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Transitive and Inherited Subgroups

JOSEPH PETRILLO

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Abstract

The primary purpose of this work is to consider the notions of transitivity and inheritance not only as properties of subgroup *properties*, but as properties of subgroups themselves. We introduce and develop first principles and investigate the interplay between the new subgroup properties that result.

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1 Introduction

Recall that if U is a subnormal subgroup of a group G and V is subnormal in U, then V is subnormal in G. Observe, however, that this statement need not be true if we replace "subnormal" with "normal". This property enjoyed by subnormality is commonly known as *transitivity*, and it plays a central role in this paper. A seminal work on this concept with regard to normality is [4]. Dually, we shall be concerned with subgroup properties that are preserved or "inherited" in overgroups in the sense that if V is an α-subgroup of G, and $V \leqslant U \leqslant G$, then V is an α-subgroup of U. For instance, both subnormality and normality are preserved in subgroups, as is the property of being a Sylow subgroup. The cover-avoidance property (or CAP, for short), however, is not necessarily preserved in overgroups (see [3]). There is a precedent for studying the concept of inheritance; references include [6] and [7].

The purpose of this work is to reinterpret global properties, such as transitivity, on the subgroup level to create new subgroup properties. We introduce and develop first principles and investigate the interplay between the new subgroup properties that result.

All groups considered in this paper are finite. If α and β are subgroup properties in a group G such that $U\alpha G$ implies $U\beta G$, then we use the shorthand notation $\alpha \Rightarrow \beta$ to denote this. We say that α is *reflexive* in G if $U\alpha U$ for all subgroups U of G. All other undefined group theoretic notation and terms are standard and may be found in [1].

2 Transitivity and generalizations

A subgroup property is said to be *transitive* if $V\alpha U$ and $U\alpha G$ imply that $V\alpha G$ for all $V \leqslant U \leqslant G$. It is interesting to note that although normality is not transitive in general, it does possess a type of transitivity through other subgroup properties. For instance, in a solvable group G, a normal subgroup of a well-placed subgroup is a normal subgroup of G (see Proposition 6.5 of [1]). Examples such as this lead naturally to a generalization of transitivity in which the property α is preserved as we pass through overgroups possessing a possibly different property β.

Definition 2.1 Let α and β be subgroup properties. Then α is said to be *transitive through* β*-subgroups* if for all $V \leqslant U \leqslant G$, the conditions $V\alpha U$ and $U\beta G$ together imply that $V\alpha G$.

Referring to the example in the previous paragraph, we see that the cover-avoidance property in a solvable group is transitive through critical subgroups. Notice that the special case $\alpha = \beta$ conveniently yields the familiar definition of transitivity.

On a local level, one may wish to impose a sort of transitivity on subgroups rather than on properties of subgroups. This becomes evident when considering, for example, that normal subgroups of a direct factor of a direct product are also normal in the product. We might say that direct factors are *normal-transitive*.

Definition 2.2 Let U be a subgroup of the group G, and let α be a subgroup property. Then U is called an *α-transitive subgroup* of G if each α-subgroup of U is an α-subgroup of G.

Observe that an α-transitive subgroup is, in turn, an α-subgroup when α is reflexive, or, more specifically, in cases where there exists a group G and a subgroup U of G such that $U\alpha U$.

The definitions of "transitive through" and "α-transitive" were introduced by the author in [2] and studied further in [3]. The latter includes the development of first principles for the purpose of studying of normal-transitive and CAP-transitive subgroups.

The next proposition makes a few connections between the concepts discussed thus far.

Proposition 2.3 *The following statements are equivalent in a group* G.

(1) α *is transitive in* G.

(2) α *is transitive through* α*-subgroups of* G.

(3) α*-transitivity is transitive through* α*-subgroups of* G.

(4) *Each* α*-subgroup of* G *is* α*-transitive in* G.

