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On Finite 2-Groups with the Non-Dedekind Metacyclic Norm of Abelian Non-Cyclic Subgroups

T.D. Lukashova — F.M. Lyman

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Dedicated to the 70th birthday of prof. Leonid A. Kurdachenko

Abstract

The authors examine the relations between the properties of a group G and its norm N_G^A of Abelian non-cyclic subgroups. In this paper the properties of finite 2-groups with the cyclic center and the metacyclic non-Dedekind norm of Abelian non-cyclic subgroups, are studied. The complete description of such groups is obtained.

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1 Introduction

One of the productive directions in group theory is the study of the influence of some characteristic subgroups (center, derived subgroup, Frattini subgroup etc.) on the structure of the whole group. Such characteristic subgroups include different Σ-norms of a group.

Recall, that the intersection of the normalizers of all subgroups of a system Σ, provided that Σ contains all subgroups of a group G with some property (for example, Σ is a system of all Abelian, all non-Abelian, all non-cyclic subgroups of a group) is called a *Σ-norm* of a group G. It is clear that every Σ-norm of a group contains the center of a group and normalizes all subgroups of the system Σ (assuming $\Sigma \neq \varnothing$).

While studying of Σ-norms and their influence on the structure of a group, a number of questions regarding to the choice of the system Σ and the restrictions that these Σ-norms satisfied arise. If the structure of Σ-norm and the nature of its embedding to a group are known, in most of cases it is possible to describe the properties and structure of a group itself. In the most researches such problem solved for the groups which coincide with their Σ-norms, that is, groups in which each subgroup of Σ is a normal subgroup of a group [5, 12, 16, 24].

The first situation, when a Σ-norm is a proper subgroup of a group was studied by R. Baer [1] in 1935 for the system Σ of all subgroups of a group. He called it the *norm of a group* G and denoted by $N(G)$. It should be noted, that interest for the norm $N(G)$ still has been not decreased as evidenced by the findings [3, 11, 22, 26, 25].

Narrowing the system Σ of subgroups, it's possible to get different Σ-norms, which can be considered as generalizations of Baer's norm $N(G)$. Among such generalized norms let's point out the norm of subnormal subgroups of a group or Wielandt subgroup [2, 27], A-norm [10], the metanorm [6, 7] and the non-cyclic norm N_G of a group [13, 17]. If Σ is the system of all Abelian non-cyclic subgroups, then the Σ-norm is called *the norm of Abelian non-cyclic subgroups of a group* G and denoted by N_G^A.

In this article the authors continue the investigation of 2-groups with the non-Dedekind norm of Abelian non-cyclic subgroups, initiated in [18]–[21]. In particular, in [18] the complete description of infinite locally finite 2-groups with such restriction on the norm N_G^A was obtained. The structure of finite 2-groups in which the norm N_G^A is a non-metacyclic non-Dedekind subgroup was investigated in [19, 20]. Finally, the finite 2-groups with the non-cyclic center and the non-Dedekind norm N_G^A were characterized in [21].

The purpose of this paper is to study finite 2-groups with the cyclic center and the metacyclic non-Dedekind norm N_G^A.

2 Preliminary results

The norm of Abelian non-cyclic subgroups of a group G (see [14]) is the intersection of the normalizers of all Abelian non-cyclic subgroups of a group G (assuming that the system of such subgroups is non-empty).

Clearly, all Abelian non-cyclic subgroups are normal in a group G which contains at least one Abelian non-cyclic subgroup and coincides with the norm N_G^A. Non-Abelian groups with this property were studied in [16] and were called $\overline{HA}$-groups ($\overline{HA}_2$-groups in the case of 2-groups). Therefore, if the norm N_G^A of a finite 2-group is non-Dedekind, then it is either $\overline{HA}_2$-group or non-Dedekind group without Abelian non-cyclic subgroups. In the latter case, by Theorem 1 of [15] the group also does not contain Abelian non-cyclic subgroups. Therefore, we assume that N_G^A contains an Abelian non-cyclic subgroup and is $\overline{HA}_2$-group.

Taking into account the description of finite $\overline{HA}_2$-groups (see [16]), we obtain the following characterization of metacyclic non-Dedekind norm N_G^A.

Lemma 2.1 *The norm* N_G^A *of Abelian non-cyclic subgroups of a finite 2-group* G *is metacyclic and non-Dedekind if and only if* N_G^A *is a group of one of the following types:*

1) $N_G^A = \langle a \rangle \leftthreetimes \langle b \rangle$, $|a| = 2^n$, $|b| = 2^m$, $n \geqslant 2$, $m \geqslant 1$, $[a, b] = a^{2^{n-1}}$;
2) $N_G^A = \langle a \rangle \langle b \rangle$, $|a| = 2^n, n > 2$, $|b| = 8$, $b^4 = a^{2^{n-1}}$, $b^{-1}ab = a^{-1}$.

Further we need the description of finite non-metacyclic 2-groups whose proper subgroups are metacyclic. Finite p-groups with such a property are studied in [4]. As a corollary of the main result in [4], we obtain the following statement.

Lemma 2.2 *Let* G *be a finite non-metacyclic 2-group. Each proper subgroup of* G *is metacyclic if and only if* G *is a group of one of the following types:*

1) $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, |a| = |b| = |c| = 2$;
2) $G = H \times \langle b \rangle, H = \langle h_1, h_2 \rangle, |h_1| = |h_2| = 4, |b| = 2, h_1^2 = h_2^2 = [h_1, h_2]$;
3) $G = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle, |a| = 4, |b| = |c| = 2, [a, c] = 1, [b, c] = a^2$;
4) $G = (\langle a \rangle \times \langle b \rangle)\langle c \rangle, |a| = |b| = |c| = 4, c^2 = a^2b^2, [c, a] = a^2, [c, b] = c^2$.

Let's consider some properties of the norm N_G^A of Abelian non-cyclic subgroups in some 2-groups.

