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Some Remarks about Groups of Finite Special Rank [1]

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Dedicated to Prof. Hermann Heineken on his eightieth birthday

Abstract

The paper presents some results about groups of finite special and section ranks. For instance, among others, it was proved that if every locally (soluble minimax) subgroup of a generalized radical group G has finite special rank, then G has finite special rank.

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1 Introduction

We say that a group G has *finite special rank* $r(G) = r$ if every finitely generated subgroup of G can be generated by at most r elements and there exists a finitely generated subgroup generated exactly by r elements. The concept of special rank has been introduced by A.I. Maltsev in the paper [12]. As it can be seen from the definition, the concept of special rank seems to be a natural analogue of the vector

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space dimension concept. Therefore, it is not surprising that this topic is very popular and useful. There is a huge array of articles examining the properties of groups of finite special rank, their relationship, and their influence on the structure of the group. The most general result on the structure of the groups of finite special rank looks as follows. But before its formulation, we recall some definitions.

A group G is called *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite. It easily follows from its definition that a generalized radical group G either has an ascendant locally nilpotent subgroup or an ascendant locally finite subgroup. In the first case, the locally nilpotent radical of G is non-identity. In the second case, G contains a non-identity normal locally finite subgroup, so the maximal normal locally finite subgroup of G (*the locally finite radical of* G) is non-identity. Thus every generalized radical group has an ascending series of normal subgroups with locally nilpotent or locally finite factors. Therefore *every generalized radical group is hyper (locally nilpotent or locally finite)*. We also recall that *a periodic locally generalized radical group is locally finite*.

A group G is said to have 0*–rank* $r_0(G) = r$ if G has an ascending series whose factors are either infinite cyclic or periodic and if the number of infinite cyclic factors is exactly r. If G has an ascending series with periodic and infinite cyclic factors and the set of infinite cyclic factors are infinite, then we will say that the group G has *infinite* 0*–rank*. Otherwise we will say that G *has no* 0*–rank*. It is possible to prove that *a group* G *has finite* 0*–rank if and only if* G *has a finite subnormal series whose factors are either infinite cyclic or periodic and the set of infinite cyclic factors is finite*. In some papers, the 0–rank of a group G is also called *the torsion-free rank* of G.

The structure of locally generalized radical groups of finite special rank (SR) is given by the following result.

Theorem SR *Let* G *be a locally generalized radical group of finite special rank* r. *Then* G *has normal subgroups* V *and* D *such that* $V \leqslant D$, V *is hypercentral,* D/V *is abelian and* G/D *is finite. In particular,* G *is generalized radical, and furthermore, it is almost hyperabelian. Moreover,* $\mathrm{Tor}(V)$ *is a direct product of its Chernikov Sylow* p*–subgroups,* $V/\mathrm{Tor}(V)$ *is nilpotent,* $D/\mathrm{Tor}(V)$ *has finite* 0*–rank at most* r. *In particular,* G *has finite* 0*–rank* $r_0(G) \leqslant r$.

Here and elsewhere $\mathrm{Tor}(G)$ denotes the largest periodic normal subgroup of G (*the periodic part of* G). We note that if G is locally nilpotent, then $\mathrm{Tor}(G)$ contains all elements of finite order.

Corollary *A locally generalized radical group of finite special rank is almost hyperabelian.*

The fact that a locally soluble group of finite special rank is hyperabelian has been proved by V.S. Charin [3] (more precisely, Charin proved that a locally soluble group of finite special rank is (locally nilpotent)–by–soluble, and this implies that it is hyperabelian). The fact, that a locally (soluble–by–finite) group of finite special rank is almost hyperabelian has been proved by N.S. Chernikov [4].

M.R. Dixon, M.J. Evans and H. Smith in their paper [7] have studied the influence of locally soluble subgroups on the structure of locally (soluble–by–finite) groups. They have proved that *if every locally soluble subgroup of a locally (soluble–by–finite) group* G *has finite special rank then* G *has finite special rank*. The first main result of the current paper is the following generalization of the mentioned result. We recall that a group G is called *locally (soluble minimax)* if every finitely generated subgroup of G is soluble minimax.