Proof — Statements (1), (2), and (4) are simply re-wordings of the definition of transitivity and hence are equivalent. To see that (1) implies (3), assume α is transitive, $U\alpha G$, and V is α-transitive in U with $W\alpha V$. We have $W\alpha U$ by α-transitivity and $W\alpha G$ by transitivity. This proves that V is α-transitive in G. Finally, to see that (3) implies (4), let $U\alpha G$ and observe that U is α-transitive in U with $U\alpha G$. By (3), U is α-transitive in G. □

In this paper, we intend to take these ideas one step further. As a motivating example, consider a critical subgroup U of a solvable group G. By [1], each normal subgroup of U is a CAP-subgroup of G.

Definition 2.4 Let U be a subgroup of the group G, and let α and β be subgroup properties. Then U is called an *$\alpha\beta$-transitive subgroup* of G, denoted $U(\alpha \uparrow \beta)G$, if each α-subgroup of U is a β-subgroup of G.

A few remarks about Definition 2.4 are worth noting. First, observe that $\alpha\alpha$-transitivity is equivalent to α-transitivity. Second, if $U(\alpha \uparrow \beta)G$ and $U\alpha U$, then $U\beta G$. This is certainly true in the general case where α is a reflexive subgroup property. Third, $U(\alpha \uparrow \leqslant)G$ is universally true for any group G and $U \leqslant G$. Last, $(\alpha \uparrow \beta)$ and $(\beta \uparrow \alpha)$ are apparently distinct subgroup properties, hence α and β do not "commute" in Definition 2.4. To see this, let G be the alternating group A_4, and let U be the subgroup of G of order 4. Evidently, U is a $(\trianglelefteq \uparrow \leqslant)$-subgroup of G, however U is not a $(\leqslant \uparrow \trianglelefteq)$-subgroup of G.

3 Inheritance and generalizations

The organization of this section is similar to that of the previous except we shall be concerned with subgroup properties that are preserved as we drop down into intermediate subgroups. If $V\alpha G$ implies that $V\alpha U$ for all $V \leqslant U \leqslant G$, then we shall call α an *inherited* subgroup property. Therefore, we can say that both subnormality and normality are inherited subgroup properties.

Consider now the following analogues of "transitive through" and "α-transitive".

Definition 3.1 Let α and β be subgroup properties. Then α is said to be *inherited in* β*-subgroups* if for all $V \leqslant U \leqslant G$, the conditions $V\alpha G$ and $U\beta G$ together imply that $V\alpha U$.

Definition 3.2 Let U be a subgroup of the group G, and let α be a subgroup property. Then U is called an α*-inherited subgroup* of G if each α-subgroup of G contained in U is an α-subgroup of U.

Note that if U is α-inherited in G and $U\alpha G$, then $U\alpha U$.

Proposition 3.3 *The following statements are equivalent in a group* G.

(1) α *is inherited in* G.

(2) α *is inherited in subgroups of* G.

(3) α*-inheritance is transitive through subgroups of* G.

(4) *Each subgroup of* G *is* α*-inherited in* G.

Proof — The equivalence of statements (1), (2), and (4) follows from the fact that these are restatements of the definition of inheritance. To see that (3) implies (4), assume that α-inheritance is transitive through subgroups of G, and let $U \leqslant G$. Since U is α-inherited in $U \leqslant G$, it follows that U is α-inherited in G. The proof will be complete once we prove that (4) implies (3). To this end, suppose U is α-inherited in $V \leqslant G$. By (4), U is α-inherited in G, and we are done. □

The definitions of "inheritance in" and "α-inheritance" were studied in [2] and [3] as "persistent in" and α-persistence". The reason for the change in terminology is so as not to confuse our current work with the notion of "normal persistence" defined by Wielandt [5]. As we did with transitivity in the last section, we can further generalize these new definitions.

Definition 3.4 Let U be a subgroup of the group G, and let α and β be subgroup properties. Then U is called an *$\alpha\beta$-inherited subgroup* of G, denoted $U(\alpha \downarrow \beta)G$, if each β-subgroup of G contained in U is an α-subgroup of U.