Lemma 2.3 *If a locally finite 2-group* G *contains a normal cyclic subgroup* $\langle g\rangle$ *of order* 4 *and does not contain an elementary Abelian subgroup of order* 8*, then* $g \in N_G^A$.

Proof — Let A be an arbitrary Abelian non-cyclic subgroup of G. By the condition of lemma a group G doesn't contain an elementary Abelian subgroup of order 8 and $g^2 \in Z(G)$. Therefore $g^2 \in A$. Then

$$[g, x] \in \langle g^2\rangle \subset A$$

for an arbitrary element $x \in A$. Hence A is g-admissible subgroup and $g \in N_G^A$. The lemma is proved. □

Lemma 2.4 (see [15], Lemma 4) *If the norm* N_G^A *of Abelian non-cyclic subgroups of a finite 2-group* G *is non-Dedekind, then* G *does not contain any elementary Abelian subgroups of order* 8.

The next statement is the direct corollary of Lemmas 2.3 and 2.4.

Corollary 2.5 *Let* G *be a finite 2-group with the non-Dedekind norm* N_G^A *of Abelian non-cyclic subgroups. If* G *contains a normal generalized quaternion group*

$$H = \langle h_1, h_2\rangle, |h_1| = 2^n,\ n \geqslant 3, |h_2| = 4, h_1^{2^{n-1}} = h_2^2, h_2^{-1}h_1h_2 = h_1^{-1},$$

then $h_1^{2^{n-2}} \in N_G^A$.

Further we need the following result by M. Drushlyak [8].

Lemma 2.6 *Let* G *be a locally finite 2-group with non-Dedekind norm* N_G^A *of Abelian non-cyclic subgroups. If the center* $Z(N_G^A)$ *of the norm* N_G^A *is cyclic, then the central involution* a *belongs to every cyclic subgroup of composite order of a group* G.

Let's denote the lower layer of G (the subgroup generated by all elements of a prime order of a group G) by $\omega(G)$.

Lemma 2.7 *Let* G *be a finite 2-group with metacyclic and non-Dedekind norm* N_G^A *of Abelian non-cyclic subgroups. If the norm* N_G^A *is different from a group of the type 1) of Lemma 2.1 for* $m = 1$*, then* N_G^A *contains all involutions of* G *and* $\omega(N_G^A) = \omega(G)$.

Proof — Suppose, contrary to the condition of the lemma, that G contains the involution $x \notin \omega(N_G^A)$. If the center $Z(G)$ of a group G is non-cyclic, then $\langle x\rangle\, \omega(N_G^A)$ is an elementary Abelian group of order 8, which contradicts Lemma 2.4. Therefore the center of G is cyclic.

Let N_G^A be a group of one of the type 1) for $m > 1$ or 2) of Lemma 2.1. By the condition

$$\big[\langle x\rangle, \omega(N_G^A)\big] \subset \langle x, a^{2^{n-1}}\rangle \cap N_G^A = \langle a^{2^{n-1}}\rangle,$$

where $a^{2^{n-1}} \in N_G^A$ is the central involution of a group, we have $\big[\langle x\rangle, (N_G^A)^2\big] = E$. But then $\big[\langle x\rangle, \omega(N_G^A)\big] = E$ and G contains an elementary Abelian subgroup of order 8, which contradicts Lemma 2.4. The lemma is proved. □

Let us consider another property of finite 2-groups that will be used later.

Lemma 2.8 *Let G be a finite non-Abelian 2-group in which all elements of order not exceeding 4 are contained in the normal subgroup*

$$H = \langle a\rangle \times \langle b\rangle,$$

where $|a| = 4$, $|b| = 2$, $\langle a\rangle \lhd G$ and $b \notin Z(G)$. Then G is $\overline{HA}_2$-group of one of the following types:

1) $G = \langle g\rangle \rtimes \langle b\rangle$, $|g| = 2^n \geqslant 8, |b| = 2, [g, b] = g^{2^{n-1}} = a^2$;
2) $G = \langle g\rangle\langle h\rangle$, $|g| = 2^n \geqslant 8, |h| = 8, g^{2^{n-1}} = h^4 = a^2$, $h^{-1}gh = g^{-1}$.

Proof — Let G satisfies the conditions of the lemma. Then G has no any quaternion subgroup and every non-cyclic subgroup of G contains the subgroup $\omega(G) = \langle a^2\rangle \times \langle b\rangle$.

Since the factor group $G/\omega(G)$ has a unique involution, it is either a cyclic or a quaternion 2-group. If $G/\omega(G)$ is a cyclic group, then $G' \subseteq \omega(G)$ and all Abelian non-cyclic subgroups are normal in G. Therefore, G is a non-Hamiltonian $\overline{HA}_2$-group. By the description of such groups (see [16]) G is a group of the type 1) of this lemma.

Let $G/\omega(G)$ be the quaternion group of order 8. It is easy to prove that in this case all Abelian non-cyclic subgroups are normal in G and G is a non-Hamiltonian $\overline{HA}_2$-group of the type 2) of lemma for $n = 3$.

Now let $G/\omega(G)$ be a generalized quaternion group of order greater than 8. By Lemma 2.2 a group G does not contain non-metacyclic subgroups, in which all proper subgroups are metacyclic. Therefore, G is metacyclic and

$$G = \langle g \rangle \langle h \rangle, \langle g \rangle \lhd G.$$

Suppose $|g| < 8$ or $|h| < 8$. Then in a generalized quaternion group

$$G/\omega(G) = \overline{G} \simeq \langle \overline{g} \rangle \langle \overline{h} \rangle$$

at least one of the elements $\overline{g}$ or $\overline{h}$ is of order 2, which is impossible. Thus, $|g| \geqslant 8$ and $|h| \geqslant 8$.