Theorem A *Let* G *be a locally generalized radical group. If every locally (soluble minimax) subgroup of* G *has finite special rank then* G *has finite special rank.*

Corollary A1 *Let* G *be a locally generalized radical group. If every locally soluble subgroup of* G *has finite special rank, then* G *has finite special rank.*

Another major result of the current paper is associated with the family of abelian subgroups. The first important result in this area is concerned with locally nilpotent groups. From Theorem 5 of the paper of S.N. Chernikov [5] and Theorem 5 of the paper of A.I. Maltsev [14] it follows that *a locally nilpotent group whose abelian subgroups have finite special rank, has finite special rank itself*. M.I. Kargapolov [10] has proved that *a soluble group, whose abelian subgroups have finite special rank, has finite special rank*. R. Baer and H. Heineken [1] extended this result to radical groups. At the same time, Yu.I. Merzlyakov [16] has constructed a locally polycyclic group of infinite special rank whose abelian subgroups have finite special rank. Thus, there is a border of qualitative transition between radical groups and locally soluble groups. In the current paper, we tried to delineate this border more precisely. We rely ourselves on the observation that in the group constructed by Merzlyakov the ranks of chief factors are unbounded.

Let k be a positive integer. A group G is called k*–generalized radical* if G has an ascending series of normal subgroups whose factors are locally nilpotent or locally finite groups of special rank k. We have

Theorem B *Let* k *be a positive integer and* G *be a locally* k*–generalized radical group. If every abelian subgroup of* G *has finite special rank then* G *has finite special rank.*

Corollary B1 *Let* k *be a positive integer and* G *be a group. Suppose that* G *has an ascending series whose factors are locally* k*–generalized radical group. If every abelian subgroup of* G *has finite special rank then* G *has finite special rank.*

Let us now consider analogues of the above results for the other ranks. Let p be a prime. We say that a group G has *finite section* p*–rank* $sr_p(G) = r$ if every elementary abelian p–section of G is finite of order at most p^r, and there is an elementary abelian p–section A/B of G such that $|A/B| = p^r$. We say that a group G has *finite section rank* if $sr_p(G)$ is finite for each prime number p. We can slightly concretize this definition. Let σ be a function from the set $\mathbb{P}$ of all primes in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We say that a group G has *finite section rank* σ if $sr_p(G) = \sigma(p)$ for every prime p.

A group G is said to be *a group of finite abelian subgroup rank* if every elementary abelian section of G is finite [1]. We note that *if the elementary abelian sections of an abelian group* A *are finite then* A *has finite section rank*. However, in general, this can fail. For example, if G is the group constructed by Merzlyakov [16], then every elementary abelian section of G is finite, but, for every prime p, the orders of the elementary abelian p–sections of G are not bounded.

A group G is called *nearly radical* if G has an ascending series whose factors are locally nilpotent or finite. In the paper [1], the term generalized radical groups was used for such groups, but we have used it for a much wider class of groups.

Theorem C *Let* G *be a locally nearly radical group. If every locally (soluble minimax) subgroup of* G *has finite section rank then* G *is almost radical and has finite section rank.*

Corollary C1 *Let* G *be a locally (soluble–by–finite) group. If every locally (soluble minimax) subgroup of* G *has finite section rank then* G *is almost radical and has finite section rank.*

Corollary C2 *Let* G *be a locally (soluble–by–finite) group. If every locally soluble subgroup of* G *has finite section rank then* G *is almost radical and has finite section rank.*

