Advances in Group Theory and Applications
© 2021 AGTA - www.advgrouptheory.com/journal
12 (2021), pp. 19–28 ISSN: 2499-1287
DOI: 10.32037/agta-2021-008

A Note on Cellular Automata

Mohammad Shahryari

(*Received June 8, 2020; Accepted July 3, 2020 — Communicated by F. de Giovanni*)

Dedicated to Mahmut Kuzucuoğlu on the occasion of his 60^{th} birthday

Abstract

In this note, we give a new totally topological definition of cellular automata over groups. We show that every continuous self-map of the shift space A^G can be represented as a G-sequence of cellular automata. In the case of a finitely generated group G and a finite alphabet A, we prove that the set of all cellular automata is discrete in the uniform metric space of continuous self-maps of A^G.

Mathematics Subject Classification (2020): 37B10, 37B15, 20F38
Keywords: cellular automata; symbolic dynamic; dynamical system

1 Introduction

Consider a topological space X which is compact, Hausdorff, and totally disconnected. If $T : X \to X$ is a continuous map, then the pair (X, T) is called a *symbolic dynamical system*. It has been in the center of attentions to study the limit behavior of subsets of X under the iterated action of T. One of the most important cases for us is the case when X is a full shift space over a group. Let G be a group and A be a set. Elements of A are called *letters* and elements of G are *cells*. The set A is also called *alphabet*. Consider the space A^G which is the set of all maps $G \to A$. If we consider the set A as a discrete topological space, then clearly A^G will be a Hausdorff and totally disconnected space where we equip A^G with the product topology (prodiscrete topology). In the case when A is finite, this space is also compact. In General, A^G is called the set of *configurations*. The group G acts on A^G by shift, i.e. for any $x \in A^G$ and $g \in G$, we have the new map $g \cdot x$, which sends any element $h \in G$ to the new symbol $x(g^{-1}h)$. Recall that a *cellular automaton* over G with alphabet set A is a map

$$T : A^G \to A^G$$

such that there exists a finite subset $S \subseteq G$ and a function $\mu : A^S \to A$ with the following property: for all $x \in A^G$ and all $g \in G$, we have

$$T(x)(g) = \mu((g^{-1} \cdot x)_{|S}),$$

where $|_S$ denotes the restriction. Any such a set S is a called a *memory set* and μ is called a *local defining function*. Let $CA(G, A)$ be the set of all such cellular automata. This set is a monoid with the ordinary composition of mappings.

Let $\mathcal{U}(A^G, A)$ be the set of all uniformly continuous functions from A^G to A. Define a binary operation $*$ on this set by

$$(f_1 * f_2)(x) = f_1((f_2(g^{-1} \cdot x))_{g \in G}).$$

This binary operation can be described in other form. For any function $f : A^G \to A$, we define a new map $T_f : A^G \to A^G$, by the rule

$$T_f(x)(g) = f(g^{-1} \cdot x).$$

Then, it can be easily seen that

$$(f_1 * f_2) = f_1 \circ T_{f_2},$$

where $\circ$ denotes the composition of functions.

Soon, we shall see that $\mathcal{U}(A^G, A)$ is a monoid. The identity of this monoid is the projection map $p_1 : A^G \to A$ defined by $p_1(x) = x(1)$, where 1 is the identity of G.

In this article, we show that two monoids $CA(G, A)$ and $\mathcal{U}(A^G, A)$ are isomorphic, and hence, every cellular automaton is in fact a uniformly continuous map $A^G \to A$. This gives us a totally topological definition of a cellular automaton. We will use this fact to give short proofs of some known theorems of the theory of cellular automata, for example, a generalization of the theorem of Curtis and Hedlund which says that a cellular automaton is just a uniformly continuous map $T : A^G \to A^G$ which is G-*equivariant*, i.e. for every configuration x and any group element g, we have $T(g \cdot x) = g \cdot T(x)$. This generalization is obtained in [3]. We also reprove the existence of the minimal memory set.