For example, if α is the cover-avoidance property and β is the property of being a subgroup, then a supersolvable subgroup U of a group G is $\alpha\beta$-inherited, or $U(\mathrm{CAP} \downarrow \leqslant)G$ (of course, this is because every subgroup of a supersolvable group is a CAP-subgroup).

Let us make similar observations about Definition 3.4 as we did with Definition 2.4. Notice first that $\alpha\alpha$-inheritance is equivalent to α-inheritance. Second, if $U(\alpha \downarrow \beta)G$ and $U\beta G$, then $U\alpha U$. Third, $U(\leqslant \downarrow \beta)G$ is true for any group G and any $U \leqslant G$. Finally, we conclude that $(\alpha \downarrow \beta)$ and $(\beta \downarrow \alpha)$ are distinct subgroup properties. Indeed, let $G = A_4$, let U be the subgroup of G of order 4, and let α and β be the properties of subnormality and normality, respectively. Then U is $\alpha\beta$-inherited in G but not $\beta\alpha$-inherited in G.

4 Cancellation and expansion laws

In this section, we present two of our main theorems. The first is comprised of a collection of "cancellation laws" which demonstrate the interplay between $\alpha\beta$-transitivity and $\alpha\beta$-inheritance in the context of two comparable subgroups V and U. In particular, these laws pertain directly to the smaller subgroup V.

Theorem 4.1 (Cancellation Laws) *Let α, β, and γ be subgroup properties, and let $V \leqslant U \leqslant G$.*

(1) *If* $V(\alpha\uparrow\gamma)U$ *and* $U(\gamma\uparrow\beta)G$, *then* $V(\alpha\uparrow\beta)G$.

(2) *If* $V(\alpha\uparrow\gamma)G$ *and* $U(\beta\downarrow\gamma)G$, *then* $V(\alpha\uparrow\beta)U$.

(3) *If* $V(\alpha\downarrow\gamma)U$ *and* $U(\gamma\downarrow\beta)G$, *then* $V(\alpha\downarrow\beta)G$.

(4) *If* $V(\alpha\downarrow\gamma)G$ *and* $U(\beta\uparrow\gamma)G$, *then* $V(\alpha\downarrow\beta)U$.

Proof — To prove (1), let $W\alpha V$. Since $V(\alpha\uparrow\gamma)U$, it follows that $W\gamma U$, and since $U(\gamma\uparrow\beta)G$, it follows that $W\beta G$. This means $V(\alpha\uparrow\beta)G$. We prove (2) by again choosing $W\alpha V$. The assumption $V(\alpha\uparrow\gamma)G$ implies that $W\gamma G$, hence $W\beta U$ since $U(\beta\downarrow\gamma)G$. That is, $V(\alpha\uparrow\beta)U$. For (3), suppose $W\beta G$ contained in V. If $U(\gamma\downarrow\beta)G$, then $W\gamma U$. Under the assumption $V(\alpha\downarrow\gamma)U$, we conclude that $W\alpha V$. Therefore, $V(\alpha\downarrow\beta)G$. The proof of statement (4) is similar. Indeed, if $W\beta U$ contained in V and $U(\beta\uparrow\gamma)G$, then $W\gamma G$. But $V(\alpha\downarrow\gamma)G$ implies that $W\alpha V$, hence $V(\alpha\downarrow\beta)U$. □

The next two corollaries give some insight into the transitivity and inheritance of $\alpha\beta$-transitivity and $\alpha\beta$-inheritance. They also bring to light the impact the second argument β has on these properties.