In the paper [15], Yu.I. Merzlyakov has proved that *if every abelian subgroup of a locally soluble group* G *has special rank at most* k*, where* k *is a fixed positive integer, then* G *has finite special rank*. On the other hand, Yu.M. Gorchakov [9] has proved that *if every abelian subgroup of a periodic locally soluble group* G *has finite special rank then* G *has finite special rank*. In this connection, we note that the group constructed in [16] is torsion-free. All this suggests that the reason for an unlimited increase of the special rank lies precisely in the 0–rank. It was justified in the paper of M.R. Dixon, M.J. Evans and H. Smith [7], where it has been proved that *if the* 0*–ranks of abelian subgroups of a locally (soluble–by–finite) group* G *are bounded and the special ranks of abelian subgroups are finite, then* G *has finite special rank*. We have extended this result to a larger class and got its counterpart for section rank.

Theorem D *Let* G *be a locally nearly radical group. Suppose that there exists a positive integer* k *such that* $r_0(A) \leqslant k$ *for every abelian subgroup* A *of* G*. If every abelian subgroup of* G *has finite section rank, then* G *has finite section rank and* $r_0(G) \leqslant k(k+1)$.

Corollary D1 *Let* G *be a locally (soluble–by–finite) group. Suppose that there exists a positive integer* k *such that* $r_0(A) \leqslant k$ *for every abelian subgroup* A *of* G*. If every abelian subgroup of* G *has finite section rank, then* G *has finite section rank and moreover* $r_0(G) \leqslant k(k+1)$.

Corollary D2 *Let* G *be a locally generalized radical group. Suppose that there exists a positive integer* k *such that* $r_0(A) \leqslant k$ *for every abelian subgroup* A *of* G*. If every abelian subgroup of* G *has finite special rank, then* G *has finite special rank and moreover* $r_0(G) \leqslant k(k+1)$.

2 Proof of Theorems SR and A

We will first prove Theorem SR. We will use [8, Theorem E], a result which concerns with groups of finite section p–rank. We will also need some information about the structure of soluble irreducible groups. Recall that A.I. Maltsev [14, Theorem 1] proved that *a soluble irreducible subgroup* G *of* $GL_n(F)$*, where* F *is a field, includes an abelian normal subgroup* A *of finite index dividing* $\mu(n)$*, where* $\mu(n) = n!(n^2(n^2)!)^n$. On the other hand, let A be an abelian torsion-free group and G be an automorphisms group of A. We say that A *is rationally irreducible with respect to* G or A *is* G*-rationally irreducible* if for every non-identity G–invariant subgroup B of A the factor-group

A/B is periodic. We remark that G *acts rationally irreducible on* A *if and only if* G *is irreducible as a group of linear transformations of the vector space* $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof of Theorem SR — We have $sr_p(G) \leqslant r$ for all prime p. By [8, Theorem E], G has a finite series of normal subgroups

$$T := \mathrm{Tor}(G) \leqslant L \leqslant K$$

such that L/T is torsion-free nilpotent, K/L is a finitely generated abelian group and G/K is finite. Moreover, for every prime p, every Sylow p–subgroup of G is Chernikov. By the main result of the paper of V.V. Belyaev [2], T includes a locally soluble normal subgroup S having finite index in T. Without loss of generality we may assume that S is a maximal locally soluble normal subgroup of T. In particular, S is G–invariant. Put $Z = C_G(T/S)$. Since T/S is finite, G/Z is finite and hence $G/(K \cap Z)$ is finite. By the choice of S, the factor-group T/S is semisimple, so that

$$(T \cap Z)/S = T/S \cap Z/S = \langle 1 \rangle.$$

It follows that

$$(L \cap Z)/S = (L \cap Z)/(T \cap Z) = (L \cap Z)/(L \cap Z \cap T) \simeq (L \cap Z)T/T$$

is a torsion-free nilpotent group. Similarly, $(K \cap Z)/(L \cap Z)$ is a finitely generated torsion-free abelian group. In order to avoid new designations, we assume that $G = Z$, that is T is locally soluble. By [3], T is hyperabelian. Since T has finite special rank, T has an ascending series of G–invariant subgroups