One of the other applications of the isomorphism between two monoids $CA(G, A)$ and $\mathcal{U}(A^G, A)$ is the study of continuous maps $A^G \to A^G$. In the case when A is finite, we will consider the space of continuous maps $C(A^G, A^G)$ and we will show that in some sense it is $CA(G, A)^G$. If we assume further that G is a finitely generated group, we can consider $C(A^G, A^G)$ with its *uniform metric*. In this case, we prove that the $CA(G, A)$ is a discrete subspace of $C(A^G, A^G)$, but the situation for uncountable groups or infinite alphabets remains unknown.

All of our notations in this note are the same as [2]. For topological definitions, the reader can see [1].

2 The isomorphism

We need some material from the theory of *uniform structures* and the reader can consult [1] for basic definitions. In the case when the alphabet set A is finite, there is no need to uniform structures and the property of being uniformly continuous will be the same as being continuous. For any $T \in CA(G, A)$, we define a new map $f_T : A^G \to A$ by

$$f_T(x) = T(x)(1).$$

Theorem 2.1 *The map* $T \mapsto f_T$ *is an isomorphism between the monoids* $CA(G, A)$ *and* $\mathcal{U}(A^G, A)$.

Proof — We first show that f_T is uniformly continuous. Suppose S is a memory set for T, that is $S \subseteq G$ is finite and if $x, y \in A^G$ satisfy $x_{|_S} = y_{|_S}$, then $T(x)(1) = T(y)(1)$. Now, put

$$W_S = \{(x, y) \in A^G \times A^G : \ x_{|_S} = y_{|_S}\}.$$

Let $\mathcal{U}$ denote the pro-discrete uniform structure over A^G. We know that $W_S \in \mathcal{U}$. Consider the map $f_T \times f_T : A^G \times A^G \to A \times A$ defined by

$$(f_T \times f_T)(x, y) = (f_T(x), f_T(y)).$$

We have the implication

$$x_{|_S} = y_{|_S} \Rightarrow T(x)(1) = T(y)(1),$$

and this means that $W_S \subseteq (f_T \times f_T)^{-1}(\Delta_A)$, where Δ_A is the diagonal of $A \times A$. This shows that

$$(f_T \times f_T)^{-1}(\Delta_A) \in \mathcal{U},$$

and hence f_T is uniformly continuous. Now, for arbitrary automata T_1 and T_2, we have

$$\begin{aligned} f_{T_1 \circ T_2}(x) &= (T_1 \circ T_2)(x)(1) = T_1(T_2(x))(1) = f_{T_1}(T_2(x)) \\ &= f_{T_1}((T_2(x)(g))_{g \in G}) = f_{T_1}((g^{-1} \cdot T_2(x)(1))_{g \in G}) \\ &= f_{T_1}((T_2(g^{-1} \cdot x)(1))_{g \in G}) = f_{T_1}((f_{T_2}(g^{-1} \cdot x))_{g \in G}) = (f_{T_1} * f_{T_2})(x). \end{aligned}$$

This shows that the map $T \mapsto f_T$ is a homomorphism. Note that if $f_{T_1} = f_{T_2}$, then for any $x \in A^G$, we have $T_1(x)(1) = T_2(x)(1)$, and since T_1 and T_2 are G-equivariant, so $T_1 = T_2$, proving that the map is injective.