Corollary 4.2 *Let* α *and* β *be subgroup properties.*

(1) $\alpha\beta$*-transitivity is transitive through* β*-transitive subgroups.*

(2) $\alpha\beta$*-transitivity is inherited in* β*-inherited subgroups.*

(3) $\alpha\beta$*-inheritance is transitive through* β*-inherited subgroups.*

(4) $\alpha\beta$*-inheritance is inherited in* β*-transitive subgroups.*

Proof — These statements are direct consequences of the cancellation laws by setting $\gamma=\beta$. □

Corollary 4.3 *Let* α *be a subgroup property.*

(1) α*-transitivity is transitive.*

(2) α*-transitivity is inherited in* α*-inherited subgroups.*

(3) α*-inheritance is transitive.*

(4) α*-inheritance is inherited in* α*-transitive subgroups.*

Proof — These statements are direct consequences of Corollary 4.2 by setting $\beta = \alpha$. □

By taking $\gamma = \alpha$ in Theorem 4.1, we discover additional results showing the impact β has on α-transitive and α-inherited subgroups that are contained in generalized transitive and inherited overgroups involving β.

Corollary 4.4 *Let α and β be subgroup properties and let $V \leqslant U \leqslant G$.*

(1) *If* $V(\alpha \uparrow \alpha)U$ *and* $U(\alpha \uparrow \beta)G$, *then* $V(\alpha \uparrow \beta)G$.

(2) *If* $V(\alpha \uparrow \alpha)G$ *and* $U(\beta \downarrow \alpha)G$, *then* $V(\alpha \uparrow \beta)U$.

(3) *If* $V(\alpha \downarrow \alpha)U$ *and* $U(\alpha \downarrow \beta)G$, *then* $V(\alpha \downarrow \beta)G$.

(4) *If* $V(\alpha \downarrow \alpha)G$ *and* $U(\beta \uparrow \alpha)G$, *then* $V(\alpha \downarrow \beta)U$.

The second main theorem consists of what we will refer to as "expansion laws". By applying the appropriate interpretations, we may informally consider this theorem to be either a restatement of the definitions of $\alpha\beta$-transitivity and $\alpha\beta$-inheritance in symbolic form, or a corollary of the cancellation laws expressed in a global setting. The supplied proof, however, is from first principles.

Theorem 4.5 (Expansion Laws) *Let α, β, and γ be subgroup properties.*

(1) $(\alpha \uparrow \beta) \Rightarrow ((\gamma \uparrow \alpha) \uparrow (\gamma \uparrow \beta))$.

(2) $(\alpha \uparrow \beta) \Rightarrow ((\gamma \downarrow \alpha) \downarrow (\gamma \downarrow \beta))$.

(3) $(\alpha \downarrow \beta) \Rightarrow ((\gamma \uparrow \alpha) \downarrow (\gamma \uparrow \beta))$.

(4) $(\alpha \downarrow \beta) \Rightarrow ((\gamma \downarrow \alpha) \uparrow (\gamma \downarrow \beta))$.

Proof — In order to prove statements (1) and (2) we assume $U(\alpha \uparrow \beta)G$. We wish to show that

$$U((\gamma \uparrow \alpha) \uparrow (\gamma \uparrow \beta))G.$$

On one hand, if $V(\gamma \uparrow \alpha)U$ and $W\gamma V$, then $W\alpha U$. By assumption, $W\beta G$. That is, $V(\gamma \uparrow \beta)G$ which proves (1). On the other hand, if $V(\gamma \downarrow \beta)G$ such that $V \leqslant U$, then for any $W\alpha U$ with $W \leqslant V$ we have that $W\beta G$ since $U(\alpha \uparrow \beta)G$. However $W\beta G$ implies $W\gamma V$ by the assumption $V(\gamma \downarrow \beta)G$, and so $V(\gamma \downarrow \alpha)U$. This proves (2).

The proofs of statements (3) and (4) are similar to those of (1) and (2), so assume $U(\alpha \downarrow \beta)G$ in both parts. For (3), we let $V(\gamma \uparrow \beta)G$ in U and must prove that $V(\gamma \uparrow \alpha)U$. If $W\gamma V$, then our assumptions yield $W\beta G$ and then $W\alpha U$. Hence $V(\gamma \uparrow \alpha)U$. To prove (4), let $V(\gamma \downarrow \alpha)U$ and we show $V(\gamma \downarrow \beta)G$. To this end let $W\beta G$ contained in V. By our assumptions, $W\alpha U$ and $W\gamma V$, which completes the proof. □

5 Cases in which one property implies the other

For some subgroup properties, it may be the case that $\alpha \Rightarrow \beta$. (For instance, normal subgroups are subnormal.) Keep in mind, however, that $\alpha \Rightarrow \beta$ could refer to inherited properties of α and β on a global level, or it could mean that $U\alpha G \Rightarrow U\beta G$ locally for a particular subgroup U or even in a particular group G. The results of this section are expressed on a global level, but of course the statements can be reinterpreted locally if the reader wishes to do so.