$$\langle 1 \rangle = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_\alpha \leqslant H_{\alpha+1} \leqslant \cdots H_\gamma = T$$

whose factors are G–chief finite. Since L/T is torsion-free nilpotent, it has a series of G–invariant subgroups whose factors are abelian of finite rank. We can state the same about the factor K/L. Then we can construct a finite series of G–invariant subgroups

$$T = A_0 \leqslant A_1 \leqslant \cdots \leqslant A_n = K$$

whose factors are torsion-free abelian and rationally G–irreducible. We put $H_{\gamma+1} = A_1, \ldots, H_{\gamma+n} = K$. Finally, since G/K is a finite soluble group, it has a finite series of G–invariant subgroups

$$K = B_0 \leqslant B_1 \leqslant \cdots \leqslant B_t = G$$

whose factors are finite abelian and G–chief. Now we put

$$H_{\gamma+n+1} = B_1, \ldots, H_{\gamma+n+t} = B_t = G.$$

Consider an arbitrary factor $H_{\alpha+1}/H_\alpha$. If $\alpha < \gamma$ or $\alpha \geqslant \gamma + n$, then this factor is finite and G–chief. Being abelian, it is an elementary abelian p–group for some prime p. Moreover, its order is at most p^r. Therefore we may think of $G/C_G(H_{\alpha+1}/H_\alpha)$ as a subgroup of $GL_r(\mathbb{F}_p)$. Being finite, this factor-group is soluble. As we remarked above, $G/C_G(H_{\alpha+1}/H_\alpha)$ have to include an abelian subgroup $U_{\alpha+1}/C_G(H_{\alpha+1}/H_\alpha)$ such that $G/U_{\alpha+1}$ is finite of order at most $\mu(r)$. Let now $\gamma \leqslant \alpha < \gamma + n$. In this case, $G/C_G(H_{\alpha+1}/H_\alpha)$ can be considered as an irreducible subgroup of $GL_r(\mathbb{Q})$. Since G/T is soluble, $G/C_G(H_{\alpha+1}/H_\alpha)$ is soluble too. Again, $G/C_G(H_{\alpha+1}/H_\alpha)$ have to include an abelian normal subgroup $U_{\alpha+1}/C_G(H_{\alpha+1}/H_\alpha)$ such that $G/U_{\alpha+1}$ is finite of order at most $\mu(r)$.

Let

$$V = \bigcap_{\alpha<\gamma+n+t} C_G(H_{\alpha+1}/H_\alpha).$$

Clearly, $V = C_G((H_{\alpha+1} \cap V)/(H_\alpha \cap V))$ for every $\alpha < \gamma + n + t$. It follows that V has an ascending central series and so V is hypercentral.

Let

$$D = \bigcap_{\alpha<\gamma+n+t} U_{\alpha+1}.$$

By the Remak's theorem, we obtain an embedding

$$D/V \hookrightarrow \mathrm{Cr}_{\alpha<\gamma+n+t} DC_G(H_{\alpha+1}/H_\alpha)/C_G(H_{\alpha+1}/H_\alpha)$$

$$\leqslant \mathrm{Cr}_{\alpha<\gamma+n+t} U_{\alpha+1}/C_G(H_{\alpha+1}/H_\alpha).$$

This shows that D/V is abelian. Finally, using again Remak's theorem, we obtain the embedding

$$G/D \hookrightarrow \mathrm{Cr}_{\alpha<\gamma+n+t} G/U_{\alpha+1}.$$

In other words, G/D is a subgroup of a Cartesian product of finite groups whose orders are at most $\mu(b)$. Being a bounded group of finite special rank, G/D is finite.

The fact that a p–group of finite special rank is Chernikov, where p is a prime, was proved by N.N. Myagkova [17]. The fact that a torsion-free locally nilpotent group of finite special rank is nilpotent follows from [14, Corollary to Theorem 5].