Now, suppose that $f \in \mathcal{U}(A^G, A)$. Define a map

$$T : A^G \to A^G$$

by $T(x)(g) = f(g^{-1} \cdot x)$. First, note that T is G-equivariant: for any $h \in G$, we have

$$\begin{aligned} T(h \cdot x)(g) &= f(g^{-1} \cdot (h \cdot x)) = f((h^{-1}g)^{-1} \cdot x) \\ &= T(x)(h^{-1}g) = (h \cdot T(x))(g). \end{aligned}$$

Since f is uniformly continuous, we have $(f \times f)^{-1}(\Delta_A) \in \mathcal{U}$. On the other hand, we know that the set

$$\{W_\Omega : \ \Omega \subseteq G, |\Omega| < \infty\}$$

is a basis for the pro-discrete uniform structure over A^G. Hence there exists a finite subset $S \subseteq G$ such that $W_S \subseteq (f \times f)^{-1}(\Delta_A)$. In other words

$$x_{|S} = y_{|S} \Rightarrow f(x) = f(y) \Rightarrow T(x)(1) = T(y)(1).$$

This shows that T is a cellular automaton with the memory set S. Clearly $f_T = f$ and this shows that the map $T \mapsto f_T$ is surjective. Therefore, we proved that $\mathcal{U}(A^G, A)$ is a monoid and it is isomorphic to $CA(G, A)$. □

As a result, we now have a very easy definition of a cellular automaton: any uniformly continuous map $f : A^G \to A$ is a cellular automaton! As an application, we reprove the theorem of Curtis and Hedlund (see [2] and [3]).

Corollary 2.2 *Let* A *be finite and* $T : A^G \to A^G$ *be continuous and* G*-equivariant. Then* T *is a cellular automaton. The converse is also true.*

Proof — Define a map $f_T : A^G \to A$ by $f_T(x) = T(x)(1)$. Note that $f_T = p_1 \circ T$, so it is continuous. Since A^G is compact, so f_T is uniformly continuous and hence $f_T \in \mathcal{U}(A^G, A)$. This shows that there exists a cellular automaton T_0 such that $f_T = f_{T_0}$. But since T is G-equivariant, it can be easily seen that $T_0 = T$, proving that T is a cellular automaton. □

As another application, we prove the existence of the *minimal memory set* for a cellular automaton.

Corollary 2.3 *Let* T *be a cellular automaton and* S *and* S' *be two memory sets for* T*. Then* $S \cap S'$ *is also a memory set.*

Proof — Let $f : A^G \to A$ be the corresponding uniformly continuous mapping and $V = (f \times f)^{-1}(\Delta_A)$. We know that a finite set $\Omega \subseteq G$ is a memory set for T if and only if $W_\Omega \subseteq V$. Clearly $V \circ V \subseteq V$. Let $(x, y) \in W_{S \cap S'}$ and choose $z \in A^G$ such that $(x, z) \in W_S$ and $(z, y) \in W_{S'}$. This shows that $(x, y) \in W_S \circ W_{S'}$. Hence

$$W_{S \cap S'} \subseteq W_S \circ W_{S'} \subseteq V \circ V \subseteq V.$$

This proves that $S \cap S'$ is a memory set. □

Before closing this section, we must say that a similar statement is true for any arbitrary subshift $X \subseteq A^G$, after a small modification: Let $\mathcal{U}_0(X, A)$ be the set of all uniformly continuous functions $f : X \to A$, with the further property that

$$(f(g^{-1} \cdot x))_{g \in G} \in X,$$

for all $x \in X$. Then we can define the binary operation

$$(f_1 * f_2)(x) = f_1((f_2(g^{-1} \cdot x))_{g \in G}).$$

on the set $\mathcal{U}_0(X, A)$ and it becomes a monoid again. We can prove then the next theorem.