We refer to the statements in the next theorem as the "substitution laws".

Theorem 5.1 (Substitution Laws) *Let α, β, and γ be subgroup properties.*

(1) *If $\gamma \Rightarrow \alpha$, then $(\alpha \uparrow \beta) \Rightarrow (\gamma \uparrow \beta)$.*

(2) *If $\beta \Rightarrow \gamma$, then $(\alpha \uparrow \beta) \Rightarrow (\alpha \uparrow \gamma)$.*

(3) *If $\alpha \Rightarrow \gamma$, then $(\alpha \downarrow \beta) \Rightarrow (\gamma \downarrow \beta)$.*

(4) *If $\gamma \Rightarrow \beta$, then $(\alpha \downarrow \beta) \Rightarrow (\alpha \downarrow \gamma)$.*

Proof — Assume $U(\alpha \uparrow \beta)G$ and $\gamma \Rightarrow \alpha$, and let $V\gamma U$. It follows that $V\alpha U$, and so $V\beta G$. This proves statement (1). If $U(\alpha \uparrow \beta)G$ and $\beta \Rightarrow \gamma$, then $V\alpha U$ implies $V\beta G$ and so $V\gamma G$. Hence, statement (2) is proved. Now let $V\beta G$ such that $V \leqslant U$. If $U(\alpha \downarrow \beta)G$ and $\alpha \Rightarrow \gamma$, then $V\alpha U$ which in turn implies $V\gamma U$. This proves (3). To prove (4), assume $U(\alpha \downarrow \beta)G$ and $\gamma \Rightarrow \beta$. If $V\gamma G$ with $V \leqslant U$, then by assumption, $V\beta G$. Therefore, $V\alpha U$. This completes the proof. □

Corollary 5.2 *Let α and β be subgroup properties such that $\beta \Rightarrow \alpha$.*

(1) $(\alpha \uparrow \beta) \Rightarrow (\alpha \uparrow \alpha)$.

(2) $(\alpha \uparrow \beta) \Rightarrow (\beta \uparrow \beta)$.

(3) $(\alpha \uparrow \beta) \Rightarrow (\beta \uparrow \alpha)$.

Proof — We set $\gamma = \alpha$ in Theorem 5.1 (2) to obtain $(\alpha \uparrow \beta) \Rightarrow (\alpha \uparrow \alpha)$, and we set $\gamma = \beta$ in Theorem 5.1 (1) to obtain $(\alpha \uparrow \beta) \Rightarrow (\beta \uparrow \beta)$. By applying Theorem 5.1 (2) with $\alpha = \beta$ and $\gamma = \alpha$, we further conclude that $(\beta \uparrow \beta) \Rightarrow (\beta \uparrow \alpha)$, hence $(\alpha \uparrow \beta) \Rightarrow (\beta \uparrow \beta) \Rightarrow (\beta \uparrow \alpha)$. □

The proof of Corollary 5.3 below is similar to that of Corollary 5.2 except we proceed by applying the appropriate substitutions for α, β, and γ in parts (3) and (4) of Theorem 5.1.

Corollary 5.3 *Let α and β be subgroup properties such that $\alpha \Rightarrow \beta$.*

(1) $(\alpha \downarrow \beta) \Rightarrow (\alpha \downarrow \alpha)$.

(2) $(\alpha \downarrow \beta) \Rightarrow (\beta \downarrow \beta)$.

(3) $(\alpha \downarrow \beta) \Rightarrow (\beta \downarrow \alpha)$.