If $|h| > 8$, then $\langle \overline{h} \rangle \lhd \overline{G}$ by the structure of the factor group $\overline{G}$. On the other hand, by the proved above $\langle \overline{g} \rangle \lhd \overline{G}$, which is impossible in a generalized quaternion group of order greater than 8. Hence, $|h| = 8$.

Let $|\overline{G}| = 2^n \geqslant 16$. Then $|G| = 2^{n+2}, |g| = 2^n$ and $h^4 = g^{2^{n-1}}$. In the factor group $\overline{G} = G/\omega(G)$ the following equality takes place

$$h^{-1} g h \, \omega(G) = g^{-1} \omega(G).$$

Considering that $\langle g \rangle \lhd G$, we have $h^{-1} g h = g^s$, where $g^{-1}\omega(G)$ is equal to $g^s \omega(G)$ and g^{s+1} belongs to $\omega(G)$.

If $s = -1$, then all Abelian subgroups are normal in a group G. Hence G is a $\overline{HA}_2$-group of the type 2) of this lemma for $n > 3$. Let $s \neq -1$. Then $s = -1 + 2^{n-1}$ and $[g, h^2] = 1$. Since

$$h^{-1}(gh^2)h = (gh^2)^{-1},$$

we have

$$G = \langle gh^2 \rangle \langle h \rangle,$$

where $h^4 = (gh^2)^{2^{n-1}}$. So, G is a group of the type 2) of this lemma. □

3 The main results

In this section the finite 2-groups with the cyclic center and the metacyclic non-Dedekind norm N_G^A of Abelian non-cyclic subgroups are studied. Their structure is described in the following theorem.

Theorem 3.1 *An arbitrary finite 2-group with the cyclic center and the non-Dedekind metacyclic norm* N_G^A *of Abelian non-cyclic subgroups is a group of one of the following types:*

1) $G = \langle a\rangle\langle b\rangle, |a| = 2^n, n > 2, |b| = 8, b^4 = a^{2^{n-1}}, b^{-1}ab = a^{-1}$, $N_G^A = G$;
2) $G = \langle a\rangle \leftthreetimes \langle b\rangle, |a| = 2^n, |b| = 2^m, n \geqslant 2, m \geqslant 1, [a, b] = a^{2^{n-1}}$, $N_G^A = G$;
3) $G = \langle y\rangle \leftthreetimes \langle b\rangle, |y| = 8, |b| = 2, [y, b] = y^2; N_G^A = \langle y^2\rangle \leftthreetimes \langle b\rangle$;
4) $G = (H \times \langle b\rangle)\langle a\rangle, H = \langle h_1, h_2\rangle, |h_1| = 2^k > 4, h_1^{2^{k-1}} = h_2^2$, $a^2 = h_1^{2^{k-2}}$, $h_2^{-1}h_1h_2 = h_1^{-1}, |b| = 2$, $[a, h_1] = a^4$, $[a, h_2] = b$, $[a, b] = a^4$; $N_G^A = \langle a\rangle \leftthreetimes \langle b\rangle$;
5) $G = \langle y\rangle\langle b\rangle$, $\langle y\rangle \cap \langle b\rangle = E$, $|y| = 2^k$, $k \geqslant 4$, $|b| = 2^m, m \geqslant 2$, $[y, b] = y^{2^{k-m}s}b^{2^{m-1}t}$, $(s, 2) = 1$, $t \in \{0, 1\}$; $N_G^A = \langle y^{2^{m-1}}\rangle \leftthreetimes \langle b\rangle$.

Proof — The sufficiency of the conditions of the theorem can be verified directly. Let us prove their necessity. Let G and its norm of Abelian non-cyclic subgroups satisfy the conditions of the theorem. Then N_G^A is a group of one of the types 1) or 2) of Lemma 2.1. If $G = N_G^A$, then G is a group of the type 1) or 2) of this theorem.

Let's continue the proof in Lemmas 3.2–3.7, depending on the structure of the norm N_G^A.

Lemma 3.2 *If a finite 2-group* G *has the norm* N_G^A *of Abelian non-cyclic subgroups of the type*

$$N_G^A = \langle a\rangle\langle b\rangle,$$

where $|a| = 2^n, n > 2, |b| = 8, a^{2^{n-1}} = b^4, b^{-1}ab = a^{-1}$, *then all Abelian non-cyclic subgroups are normal in* G *and* $G = N_G^A$.

Proof — Let G have the norm N_G^A of Abelian non-cyclic subgroups of the mentioned type. Then the lower layer of the norm

$$\omega(N_G^A) = \langle a^{2^{n-1}}\rangle \times \langle a^{2^{n-2}}b^2\rangle$$

is an elementary Abelian group of order 4 with $a^{2^{n-1}} = a_1 \in Z(N_G^A)$ and $a^{2^{n-2}}b^2 = a_2 \notin Z(N_G^A)$. The subgroup $\omega(N_G^A)$ contains all involutions of this group by Lemma 2.7. Let's prove that N_G^A also contains all the elements of order 4 of this group.

Put

$$H = \langle a^{2^{n-2}} \rangle\, \omega(N_G^A) = \langle a^{2^{n-2}} \rangle \times \langle a_2 \rangle.$$

Then $H \lhd G$ as the product of two characteristic subgroups of N_G^A.