Theorem 2.4 *There is a natural isomorphism between $\mathcal{U}_0(X, A)$ and the monoid of all cellular automata $X \to X$.*

3 The structure of continuous maps

In this section, we will show that every dynamical system of the form (A^G, T) is in fact a unique G-sequence of cellular automata. As we saw in the previous section, every uniformly continuous map $A^G \to A$ has a unique decomposition of the form $p_1 \circ T$, where p_1 is the projection map $x \mapsto x(1)$ and T is a cellular automaton. In other words, we have

$$\mathcal{U}(A^G, A) = \{p_1 \circ T : \ T \in CA(G, A)\},$$

and the binary operation in this monoid is given by

$$(p_1 \circ T_1) * (p_1 \circ T_2) = p_1 \circ (T_1 T_2).$$

From now on, we assume that A is finite so every continuous map on A^G is automatically uniformly continuous. Let $C(A^G, A^G)$ be the space of all continuous functions $A^G \to A^G$, which is a monoid under composition of maps. For any $T \in C(A^G, A^G)$, the map

$$p_1 \circ T : A^G \to A$$

is continuous and hence there exists a unique cellular automaton T^* such that

$$p_1 \circ T = p_1 \circ T^*.$$

It is easy to see that the map T^* has the rule $T^*(x)(g) = T(g^{-1} \cdot x)(1)$. Note that for any two maps $T_1, T_2 \in C(A^G, A^G)$, we have

$$p_1 \circ (T_1 T_2) = p_1 \circ (T_1 T_2)^*.$$

On the other hand

$$(p_1 \circ T_1) * (p_1 \circ T_2) = (p_1 \circ T_1^*) * (p_1 \circ T_2^*) = p_1 \circ (T_1^* T_2^*).$$

This shows that $(T_1 T_2)^* = T_1^* T_2^*$, and therefore the map $T \mapsto T^*$ is a retraction of the monoid $C(A^G, A^G)$ onto the sub-monoid $CA(G, A)$.

Theorem 3.1 *Every continuous map* $T : A^G \to A^G$ *can be represented in a unique way as a sequence* $(T_g)_{g \in G}$, *where every* T_g *is a cellular automaton.*

Proof — Consider a fixed element $g \in G$. Let $p_g : A^G \to A$ be the projection map $x \mapsto x(g)$. Then the function $p_g \circ T$ is continuous and hence there exists a unique cellular automaton T_g such that

$$p_g \circ T = p_1 \circ T_g.$$

This means that for any x and g, we have

$$T(x)(g) = T_g(x)(1).$$

So, we can define a map

$$\Delta : CA(G, A)^G \to C(A^G, A^G),$$

which assigns the sequence $(T_g)_g$ and the map T to each other. As we saw this map is onto. It is also injective, since if we assume that $\Delta((T_g)_g) = \Delta((S_g)_g)$, then for any g we have $p_1 \circ T_g = p_1 \circ S_g$ and hence for any x, the equality $T_g(x)(1) = S_g(x)(1)$ is valid. Therefore, for arbitrary $h \in G$, we have

$$\begin{aligned} T_g(x)(h) &= (h^{-1} \cdot T_g(x))(1) = T_g(h^{-1} \cdot x)(1) = S_g(h^{-1} \cdot x)(1) \\ &= (h^{-1} \cdot S_g(x))(1) = S_g(x)(h). \end{aligned}$$

This shows that the map is injective. □

Hence, we can identify $C(A^G, A^G)$ with $CA(G, A)^G$. In fact, by this identification, we have

$$C(A^G, A^G) = CA(G, A)^G = C(A^G, A)^G.$$

Now, we focus on the case where G is finitely generated. Consider a finite generating set S for the group G, which is symmetric (that is,

$S^{-1} = S$). Then clearly we have the word metric on G with respect to S. This means that the distance between two distinct elements g and h is the length of the shortest group word $s_1 s_2 \dots s_m$, such that

$$gh^{-1} = s_1 s_2 \dots s_m$$

and all s_i belong to S. Let $B(r)$ be the closed ball of radius r in G, i.e.

$$B(r) = \{g \in G : \text{distance between } g \text{ and } 1 \text{ is at most } r\}.$$

Clearly, we have

$$B(0) \subseteq B(1) \subseteq B(2) \subseteq \dots, \text{ and } G = \bigcup_{r=1}^{\infty} B(r).$$