We observed earlier that if $U(\alpha \uparrow \beta)G$ and $U\alpha U$, then $U\beta G$. More generally, if α is reflexive, then $(\alpha \uparrow \beta) \Rightarrow \beta$. Let us formally record this fact as a lemma for easy reference.

Lemma 5.4 *If α and β are subgroup properties such that α is reflexive, then $(\alpha \uparrow \beta) \Rightarrow \beta$. In particular, $(\alpha \uparrow \alpha) \Rightarrow \alpha$.*

Since reflexivity can be expressed as a conditional statement, it makes sense to investigate its impact on $\alpha\beta$-transitivity and $\alpha\beta$-inheritance in light of the substitution laws of Theorem 5.1. In fact, Lemma 5.4 and the substitution laws yield the following two corollaries.

Corollary 5.5 *Let α and β be subgroup properties such that α is reflexive.*

(1) $(\alpha \uparrow \beta) \Rightarrow ((\alpha \uparrow \alpha) \uparrow \beta)$.

(2) $(\alpha \downarrow \beta) \Rightarrow (\alpha \downarrow (\alpha \uparrow \beta))$.

Corollary 5.6 *Let α and β be subgroup properties such that β is reflexive.*

(1) $(\alpha \uparrow \beta) \Rightarrow ((\beta \uparrow \alpha) \uparrow \beta)$.

(2) $(\alpha \downarrow \beta) \Rightarrow (\alpha \downarrow (\beta \uparrow \beta))$.

6 Cases in which one or both properties are transitive

In this section we derive numerous necessary conditions for $\alpha\beta$-transitivity and $\alpha\beta$-inheritance by imposing transitivity on α or β or both. First we consider the cases in which α is transitive and in which α is both transitive and reflexive.

Proposition 6.1 *Let α and β be subgroup properties such that α is transitive.*

(1) $(\alpha \uparrow \beta) \Rightarrow (\alpha \uparrow (\alpha \uparrow \beta))$.

(2) $(\alpha \downarrow \beta) \Rightarrow ((\alpha \uparrow \alpha) \downarrow \beta)$.

Proof — To prove statement (1), let $\gamma = \alpha$ in Theorem 4.5 (1) to obtain

$$(\alpha \uparrow \beta) \Rightarrow ((\alpha \uparrow \alpha) \uparrow (\alpha \uparrow \beta)).$$

If α is transitive, then $\alpha \Rightarrow (\alpha \uparrow \alpha)$ by Proposition 2.3, therefore

$$((\alpha \uparrow \alpha) \uparrow (\alpha \uparrow \beta)) \Rightarrow (\alpha \uparrow (\alpha \uparrow \beta))$$

by Theorem 5.1 (1). Statement (2) follows immediately from Theorem 5.1 (3) by setting $\gamma = (\alpha \uparrow \alpha)$. $\square$

Corollary 6.2 *Let α and β be subgroup properties such that α is both transitive and reflexive.*

(1) $(\alpha \uparrow \beta) \Leftrightarrow ((\alpha \uparrow \alpha) \uparrow \beta) \Leftrightarrow (\alpha \uparrow (\alpha \uparrow \beta))$.

(2) $(\alpha \downarrow \beta) \Leftrightarrow ((\alpha \uparrow \alpha) \downarrow \beta) \Rightarrow (\alpha \downarrow (\alpha \uparrow \beta))$.

Proof — Let us first focus our attention on statement (1). In light of Corollary 5.5 (1) and Proposition 6.1 (1), all that remains to prove is that $((\alpha \uparrow \alpha) \uparrow \beta) \Rightarrow (\alpha \uparrow \beta)$ and $(\alpha \uparrow (\alpha \uparrow \beta)) \Rightarrow (\alpha \uparrow \beta)$. However, both statements follow from Lemma 5.4 and the substitution laws of Theorem 5.1. For statement (2), similar reasoning confirms that $((\alpha \uparrow \alpha) \downarrow \beta) \Rightarrow (\alpha \downarrow \beta)$. $\square$

Next we consider $\alpha\beta$-transitivity and $\alpha\beta$-inheritance for the cases in which β is transitive and in which β is both transitive and reflexive.