Suppose that there exist an element $y \in G \backslash H$, $|y| = 4$. Since the center of the norm N_G^A is cyclic, the central involution a_1 belongs to each cyclic subgroup of composite order by Lemma 2.6. Thus $y^2 = a_1 \in H$. Therefore, the group

$$G_1 = \langle y \rangle H$$

has order 16 and the factor groups $G_1/\omega(N_G^A)$ and $G_1/\langle a^{2^{n-2}} \rangle$ are Abelian. Thus

$$G_1' \subseteq \omega(N_G^A) \cap \langle a^{2^{n-2}} \rangle = \langle a_1 \rangle.$$

If $[y, a^{2^{n-2}}] = 1$, then $|ya^{2^{n-2}}| = 2$, which contradicts Lemma 2.7. So,

$$[y, a^{2^{n-2}}] = y^2 = a_1.$$

Let $A = \langle y \rangle\, \omega(N_G^A)$. Then $|A| = 8$ and $\omega(N_G^A)$ contains all involutions of this group by Lemma 2.7. Hence, the subgroup A is Abelian. Since A is N_G^A-admissible subgroup,

$$[\langle y \rangle, N_G^A] \subseteq A \cap N_G^A = \omega(N_G^A).$$

So, $a^{-1}ya = ya_1^{\alpha}a_2^{\beta}$, $\alpha, \beta \in \{0, 1\}$. Then $a^{-2}ya^2 = ya_1^{2\alpha}a_2^{2\beta} = y$ and $[y, a^{2^{n-2}}] = 1$, which contradicts the proved above. Hence, $y \in H \subset N_G^A$. Then the subgroup H satisfies all the conditions of Lemma 2.8 and G is $\overline{HA}_2$-group of one of the types 1) or 2) of Theorem 3.1. In both cases we have $G = N_G^A$ and G is a group of type 1) of Theorem 3.1. The lemma is proved. □

Further we consider finite 2-groups with the norm of Abelian non-cyclic subgroups of the type 1) of Lemma 2.1:

$$N_G^A = \langle a \rangle \rtimes \langle b \rangle, |a| = 2^n, |b| = 2^m, n \geqslant 2, m \geqslant 1, [a, b] = a^{2^{n-1}},$$

where $\omega(N_G^A) \not\subset Z(G)$.

Clearly, that the center of a group G with such a norm N_G^A for $m > n \geqslant 2$ is non-cyclic. Therefore, we assume further that $n \geqslant m > 1$ or $n > m = 1$.

Lemma 3.3 *If the norm* N_G^A *of Abelian non-cyclic subgroups of a fi-*

nite 2-group G *is a group of the type*

$$N_G^A = \langle a \rangle \rtimes \langle b \rangle, |a| = 4, |b| = 2, [a, b] = a^2,$$

and $N_G^A \neq G$*, then* G *is a group of the type 3) of Theorem 3.1.*

Proof — Let the norm N_G^A of Abelian non-cyclic subgroups of G be a group of order 8. Denote by $C = C_G(N_G^A)$ the centralizer of N_G^A in G. It is known that the automorphism group of the dihedral group D_8 of order 8 is isomorphic to D_8, $\mathrm{Aut}(D_8) \simeq D_8$. So, $[G : C] \leqslant 8$.

Let's consider the subgroup $G_1 = C \cdot N_G^A$. Since $N_G^A \cap C = \langle a^2 \rangle$, we obtain $|G_1/C| = 4$. Hence, in the chain of subgroups

$$G \supseteq G_1 \supseteq C$$

we get $|G/C| \leqslant 8, |G_1/C| = 4, |G/G_1| \leqslant 2$. So,

$$G = G_1\langle y \rangle = \big(C \cdot N_G^A\big)\langle y \rangle,$$

where $y^2 \in G_1$. By Lemma 2.4 a group G does not contain any elementary Abelian subgroups of order 8. So, C has a unique involution a^2 and it is a cyclic or a generalized quaternion 2-group.

Let C be a generalized quaternion 2-group of order greater than 8,

$$C = \langle h_1, h_2 \rangle,$$

where $|h_1| = 2^n, n \geqslant 3, |h_2| = 4, h_1^{2^{n-1}} = h_2^2 = a^2, h_2^{-1}h_1h_2 = h_1^{-1}$. By Corollary 2.5 $h_1^{2^{n-2}} \in N_G^A$. So

$$h_1^{2^{n-2}} \in N_G^A \cap C = Z(N_G^A),$$

which is impossible.

Suppose that C is the quaternion group of order 8. Then

$$G_1 = C \cdot N_G^A$$

is the central gluing of the quaternion group and the dihedral group of order 8. Let's prove that C contains a cyclic subgroup of order 4 which is normal in G.

Let $h_1 \in C, |h_1| = 4$. Then it follows from the condition $[h_1, y] \in C$

that $C \lhd G$. If $[h_1, y] \in \langle h_1 \rangle$, then $\langle h_1 \rangle \lhd G$ and by Lemma 2.3

$$h_1 \in N_G^A \cap C \in Z(N_G^A),$$

which is impossible. So, $[h_1, y] \notin \langle h_1 \rangle$. Hence

$$|[h_1, y]| = 4 \quad \text{and} \quad [h_1, y] = h_2,$$

where $|h_2| = 4$, $h_2 \notin \langle h_1 \rangle$. Thus, $C = \langle h_1, h_2 \rangle$, where $h_1^2 = h_2^2$, $h_2^{-1} h_1 h_2 = h_1^{-1}$ and $y^{-1} h_1 y = h_1 h_2$.

Since $y^2 \in C \cdot N_G^A$, we have $y^{-2} h_1 y^2 = h_1^m$, where $(m, 2) = 1$. On the other hand,

$$y^{-2} h_1 y^2 = y^{-1} h_1 h_2 y = y^{-1} h_1 y y^{-1} h_2 y = h_1 h_2 y^{-1} h_2 y.$$

So, $h_1^m = h_1 h_2 y^{-1} h_2 y$ and $y^{-1} h_2 y = h_2^{-1} h_1^{m-1}$. Thus,

$$y^{-1} h_2 y = h_2^{-1} \quad \text{or} \quad y^{-1} h_2 y = h_2.$$

In both cases $\langle h_2 \rangle \lhd G$ and therefore $h_2 \in Z(N_G^A)$, which contradicts the condition. This means, that C cannot be the quaternion group of order 8.