Choose in D/V a free abelian subgroup W/V such that D/W is periodic. Since G/D is finite, $X/V := (W/V)G/V$ is also free abelian and G/X is periodic. It follows that $r_0(G) = r_0(X)$. In turns out that $r_0(X) = r_0(X/\mathrm{Tor}(V))$. Finally, the fact that $r_0(X/\mathrm{Tor}(V))$ coincides with the special rank of $X/\mathrm{Tor}(V)$ was proved by D.I. Zaitsev [21]. □

Lemma 2.1 *Let G be a generalized radical group. Suppose that R is the maximal radical normal subgroup of G and L/R is the maximal locally finite normal subgroup of G/R. If G/R is infinite, then L/R is infinite.*

Proof — Suppose, for a contradiction, that G/R is infinite, but L/R is finite. Then $L/R \neq G/R$ and G/L have to include a non-identity torsion-free locally nilpotent normal subgroup K/L. Note that K/L is infinite. If $C/R := C_{G/R}(L/R)$, then G/C is finite and hence $K/L \cap C/L := U/R$ is infinite. However $C/R \cap L/R \leqslant \zeta(C/R)$ and so $C \cap L$ is radical and normal in G. It follows that $U \cap L = C \cap L = R$. Hence

$$U/R \simeq (U/R)/(U/R \cap L/R) \simeq (U/R)(L/R)/(L/R) \leqslant (K/R)/(L/R) \simeq K/L$$

and it follows that U/R is locally nilpotent. Hence U is a radical normal subgroup of G, which contradicts the choice of R. The result follows. □

Theorem 2.2 *Let G be a locally generalized radical group. If every locally radical subgroup of G has finite special rank, then G has finite special rank.*

Proof — Let F be a finitely generated subgroup of G, and let R be the maximal radical normal subgroup of F. Denote by L/R the locally finite radical of F/R. Let p be a prime and P/R be a Sylow p–subgroup of L/R. Being periodic locally generalized radical, P/R is locally finite. Being a p–subgroup, P/R is locally nilpotent. It follows that P is a radical subgroup of G/R. Then P has finite special rank, which implies that P/R is Chernikov (N.N. Myagkova [17]). Since this holds for every prime p, L/R have to include a locally soluble normal subgroup S/R of finite index (V.V. Belyaev [2]). If V is an arbitrary finitely generated subgroup of S, then V is an extension of its radical subgroup $V \cap R$ by the finite soluble group

$$V/(V \cap R) \simeq VR/R \leqslant S/R.$$

In particular, V is radical. In other words, S is a locally radical subgroup of G. Then S has finite special rank, and the Theorem SR yields that S is hyperabelian. It follows that $S \leqslant R$. It turns out that L/R is finite. By Lemma 2.1, F/R is finite. By the given conditions, R has finite special rank. Moreover Theorem SR shows that R has finite 0–rank so that $r_0(F)$ is finite.

Suppose that the 0–ranks of the finitely generated subgroups of G are unbounded. Then there are a family $\{F_n \mid n \geqslant 1\}$ of finitely generated subgroups such that

$$r_0(F_1) < r_0(F_2) < \cdots < r_0(F_n) < \cdots .$$

Let $K_1 = F_1, K_2 = \langle F_1, F_2 \rangle, K_n = \langle F_1, \cdots, F_n \rangle, n \geqslant 1$. As we saw above, being finitely generated K_n includes a radical normal subgroup R_n having finite index in K_n. Then R_n is a finitely generated subgroup of finite special rank. Since $r_0(R_n) = r_0(K_n) \geqslant r_0(F_n),\ n \geqslant 1$,

$$r_0(R_1) < r_0(R_2) < \cdots < r_0(R_n) < \cdots .$$

Since the subgroup R_n is normal in K_n for all $n \geqslant 1$, we have

$$\langle R_1, R_2, \cdots, R_n \rangle = R_1 R_2 \cdots R_n.$$

In particular, $R_1 R_2 \cdots R_n$ contains a finite series of normal subgroups whose factors are radical. It follows that the product $R_1 R_2 \cdots R_n$ is radical for all $n \geqslant 1$. Hence the subgroup

