", *Math. Z.* 264 (2010), 927–938]).

Question *Let* $G < \mathrm{CT}(\mathbb{Z})$ *be a group generated by* 3 *class transpositions, and let* m *be the least common multiple of the moduli of the residue classes interchanged by the generators of* G*. Assume that* G *does not setwisely stabilize any union of residue classes modulo* m *except for* $\emptyset$ *and* $\mathbb{Z}$*, and assume that the integers* $0, \dots, 42$ *all lie in the same orbit under the action of* G *on* $\mathbb{Z}$*. Is the action of* G *on* $\mathbb{N}_0$ *necessarily transitive?*

It is easy to see that the answer is positive for groups generated by 3 class transpositions which interchange residue classes with the same moduli (this is the case where no multiplications and no divisions occur, and the group is always finite). Transitivity on $\mathbb{N}_0$ obviously cannot occur in this case.

There is computational evidence suggesting that there is, say, "a reasonable chance" that the answer is positive in general. Note however that when replacing 42 by 41, the answer obviously gets negative since the finite group

$$G_{2,3,7} := \langle \tau_{0(2),1(2)}, \tau_{0(3),2(3)}, \tau_{0(7),6(7)} \rangle$$

acts transitively on the set $\{0, \dots, 41\}$ as well as on the set of residue classes modulo 42. Therefore if true, the assertion is sharp.

The condition that the group does not setwisely stabilize any union of residue classes modulo the least common multiple of the moduli of the residue classes interchanged by its generators is necessary, as the example

$$G_{nontrs} := \langle \tau_{0(2),1(2)}, \tau_{0(2),3(4)}, \tau_{4(9),2(15)} \rangle$$

shows: while all integers $0, \ldots, 87$ lie in the same orbit under the action of G_{nontrs}, this group stabilizes $88(90) \cup 89(90)$ setwise, and does therefore not act transitively on $\mathbb{N}_0$.

An example of a group which does act transitively is

$$G_{trs} := \langle \tau_{0(2),1(2)}, \tau_{0(3),2(3)}, \tau_{1(2),2(4)} \rangle.$$

This group acts at least 5-transitively on $\mathbb{N}_0$. Since the group

$$G_T := \langle \tau_{0(2),1(2)}, \tau_{1(2),2(4)}, \tau_{1(4),2(6)} \rangle$$

acts transitively on $\mathbb{N}_0$ if and only if the Collatz conjecture holds, a positive answer to the question would also imply the Collatz conjecture [S. Kohl: "The Collatz conjecture in a group theoretic context", `http://advgrouptheory.com/r/adv2g.php`, submitted]. On the other hand, if the Collatz conjecture holds, this would (by far!) not imply a positive answer to the question.

A positive answer to the question would mean that groups generated by 3 class transpositions are "well-behaved" in the sense that for deciding transitivity, looking at very small numbers is sufficient, and that for larger numbers "nothing can happen any more". For a discussion of the question, see `http://advgrouptheory.com/r/adv2g2.php`.

Stefan Kohl

ADV – 2Q

In order to study the involutive set-theoretic solution of the Yang-Baxter equation, W. Rump in ["Braces, radical rings, and the quantum Yang-Baxter equation", *J. Algebra* 307 (2007), 153–170] introduced a new algebraic structure called brace. A *left brace* is a set with two operations $+$ and $\circ$ such that $(B, +)$ is an abelian group, $(B, \circ)$ is a group, and $a \circ (b + c) + a = a \circ b + a \circ c$ for all $a, b, c \in B$.

It is known that the multiplicative group $(B, \circ)$ of any finite left brace is solvable (see [P. Etingof, T. Schedler and A. Soloviev: "Set-theoretical solutions to the quantum Yang-Baxter equation", *Duke Math. J.* 100 (1999), 169–209]). The class of finite groups that are the multiplicative group of a left brace, shortly Involutive Yang-Baxter Groups, includes, for instance, nilpotent groups of class at most 2, abelian-by-cyclic groups (see [F. Cedó, E. Jespers and Á. del Río: "Involutive Yang-Baxter groups", *Trans. Amer. Math. Soc.* 362 (2010), 2541–2558]) and solvable groups such that all Sylow p-subgroups are abelian (see [N. Ben David and Y. Ginosar: "On groups of I-type and involutive Yang–Baxter groups", *J. Algebra* 458 (2016), 197–206]). Moreover, F. Eisele in ["On the IYB-property in some solvable groups", *Arch. Math. (Basel)* 101 (2013), 303–318] showed, with a computer help, that any solvable group of order $\leqslant 200$ and any p-group of order < 1024 is an Involutive Yang-Baxter group.

Recently Rump in ["The brace of a classical group", *Note Mat.* 34 (2014), 115–147], presented a p-group of order p^{10} which is not an Involutive Yang-Baxter group. A detailed and comprehensive presentation of this example can be found in [D. Bachiller: "Counterexample to a conjecture about braces", *J. Algebra* 453 (2016), 160–176]. The following problem, quoted explicitly in the paper of Cedó, Jespers and del Río mentioned above, is therefore quite natural.

Question *Which finite solvable groups are Involutive Yang-Baxter groups?*

Francesco Catino

ADV – 2R

An element g of a group G is called a *(left) Engel element* if for any $x \in G$ there exists $n = n(x, g) \geqslant 1$ such that $[x,_n g] = 1$. As usual, the commutator $[x,_n g]$ is defined recursively by the rule

$$[x,_n g] = \big[[x,_{n-1} g], g\big]$$

assuming $[x,_0 g] = x$. If n can be chosen independently of x, then g is a *(left) n-Engel element*. A group G is called *Engel* if all elements of G are Engel. Gruenberg showed in ["The Engel structure of linear groups", *J. Algebra* 3 (1966), 291–303] that the set of Engel elements in a linear group is a locally nilpotent subgroup. By a linear group we understand here a subgroup of $GL(m, F)$ for some field F and a positive integer m.

We say that a group G is *almost Engel* if for every $g \in G$ there is a finite set $\mathcal{E}(g)$ such that for every $x \in G$ all sufficiently long commutators $[x,_n g]$ belong to $\mathcal{E}(g)$, that is, for every $x \in G$ there is a positive integer $n(x, g)$ such that $[x,_n g] \in \mathcal{E}(g)$ whenever $n(x, g) \leqslant n$. Thus, Engel groups are precisely the almost Engel groups for which we can choose $\mathcal{E}(g) = \{1\}$ for all $g \in G$. Almost Engel groups were introduced in [E. I. Khukhro and P. Shumyatsky: "Almost Engel compact groups", `https://arxiv.org/abs/1610.02079`], where it was proved that almost Engel compact groups are finite-by-(locally nilpotent). Later it was shown in [P. Shumyatsky: "Almost Engel linear groups", `https://128.84.21.199/abs/1610.03126`] that almost Engel linear groups are finite-by-(locally nilpotent). In view of Gruenberg's results the following question now becomes interesting.

Question *Is the set of almost Engel elements in a linear group always a subgroup?*

Pavel Shumyatsky