It remains to consider the case when $C = \langle c \rangle$ is a cyclic subgroup. Suppose that $|c| = 2^n \geqslant 4$. By Lemmas 2.3 and 2.4 $c^{2^{n-2}} \in N_G^A$. But in this case the element $c^{2^{n-2}}$ is contained in $Z(N_G^A)$, which is impossible. So, $|C| = 2$ and $C = \langle a^2 \rangle$. Further, by the conditions $C \subseteq N_G^A$, $G = (C \cdot N_G^A)\langle y \rangle$ and $y^2 \in (C \cdot N_G^A)$, we have $G = N_G^A \cdot \langle y \rangle$. So,

$$G/C \simeq N_G^A \langle y \rangle / \langle a^2 \rangle \simeq \big(\langle \overline{a} \rangle \times \langle \overline{b} \rangle\big)\langle \overline{y} \rangle,$$

where $2 \leqslant |\overline{y}| \leqslant 4$. Taking into account that G/C is isomorphic to some subgroup of the dihedral group of order 8, we conclude that $|\overline{y}| = 2$. Then from the condition $|G/\langle a \rangle| = 4$ it follows that the group $G/\langle a \rangle$ is Abelian and $G' \subseteq \langle a \rangle$.

Let $C_1 = C_G(a)$ be the centralizer of element a in G. Since $b \notin C_1$,

$$G = C_1 \leftthreetimes \langle b \rangle.$$

Then by the conditions $|C_1| = 8$ and $Z(C_1) \supseteq \langle a \rangle$ we conclude, that C_1 is an Abelian group.

Suppose, that C_1 is a non-cyclic subgroup. Then we can assume that

$$C_1 = \langle a\rangle \times \langle x\rangle, |x| = 2.$$

Taking into account that the subgroup $\langle a^2\rangle \times \langle x\rangle$ is $\langle b\rangle$-admissible, we have

$$G' \subseteq \big(\langle a^2\rangle \times \langle x\rangle\big) \cap N_G^A = \langle a^2\rangle.$$

Since every Abelian non-cyclic subgroup of G contains $\langle a^2\rangle$, it is normal in G and therefore $G = N_G^A$, which contradicts the condition. So, $C_1 = \langle y\rangle$ is a cyclic subgroup of order 8,

$$G = \langle y\rangle \leftthreetimes \langle b\rangle, y^2 = a.$$

As G contains a cyclic subgroup of index 2, Theorem 12.5.1 [9] yields that $b^{-1}yb = y^3$. Finally, we have

$$G = \langle y\rangle \leftthreetimes \langle b\rangle,$$

where $|y| = 8, |b| = 2, b^{-1}yb = y^3$. Thus, G is a group of the type 3) of Theorem 3.1. □

Lemma 3.4 *If the norm N_G^A of Abelian non-cyclic subgroups of a finite 2-group G is a group of the type*

$$N_G^A = \langle a\rangle \leftthreetimes \langle b\rangle, |a| = 2^n, n > 2, |b| = 2, b^{-1}ab = a^{-1}$$

and $G \neq N_G^A$, then G is a group of the type 4) of Theorem 3.1.

Proof — Let G have the norm of the Abelian non-cyclic subgroups of the given in the condition of the lemma type. Since $\omega(N_G^A) = \langle a^{2^{n-1}}\rangle \times \langle b\rangle$ is a characteristic subgroup of N_G^A, we have $\omega(N_G^A) \triangleleft G$.

Denote by $C = C_G\big(\omega(N_G^A)\big)$ the centraliser of $\omega(N_G^A)$ in G. Then $C \triangleleft G$ and $[G : C] = 2$. Since $a \notin C$, we can assume that

$$G = C \cdot \langle a\rangle, a^2 \in C.$$

Consider the factor group

$$\overline{C} = C/\langle b\rangle.$$

By Lemma 2.7 the subgroup $\omega(N_G^A)$ contains all involutions of its centralizer. Since the element $a_1 = a^{2^{n-1}}$ belongs to every cyclic subgroup of composite order of a group G by Lemma 2.6, $\overline{C}$ contains a unique involution. Therefore, the factor group $\overline{C}$ is a cyclic or a generalized quaternion group.

1) Let $\overline{C}$ be a cyclic 2-group. Then C is an Abelian group with a complementary subgroup $\langle b\rangle$ and $C = \langle x\rangle \times \langle b\rangle$. Since $a^2 \in C$, it follows that $a^2 \in Z(G)$ and

$$G = C \cdot \langle a\rangle = \big(\langle x\rangle \times \langle b\rangle\big)\langle a\rangle.$$

Consider the factor group

$$\widetilde{G} = G/\omega(N_G^A) \simeq \langle \widetilde{x}\rangle\langle \widetilde{a}\rangle,$$

where $\langle \widetilde{a}^2\rangle \subseteq \langle \widetilde{x}\rangle$. Since $\widetilde{G}$ has a central cyclic subgroup of index 2, $\widetilde{G}' \subseteq \langle \widetilde{a}^{2^{n-2}}\rangle$ by Theorem 12.5.1 of [9].

If $\widetilde{G}' = E$, then $G' \subseteq \omega(N_G^A)$ and $[a, x] = a_1^{\alpha} b^{\beta}$, where $\alpha, \beta \in \{0, 1\}$. So, $[a^2, x] = a_1^{\beta} = 1$, $\beta = 0$ and therefore $[a, x] \in \langle a_1\rangle$. Thus, in this case $G' \subseteq \langle a_1\rangle$ and $G = N_G^A$, which is impossible.

Now let $\widetilde{G}' = \langle \widetilde{a}^{2^{n-2}}\rangle$. Put $[a, x] = a^{2^{n-2}\alpha} b^{\beta}$, where $(\alpha, 2) = 1$ and $\beta \in \{0, 1\}$. Since $a^2 \in Z(G)$ and $G' \not\subseteq \omega(N_G^A)$, we have

$$(\alpha, 2) = (\beta, 2) = 1 \quad \text{and} \quad [a, x] = a^{\pm 2^{n-2}} b.$$

On the other hand, $x^2 b \in Z(G)$, so $x^2 b = a^2$.