$$E = \bigcup_{n \geqslant 1} R_1 R_2 \cdots R_n$$

is locally radical. In this case, E has finite special rank. Theorem SR shows that E has finite 0–rank. Then there is a positive integer $m \geqslant 1$ such that $r_0(R_m) > r_0(E)$. On the other hand, R_m is a subgroup of E and therefore $r_0(R_m) \leqslant r_0(E)$. This contradiction shows that there is a positive integer k such that $r_0(F) \leqslant k$ for every finitely generated subgroup F of G.

By [8, Proposition 2], G has finite 0–rank. By [8, Theorem A], G has normal subgroups

$$T \leqslant L \leqslant K \leqslant G$$

such that T is locally finite, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian and G/K is finite. As above, we can prove that for every prime p the Sylow p–subgroups of T are Chernikov. Applying again the main result of [2], we obtain that T includes a locally soluble normal subgroup C of finite index. Without loss of generality, we can assume that C is the maximal locally soluble normal subgroup of T. Since the product of two periodic locally soluble normal subgroups is also locally soluble, the subgroup C is characteristic in T and hence is normal in G. The choice of C yields that T/C is a finite semisimple group. Put $D = C_G(T/C)$ so that G/D is finite. Since T/C includes no abelian G–invariant subgroups, $(T/C) \cap (D/C) = \langle 1 \rangle$. Therefore

$$(K \cap D)/C = (K \cap D)/(T \cap D) = (K \cap D)/(K \cap T \cap D) =$$

$$= (K \cap D)/(K \cap D \cap T) \simeq (K \cap D)T/T \leqslant K/T,$$

which shows that $(K \cap D)/C$ is soluble. Being locally soluble, C is locally radical and so C has finite special rank. By Theorem SR, C is hyperabelian, and therefore $K \cap D$ is radical. It follows that $K \cap D$ has finite special rank. The finiteness of G/K and G/D implies that $G/(K \cap D)$ is finite. It follows that G has finite rank. □

Now the quoted result of Dixon, Evans and Smith [7] easily follows from the above theorem.

Corollary 2.3 *Let* G *be a locally (soluble–by–finite) group. If every locally soluble subgroup of* G *has finite special rank, then* G *has finite special rank.*

Proof — Let H be an arbitrary locally radical subgroup of G and F be a finitely generated subgroup of H. Firstly, F is a radical subgroup. On the other hand, F is almost soluble. Being radical, F must be soluble. It follows that H is locally soluble. Hence every locally radical subgroup of G has finite special rank. By Theorem 2.2, G has finite special rank. □

Corollary 2.4 *Let* G *be a generalized radical group. If every abelian subgroup of* G *has finite special rank, then* G *has finite special rank.*

Proof — Let R be the maximal radical normal subgroup of G and L/R be the maximal locally finite normal subgroup of G/R. Choose an arbitrary locally radical subgroup K of L. Put $D = K \cap R$. Being locally radical and locally finite, K/D is locally soluble. If A/D is an arbitrary abelian subgroup of K/D, then clearly A is a radical subgroup. It follows that A has finite special rank [1]. Hence A/D has finite special rank. It follows that K/D has finite special rank [9]. The facts that D and K/D have finite special rank imply that K has finite special rank. By Theorem 2.2, L has finite special rank. Then L/R includes a locally soluble normal subgroup S/R of finite index (V.V. Belyaev [2]). We recall that a locally soluble group having finite special rank is radical [3]. The choice of R shows that L/R must be finite. Then Lemma 2.1 implies that G/R is finite. It turns out that G has finite special rank. □

Now we are on position to prove Theorem A.