Proposition 6.3 *Let α and β be subgroup properties such that β is transitive.*

(1) $(\alpha \uparrow \beta) \Rightarrow (\alpha \uparrow (\beta \uparrow \beta))$.

(2) $(\alpha \downarrow \beta) \Rightarrow ((\beta \uparrow \alpha) \downarrow \beta)$.

Proof — Statement (1) is true by Proposition 2.3 and Theorem 5.1. For statement (2), let $\gamma = \beta$ in Theorem 4.5 (3) to get

$$(\alpha \downarrow \beta) \Rightarrow ((\beta \uparrow \alpha) \downarrow (\beta \uparrow \beta)).$$

The result follows from the transitivity of β, Proposition 2.3, and Theorem 5.1 (4). □

Suitable applications of Theorem 4.5, Corollary 5.6, and Propositions 2.3 and 6.3 yield the following corollary.

Corollary 6.4 *Let α and β be subgroup properties such that β is both transitive and reflexive.*

(1) $(\alpha \uparrow \beta) \Leftrightarrow (\alpha \uparrow (\beta \uparrow \beta)) \Rightarrow ((\beta \uparrow \alpha) \uparrow \beta)$.

(2) $(\alpha \downarrow \beta) \Leftrightarrow (\alpha \downarrow (\beta \uparrow \beta)) \Leftrightarrow ((\beta \uparrow \alpha) \downarrow \beta)$.

Recall how Corollary 4.2 succinctly expressed the transitivity and inheritance of $\alpha\beta$-transitive and $\alpha\beta$-inherited subgroups. It turns out that these properties remain unaffected by the transitivity of α and by the reflexivity of α and β. The transitivity of β, however, does yield new information.

Corollary 6.5 *Let α and β be subgroup properties such that β is transitive.*

(1) *$\alpha\beta$-transitivity is transitive through β-subgroups.*

(2) *$\alpha\beta$-inheritance is inherited in β-subgroups.*

Proof — Parts (1) and (4) of Corollary 4.2 state that

$$(\beta \uparrow \beta) \Rightarrow ((\alpha \uparrow \beta) \uparrow (\alpha \uparrow \beta)) \quad \text{and} \quad (\beta \uparrow \beta) \Rightarrow ((\alpha \downarrow \beta) \downarrow (\alpha \downarrow \beta)),$$

respectively. This corollary follows from the fact that $\beta \Rightarrow (\beta \uparrow \beta)$ when β is transitive. □

The last case we will consider here is the one in which both α and β are transitive, and a subgroup is both α and β. This situation has a most interesting effect on $\alpha\beta$-inherited subgroups.

Proposition 6.6 *Let α and β be transitive subgroup properties. Then $\alpha \wedge \beta \wedge (\alpha \downarrow \beta) \Rightarrow (\beta \uparrow \alpha)$.*

Proof — Assume $U(\alpha \downarrow \beta)G$, $U\alpha G$, and $U\beta G$. Let $V\beta U$; we want to show that $V\alpha G$. We have $V\beta U\beta G$ so $V\beta G$ by the transitivity of β. Under the assumption that $U(\alpha \downarrow \beta)G$, it follows that $V\alpha U$. But then $V\alpha U\alpha G$ implies $V\alpha G$ by the transitivity of α, and we are done. □

In words, an $\alpha\beta$-inherited subgroup which possesses both transitive properties α and β is a $\beta\alpha$-transitive subgroup.

7 Cases in which one or both properties are inherited

The goal in this final section is to consider $\alpha\beta$-transitivity and $\alpha\beta$-inheritance in the presence of inheritance. The results here are a bit richer than those in the previous section possibly due to the fact that a inherited subgroup property α necessarily forces each subgroup to be $\alpha\alpha$-inherited.