If $|a| = 8$, then $x^2 = a^2 b$, $|x| = 8$ and $G = \langle a, x\rangle$ is a $\overline{HA}_2$-group, which is impossible, because $G \neq N_G^A$. Therefore, $|a| > 8$. Then $|x| > 8$, $|xa^{-1\mp 2^{n-3}}| = 2$ and $\langle a_1, xa^{-1\mp 2^{n-3}}\rangle$ is an Abelian non-cyclic subgroup. But,

$$a^{-1} x a^{-1\mp 2^{n-3}} a = x a^{\pm 2^{n-2}} b a^{-1\mp 2^{n-3}} \notin \langle a_1, xa^{-1\mp 2^{n-3}}\rangle$$

and $a \notin N_G\big(\langle a_1,\ xa^{-1\mp 2^{n-3}}\rangle\big)$, which contradicts the definition of the norm N_G^A. Therefore, this case is impossible.

2) Let $\overline{C}$ be a generalized quaternion group

$$\overline{C} = \langle \overline{h}_1, \overline{h}_2\rangle,$$

where $|\overline{h}_1| = 2^k \geqslant 4$, $|\overline{h}_2| = 4$, $\overline{h}_1^{2^{k-1}} = \overline{h}_2^2$, $\overline{h}_2^{-1}\overline{h}_1\overline{h}_2 = \overline{h}_1^{-1}$.

Denote the preimages of the elements $\overline{h}_1$ and $\overline{h}_2$ by h_1 and h_2.

Since h_1 and h_2 are of composite order,

$$\langle h_1\rangle \cap \omega(N_G^A) = \langle h_2\rangle \cap \omega(N_G^A) = \langle a_1\rangle$$

by Lemma 2.6.

By the defining relations for $\overline{C}$, we obtain $h_2^{-1}h_1h_2 = h_1^{-1}b^m$. If $m \neq 0$, then the subgroup $\langle h_1h_2\rangle$ is of order 4 and does not contain a_1, which contradicts Lemma 2.6. So, $m = 0$, $h_2^{-1}h_1h_2 = h_1^{-1}$ and

$$C = H \times \langle b\rangle,$$

where $H = \langle h_1, h_2\rangle$ is a quaternion 2-group, $|h_1| = 2^k \geqslant 4$. It follows from the conditions $a^2 \in C$ and $\langle a^2\rangle = Z(N_G^A) \lhd G$, that $a^2 = h_1^s b^t$.

Since

$$[a, h] \in \left(N_G^A \cap \langle b, h\rangle\right) = \omega(N_G^A)$$

for an arbitrary element $h \in H\backslash\langle h_1\rangle$ and $|h| = 4$, $[a, h_2] \in \omega(N_G^A)$, $[a, h_1h_2] \in \omega(N_G^A)$. Therefore, $[\langle a\rangle, H] \subset \omega(N_G^A)$.

If H is the quaternion group of the order 8, then $|a| = 8$ and $G' \subset \omega(N_G^A)$. Since in this case every Abelian non-cyclic subgroup of G contains $\omega(N_G^A)$, it is normal in G and $G = N_G^A$, which contradicts the condition.

So, we assume that $|H| > 8$, where $|h_1| = 2^k > 4$. Since $[a, h_2] \in \omega(N_G^A)$, we have $[a^2, h_2] \in \langle a_1\rangle$. On the other hand,

$$[a^2, h_2] = [h_1^s b^t, h_2] = [h_1^s, h_2] = h_1^{-2s}.$$

Thus, $s \equiv 0 \pmod{2^{n-2}}$, $a^2 = h_1^{2^{k-2}s_1} b^t$ and

$$a^4 = (h_1^{2^{k-2}s_1} b^t)^2 = h_1^{2^{k-1}s_1} = a_1^{s_1}.$$

Finally, we have $(s_1, 2) = 1$ and $|a| = 8$ again.

Let us consider the action of the element h_1 on the element a. Since $[a, h_1] \in \omega(N_G^A)$,

$$h_1^{-1} a h_1 = a a_1^{\alpha} b^{\beta}.$$

Then $h_1^{-1}a^2h_1 = a^2a_1^{\beta}$. On the other hand, we have $h_1^{-1}a^2h_1 = a^2$, so $\beta = 0$ and $[a, h_1] = a_1^{\alpha}$, $\alpha \in \{0, 1\}$. Since $(a^{-1}h_1^{2^{k-3}s_1})^2 = a_1^l b^t$, where $l \in \{0, 1\}$, the subgroup $\langle a^{-1}h_1^{2^{k-3}s_1}\rangle$ has a composite order and does not contain the involution a_1 for $t \neq 0$, which contradicts Lemma 2.6. Therefore, $a^2 = h_1^{2^{k-2}s_1}$, where $(s_1, 2) = 1$.

Considering that $[a, h_2] \in \omega(N_G^A)$, let $[a, h_2] = a_1^m b^r$. Then by the condition

$$[a^2, h_2] = [h_1^{2^{k-2}s_1}, h_2] \neq 1,$$

we have $r \neq 0$ and $[a, h_2] = b$. Thus, in this case G is a group of the type 4) of Theorem 3.1. □

From the proof of Lemma 3.4 we obtain the following statement.

Corollary 3.5 *If the norm of Abelian non-cyclic subgroups* N_G^A *of a finite 2-group* G *is a group of the type*

$$N_G^A = \langle a \rangle \rtimes \langle b \rangle, |a| = 2^n > 8, |b| = 2, [a, b] = a^{2^{n-1}},$$

then $N_G^A = G$.

Lemma 3.6 *If the norm of Abelian non-cyclic subgroups* N_G^A *of a finite 2-group* G *is a group of the type*

$$N_G^A = \langle a \rangle \rtimes \langle b \rangle, |a| = 4, |b| = 4, [a, b] = a^2,$$

then $\omega(N_G^A) \subseteq Z(G)$.