Proof of Theorem A — By Theorem 2.2, it suffices to prove that every locally radical subgroup of G has finite special rank. Thus, without loss of generality, we may assume that G is a locally radical group. Let F be an arbitrary finitely generated subgroup of G. Then F is a radical subgroup. If A is an arbitrary abelian subgroup of F, then clearly A is locally minimax and hence A has finite special rank. It follows that F has finite special rank [1]. Then F is minimax by [11, Theorem 5.10]. Hence, G is locally minimax, and therefore, G has finite special rank. □

3 Proof of Theorem B

This result will be proved with the aid of a sequence of auxiliary statements.

Let k be a positive integer. A group G is said to be k*–radical* (k*–hyperabelian*, respectively) if G has an ascending normal series whose factors are locally nilpotent (abelian, respectively) groups of special rank at most k. As a locally nilpotent group of finite special rank is hypercentral, it follows that a k–radical group of finite special rank is hyperabelian. Our next result shows that such a group is also k–hyperabelian.

Proposition 3.1 *A group G is k–radical if and only if it is k–hyperabelian.*

Proof — Let G be a k–radical group. Then G has an ascending series of normal subgroups

$$\langle 1\rangle = L_0 \leqslant L_1 \leqslant \cdots \leqslant L_\alpha \leqslant L_{\alpha+1} \leqslant \cdots L_\gamma = G$$

whose factors are locally nilpotent groups of finite special rank at most k. If $V/U := L_{\alpha+1}/L_\alpha$ is an arbitrary factor of this series, we put $T/U := \mathrm{Tor}(V/U)$. Then

$$T/U = \mathrm{Dr}_{p \in \Pi(T/U)} T_p/U,$$

where T_p/U is the Sylow p–subgroup of T/U and $\Pi(H)$ stands to denote the set of primes occurring as the divisors of periodic elements of H. Since T_p/U has finite special rank, T_p/U is a Chernikov group (N.N. Myagkova [17]). Hence T/U has an ascending series of characteristic subgroups (and therefore normal in G) whose factors are abelian groups of special rank at most k.

The factor V/T is torsion-free locally nilpotent and, being a group of finite special rank at most k, it has to be nilpotent by [14, Corollary to Theorem 5]. Hence V/T has a finite series of characteristic subgroups whose factors are abelian groups of finite rank at most k. It follows that the above series has a refinement consisting of normal subgroups whose factors are abelian groups of finite special rank at most k, that is G is k–hyperabelian.

The converse is trivial. □

If G is a group, then by $\mathrm{dl}(G)$ we denote the length of the derived series of G. A well-known theorem of Zassenhaus [23] ensures that there exists an integer-valued function ζ such that $\mathrm{dl}(G) \leqslant \zeta(k)$ for a soluble subgroup G of $GL_k(F)$, where k is a positive integer and F is a field.

Proposition 3.2 *Let k be a positive integer and G be a k–radical group. Then G has normal subgroups $Z \leqslant K$ such that Z is hypercentral, K/Z is abelian and G/K is isomorphic to a subgroup of a Cartesian product of finite groups of order at most $\mu(k)$. Moreover, G/K is soluble and $\mathrm{dl}(G/K) \leqslant \zeta(k)$.*

Proof — Let G be a k–radical group. By Proposition 3.1, G has an ascending series of normal subgroups

$$\langle 1 \rangle = L_0 \leqslant L_1 \leqslant \cdots \leqslant L_\alpha L_{\alpha+1} \leqslant \cdots L_\gamma = G$$

whose factors are abelian groups of finite special rank at most k. If $V/U := L_{\alpha+1}/L_\alpha$ is an arbitrary factor of this series, we put $T/U := \mathrm{Tor}(V/U)$. Then