Now, it is possible to define a metric d on the shift space A^G. For every distinct configurations x and y, we have $d(x, y) = 2^{-R}$, where

$$R = \min\{r : \; x_{|_{B(r)}} \neq y_{|_{B(r)}}\}.$$

It is known that the prodiscrete topology on A^G is the same as the topology induced by the metric d. Since A^G is compact, we can define the *uniform metric* of $C(A^G, A^G)$ as follows:

$$d(T_1, T_2) = \sup_x d(T_1(x), T_2(x)).$$

Using standard arguments, one can prove that the space $C(A^G, A^G)$ is complete with respect to this metric but it is not compact in general. Instead, it can be shown that (the proof is standard diagonal argument) every sequence of elements in this space has a subsequence which converges point wise. We prove that the subspace $CA(G, A)$ is closed.

Proposition 3.2 *The subspace* $CA(G, A)$ *is closed in* $C(A^G, A^G)$.

PROOF — Consider an element $T \in C(A^G, A^G)$ which is not a cellular automaton. So, there are elements $x \in A^G$ and $s \in S$, such that $T(s \cdot x) \neq s \cdot T(x)$. This means that there exists an element g such that

$$T(s \cdot x)(g) \neq (s \cdot T(x))(g).$$

Let N be a natural number bigger that the norm of g with respect to the word metric ($N > \mathrm{distance}(g, 1)$). Let $r = 1/2^N$. Now, suppose $d(T, T') < r$. We show that T' is not a cellular automaton. Assume by contrary that T' is a cellular automaton. We have $d(T(y), T'(y)) < r$, for every configuration y (and especially for x) and hence (by the definition of the metric on A^G) we have

$$T(x)(s^{-1}g) = T'(x)(s^{-1}g).$$

This shows that

$$(s \cdot T(x))(g) = (s \cdot T'(x))(g) = T'(s \cdot x)(g) = T(s \cdot x)(g),$$

which is a contradiction. □

More surprising fact about the space of cellular automata is given in the following theorem; this space is discrete in the case of finitely generated groups and finite alphabets.

Theorem 3.3 *Let* G *be a finitely generated group and* A *be a finite alphabet set. Then* $CA(G, A)$ *is a discrete subset of* $C(A^G, A^G)$.

Proof — Consider two cellular automata T and T' from $CA(G, A)$. Let M and μ be a memory set and local defining function for T, respectively. Similarly, for T', we consider a memory set M' and a local defining function μ'. Assume that $d(T, T') \leqslant 2^{-1}$. This means that for any configuration x we have $d(T(x), T'(x)) \leqslant 2^{-1}$. In other words, we have $T(x)(1) = T'(x)(1)$. So, for any arbitrary configuration x, we have

$$\mu(x_{|_M}) = \mu'(x_{|_{M'}}).$$

Now, for any $g \in G$, we have

$$T(x)(g) = \mu(g^{-1} \cdot x_{|_M}) = \mu'(g^{-1} \cdot x_{|_{M'}}) = T'(x)(g).$$

This shows that $T(x) = T'(x)$ and hence $T = T'$. In other words, for any two distinct automata T and T', we must have $d(T, T') = 1$, and this completes the proof. □

As a final question, one may ask about the general case of the above theorem. It is also interesting to determine all cellular automata which are isolated in the metric space $C(A^G, A^G)$.

REFERENCES

[1] N. Bourbaki: "Elements of Mathematics: General Topology (Chapters 1–4)", *Springer*, Berlin (1995).

[2] T. Ceccherini-Silberstein – M. Coornaert: "Cellular Automata and Groups", *Springer*, Berlin (2010).

[3] T. Ceccherini-Silberstein – M. Coornaert: "A generalization of Curtis-Hedlund theorem", *Theor. Comp. Sci.* 400 (2008), 225–229.

M. Shahryari
Department of Mathematics
College of Science
Sultan Qaboos University
Muscat (Oman)
e-mail: m.ghalehlar@squ.edu.om