Proof — Let the norm N_G^A satisfy the conditions of the lemma. If $N_G^A = G$, then the statement of the lemma is obvious. Therefore, further assume that $N_G^A \neq G$.

By Lemma 2.7

$$\omega(N_G^A) = \langle a^{2^{n-1}} \rangle \times \langle b^{2^{m-1}} \rangle = \omega(G)$$

and therefore $\omega(N_G^A) \lhd G$. Assume that $\omega(N_G^A) \nsubseteq Z(G)$ and denote the centralizer of the lower layer $\omega(N_G^A)$ in a group G by $C = C_G(\omega(N_G^A))$. Then by Lemma 3 of [19]

$$G = C \cdot \langle y \rangle,$$

where $y^2 \in C$, $|y| > 4$. Since $N_G^A \subseteq C$ and $\omega(N_G^A) \subseteq Z(C)$, by Lemma 20 of [21] $C = N_C^A = N_G^A$ and

$$G = C \cdot \langle y \rangle = N_G^A \cdot \langle y \rangle = (\langle a \rangle \rtimes \langle b \rangle)\langle y \rangle,$$

where $|y| = 8$. By the condition $[y, b^2] \neq 1$ we have $\langle y\rangle \bigcap \omega(G) = \langle a^2\rangle$.
Let's consider the factor group

$$\overline{G} = G/\omega(G) = (\langle\overline{a}\rangle \times \langle\overline{b}\rangle)\langle\overline{y}\rangle.$$

Since $\overline{y}^2 \in \overline{C}$ and $\overline{G}$ has three involutions, $\overline{G}$ is a metacyclic group,

$$\overline{G} = \langle\overline{y}_1\rangle\langle\overline{b}_1\rangle,$$

where $|\overline{y}_1| = 4$ and $\langle\overline{y}_1\rangle \lhd \overline{G}$. Thus, by Theorem 12.5.1 of [9] we have

$$\overline{b}_1{}^{-1}\overline{y}_1\overline{b}_1 = \overline{y}_1 \quad \text{or} \quad \overline{b}_1{}^{-1}\overline{y}_1\overline{b}_1 = \overline{y}_1{}^{-1}.$$

In the first case $G' \subseteq \omega(G)$. Since all Abelian non-cyclic subgroups contain $\omega(G)$, $N_G^A = G$, which is impossible. Therefore,

$$\overline{b}_1{}^{-1}\overline{y}_1\overline{b}_1 = \overline{y}_1{}^{-1}$$

and $\overline{G}$ is the dihedral group. But then $|\overline{y}_1\overline{b}_1| = 2$ and the preimage of the element $\overline{y}_1\overline{b}_1$ is of order 4. By Lemma 3 of [19] all elements of order 4 are contained in C, which is impossible. Therefore, the assumption is false and $\omega(N_G^A) \subseteq Z(G)$. □

Lemma 3.7 *If the norm* N_G^A *of Abelian non-cyclic subgroups of a finite 2-group* G *is a group of the type*

$$N_G^A = \langle a\rangle \leftthreetimes \langle b\rangle, |a| = 2^n, |b| = 2^m, n > 2, n \geqslant m \geqslant 2, [a, b] = a^{2^{n-1}}$$

and $\omega(N_G^A) \not\subset Z(G)$, *then* G *is a group of the type 5) of Theorem 3.1.*

Proof — Let a group G and its norm N_G^A of Abelian non-cyclic subgroups satisfy the condition, $\omega(N_G^A) \not\subset Z(G)$ and $N_G^A \neq G$. By Lemma 2.7

$$\omega(N_G^A) = \langle a^{2^{n-1}}\rangle \times \langle b^{2^{m-1}}\rangle = \omega(G) \lhd G.$$

Since $\omega(G) \not\subset Z(G)$ and $\langle a^{2^{n-1}}\rangle$ is the characteristic subgroup in N_G^A, we have $a_1 = a^{2^{n-1}} \in Z(G)$ and $b_1 = b^{2^{m-1}} \notin Z(G)$.
Denote $C = C_G(\omega(G))$. By Lemma 3 of [19]

$$G = C \cdot \langle y\rangle,$$

where $y^2 \in C$, $|y| > 4$ and $N_C^A = N_G^A$. Since the subgroup C has the non-cyclic center and the non-Dedekind norm of Abelian non-cyclic subgroups, by Theorem 16 of [21] either C is a $\overline{HA_2}$-group and $C = N_C^A = N_G^A$, or C is a group of the type 2) of this theorem.

Let us prove that a group G is metacyclic. In fact, otherwise it contains a non-metacyclic subgroup H, in which all subgroups are metacyclic. Then $\exp H \leqslant 4$ by Lemma 2.2. Since the centralizer C contains all elements of order 4 of a group G, $H \subseteq C$. This contradiction proves, that

$$G = \langle g\rangle\langle h\rangle$$

is a metacyclic group with three involutions and the non-central lower layer. By the description of such groups (see Theorem 1.1.4 of [23]), we conclude that G is a group of one of the types:

1) $G = \langle g\rangle\langle h\rangle$, $|g| = 2^{\alpha}$, $|h| = 2^{\beta}$, $\langle g\rangle \cap \langle h\rangle = E$, $h^{-1}gh = g^{-1-2^{\alpha-\beta}}$, $\alpha > 2, \beta > 2, \alpha - \beta > 1$;

2) $G = \langle g\rangle\langle h\rangle$, $|g| = 2^{\alpha}$, $|h| = 2^{\beta}$, $\langle g\rangle \cap \langle h\rangle = E$, $h^{-1}gh = g^{1+2^{\alpha-\beta}}$, $\alpha > 2, \beta > 2, \alpha - \beta > 1$;

3) $G = \langle g\rangle\langle h\rangle$, $|g| = 2^{\alpha}, |h| = 2^{\beta}, h^{-1}gh = g^{1+2^{r}}$, $\langle g\rangle \cap \langle h\rangle = \langle g^{2^{l}}\rangle$, $1 < r < l < \alpha < \beta$, $Z(G) = \langle g^{2^{\alpha-r}}\rangle\langle h^{2^{\alpha-r}}\rangle$.