Proposition 7.1 *Let α and β be subgroup properties such that α is inherited. Then for any subgroup property γ, we have $(\alpha \downarrow \beta) \Rightarrow (\gamma \uparrow (\alpha \downarrow \beta))$.*

Proof — If γ is a subgroup property and α is inherited, then

$$\gamma \Rightarrow \leqslant \Rightarrow (\alpha \downarrow \alpha).$$

By Theorems 4.5 and 5.1, we have

$$(\alpha \downarrow \beta) \Rightarrow ((\alpha \downarrow \alpha) \uparrow (\alpha \downarrow \beta)) \Rightarrow (\gamma \uparrow (\alpha \downarrow \beta)).$$

The statement is proved. □

Proposition 7.2 *Let α and β be subgroup properties such that β is inherited. Then for any subgroup property γ, we have $(\alpha \uparrow \beta) \Rightarrow ((\beta \downarrow \alpha) \downarrow \gamma)$.*

Proof — Suppose γ is a subgroup property and β is inherited. It follows that $\gamma \Rightarrow \leqslant \Rightarrow (\beta \downarrow \beta)$. By Theorems 4.5 and 5.1, we have

$$(\alpha \uparrow \beta) \Rightarrow ((\beta \downarrow \alpha) \downarrow (\beta \downarrow \beta)) \Rightarrow ((\beta \downarrow \alpha) \downarrow \gamma).$$

The statement is proved. □

Proposition 7.3 *Let α and β be inherited subgroup properties. Then for any subgroup property γ, we have $(\alpha \uparrow \beta) \Rightarrow (\gamma \uparrow (\beta \downarrow \alpha))$.*

Proof — Assume $U(\alpha \uparrow \beta)G$, and let $V\gamma U$; we wish to show that $V(\beta \downarrow \alpha)G$. To this end, let $W\alpha G$ with $W \leqslant V$. It remains to prove that $W\beta V$. Since α is inherited, $W\alpha U$, and by hypothesis, $W\beta G$. The inheritance of β yields $W\beta V$ as required. □

Corollary 7.4 *Let α and β be inherited subgroup properties. Then $(\alpha \uparrow \beta) \Rightarrow (\beta \downarrow \alpha)$.*

Proof — Take $\gamma = \leqslant$ in Proposition 7.3 to get

$$U(\alpha \uparrow \beta)G \Rightarrow U(\leqslant \uparrow (\beta \downarrow \alpha))G.$$

The desired result follows from the fact that $U \leqslant U$. □

We close this paper with two corollaries analogous to Corollaries 4.2 and 6.5.

Corollary 7.5 *Let α and β be subgroup properties such that α is inherited.*

(1) *$\alpha\beta$-inheritance is transitive.*

(2) *$\alpha\beta$-inheritance is inherited in $\beta\alpha$-transitive subgroups.*

Proof — Statement (1) comes from Proposition 7.1 by setting $\gamma = (\alpha \downarrow \alpha)$. Statement (2) is obtained from Proposition 7.2 by interchanging α and β and then replacing γ with $(\alpha \downarrow \beta)$. □

Corollary 7.6 *Let α and β be subgroup properties such that β is inherited.*

(1) *$\alpha\beta$-transitivity is inherited.*

(2) *$\alpha\beta$-inheritance is transitive through subgroups.*

Proof — By replacing α with β and then γ with α in parts (3) and (4) of Theorem 4.5, we obtain

$$(\beta \downarrow \beta) \Rightarrow ((\alpha \uparrow \beta) \downarrow (\alpha \uparrow \beta)) \quad \text{and} \quad (\beta \downarrow \beta) \Rightarrow ((\alpha \downarrow \beta) \uparrow (\alpha \downarrow \beta)).$$

Both statements follow from the fact that $\leqslant \Rightarrow (\beta \downarrow \beta)$ when β is inherited. □

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Joseph Petrillo
Division of Mathematics and Computer Science
Alfred University, 1 Saxon Drive
Alfred, NY (USA)
e-mail: petrillo@alfred.edu