1) Let G be a group of the first type. Then $h^{2^{\beta-1}} \notin Z(G), h \in C$ and $g \notin C$. Taking into account the structure of the subgroup C, we conclude that $h \in N_G^A$. If $C = N_G^A$, then $C' = \langle g^{2^{\alpha-1}}\rangle$. On the other hand, in a group of the type 1) we have $G' = \langle g^2\rangle$, $C' = \langle g^4\rangle$ and therefore $\alpha = 3$, which is impossible, because otherwise $\beta = 1$.

So, C is a group of the type 2) of Theorem 16 of [21]. In this case, $|C'| = 2^{r+1}$, where r is the smallest integer such that $(g^2)^{2^r} \in N_G^A$. Then by the equality $|C'| = |\langle g^4\rangle| = 2^{\alpha-2}$, we have $r + 1 = \alpha - 2$ and the smallest power of the element g contained in N_G^A is the element $g^{2^{r+1}} = g^{2^{\alpha-2}}$ of order 4, which is impossible.

2) Let G be a group of the type 2). As in the previous case we conclude, that $h^{2^{\beta-1}} \notin Z(G)$, $g \notin C$ and $h \in N_G^A$. Let's denote the smallest power of the element g contained in N_G^A by g^{2^r}. If $C = N_G^A$, then $g^2 \in C$. Then by the condition $C' = \langle g^{2^{\alpha-1}}\rangle = \langle a_1\rangle$ we have that

$$h^{-1}g^2h = g^2g^{2^{\alpha-\beta+1}}$$

and $2^{\alpha-\beta+1} \equiv 0 \pmod{2^{\alpha-1}}$. So

$$\alpha-\beta+1 \geqslant \alpha-1, 2-\beta \geqslant 0 \quad \text{and} \quad \beta \leqslant 2.$$

By the condition of lemma, it follows that $\beta = 2$ and

$$h^{-1}gh = g^{1+2^{\alpha-2}}.$$

Therefore, G is a group of the type 5) of Theorem 3.1 for $y = g, b = h$, $k = \alpha, m = \beta = 2$ and $t = 0$.

Let $C \neq N_G^A$. Then C is a group of the type 2) of Theorem 16 of [21]. Since g^{2^r} is the smallest power of the element g contained in N_G^A and $(N_G^A)' = \langle g^{2^{\alpha-1}} \rangle$, $[g^{2^r}, h] \in \langle g^{2^{\alpha-1}} \rangle$. On the other hand, by the defining relations of a group of the type 2) we obtain

$$h^{-1}g^{2^r}h = g^{2^r}g^{1+2^{r+\alpha-\beta}}.$$

So, $2^{r+\alpha-\beta} \equiv 0 \pmod{2^{\alpha-1}}$ and $\beta \leqslant r+1$. Since by Theorem 16 of [21] $r_1 \leqslant \beta-2$ and $r_1 = r-1$ for the exponent of the smallest power $(g^2)^{2^{r_1}}$ of the element g^2 contained in N_G^A, $r-1 \leqslant \beta-2$ and $\beta \geqslant r+1$. Therefore, $\beta = r+1$ and G is a group of the type 5) of Theorem 3.1 for $y = g, b = h$, $k = \alpha, m = \beta = r+1$ and $t = 0$.

3) Let G be a group of the type 3). In this case

$$\omega(G) = \langle g^{2^{\alpha-1}} \rangle \times \langle g^{2^{l-1}} h^{-2^{\beta-\alpha+l-1}} \rangle.$$

By the cyclicity of the center $Z(G) = \langle g^{2^{\alpha-r}} \rangle \langle h^{2^{\alpha-r}} \rangle$ and the condition $\beta > \alpha$ we conclude, that $\langle g^{2^{\alpha-r}} \rangle \subseteq \langle h^{2^{\alpha-r}} \rangle$. Then

$$g^{2^{\alpha-r}} \in \langle g \rangle \cap \langle h \rangle = \langle g^{2^l} \rangle,$$

$\alpha - r \geqslant l \geqslant r+1$ and $\alpha \geqslant 2r+1$. On the other hand for $\alpha < 2r+1$ we have $[g^{2^r}, h] = 1$ and $g^{2^r} \in Z(G)$, which is impossible because $r < l$. Therefore, $\alpha = 2r+1$ and $l = r+1$.

It is clear, that the group G can be represented as

$$G = \langle h \rangle \langle gh^{-2^{\beta-\alpha}} \rangle,$$

where $\langle h \rangle \cap \langle gh^{-2^{\beta-\alpha}} \rangle = E$, $|h| = 2^\beta, |gh^{-2^{\beta-\alpha}}| = 2^{r+1}$. Let's denote $y = h$ and $b = gh^{-2^{\beta-\alpha}}$. Then

$$b^{-1}yb = y(g^{2^r}h^{-2^{\beta-\alpha+r}})h^{2^{\beta-\alpha+r}} = yb^{2^r t}y^{2^{\beta-r-1}}.$$

A norm N_G^A of this group is the subgroup $N_G^A = \langle y^{2^r}\rangle \leftthreetimes \langle b\rangle$ and G is a group of the type 5) of Theorem 3.1 for $k = \beta$ and $m = r+1$.The lemma is proved. □

The proof of the theorem is complete.

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Tetiana D. Lukashova
Department of Algebra and Mathematical Logic
Taras Shevchenko National University of Kyiv
Volodymyrska 60, Kyiv (Ukraine), 01033
e-mail: tanya.lukashova2015@gmail.com

Fedir M. Lyman
Department of Mathematics
Makarenko Sumy State Pedagogical University
Romenska str., 87, Sumy (Ukraine), 40002
e-mail: mathematicsspu@gmail.com