$$T/U = \mathrm{Dr}_{p \in \Pi(T/U)} T_p/U,$$

where T_p/U is the Sylow p–subgroup of T/U, $p \in \Pi(T/U)$. As above, since T_p/U has finite special rank, T_p/U is a Chernikov group (N.N. Myagkova [17]). Then T/U has an ascending series of G–invariant subgroups whose factors are G–chief and have special rank at most k. The factor V/T is torsion-free and has finite special rank at most k. Choose in V/T a G–invariant subgroup V_1/T whose special rank is minimal. If W/T is a non-identity G–invariant subgroup of V_1/T, then $r(W/T) = r(V_1/T)$. It follows that the factor-group V_1/W is periodic. In other words, V_1/T is a rationally G–irreducible factor. Let $X/V_1 := \mathrm{Tor}(V/V_1)$. Then X is a G–invariant subgroup of G, and clearly the factor X/T is also rationally G–irreducible. We note that the factor-group V/X is torsion-free and $r(V/X) < r(V/T)$. Repeating the argument, we obtain that V/T has a finite series of G–invariant subgroups whose factors are torsion-free, G–rationally irreducible and have special rank at most k. Since this holds for each factor of the above series, G has an ascending series of normal subgroups

$$\langle 1 \rangle = D_0 \leqslant D_1 \leqslant \cdots \leqslant D_\alpha \leqslant D_{\alpha+1} \leqslant \cdots D_\eta = G$$

whose factors either are finite abelian and G–chief of special rank at most k or torsion-free G–rationally irreducible of special rank at most k. We consider an arbitrary factor $D_{\alpha+1}/D_\alpha$ of this series.

If $D_{\alpha+1}/D_\alpha$ is finite, then this is an elementary abelian p–group of order at most p^k and it can be thought as a vector space of dimension at most k over the prime field $\mathbb{F}_p$. By the above remarked, $dl(G/C_G(D_{\alpha+1}/D_\alpha)) \leqslant \zeta(k)$. Moreover, G includes a normal subgroup K_α such that $K_\alpha \geqslant C_G(D_{\alpha+1}/D_\alpha)$ such that $K_\alpha/C_G(D_{\alpha+1}/D_\alpha))$ is abelian and G/K is finite of order at most $\mu(k)$.

Suppose that $D_{\alpha+1}/D_\alpha$ is torsion-free, G–rationally irreducible and have special rank at most k. In this case, $G/C_G(D_{\alpha+1}/D_\alpha)$ can be thought as an irreducible linear group acting on $Y = (D_{\alpha+1}/D_\alpha) \otimes_{\mathbb{Z}} \mathbb{Q}$. We have $\dim_{\mathbb{Q}}(Y) = r(D_{\alpha+1}/D_\alpha) \leqslant k$. As above, $dl(G/C_G(D_{\alpha+1}/D_\alpha)) \leqslant \zeta(k)$. Furthermore, G includes a normal subgroup K_α such that $K_\alpha \geqslant C_G(D_{\alpha+1}/D_\alpha)$ such that $K_\alpha/C_G(D_{\alpha+1}/D_\alpha))$ is abelian and G/K_α is finite of order at most $\mu(k)$.

Put

$$Z = \bigcap_{\alpha<\eta} C_G(D_{\alpha+1}/D_\alpha) \text{ and } K = \bigcap_{\alpha<\eta} K_\alpha.$$

Applying Remak's theorem, we obtain an embedding

$$G/K \hookrightarrow Cr_{\alpha<\eta} G/K_\alpha.$$

Recall that every factor G/K_α is finite of order at most $\mu(k)$. Another application of Remak's theorem gives now an embedding

$$G/Z \hookrightarrow Cr_{\alpha<\eta} K/C_G(D_{\alpha+1}/D_\alpha).$$

Since the factor-groups $K_\alpha/C_G(D_{\alpha+1}/D_\alpha))$ are all abelian, K/Z is also abelian.

Finally, the normal subgroup Z has an ascending series of normal subgroups

$$\langle 1 \rangle = D_0 \leqslant Z \cap D_1 \leqslant \cdots \leqslant Z \cap D_\alpha \leqslant Z \cap D_{\alpha+1} \leqslant \cdots Z \cap D_\eta = Z.$$

We have

$$\begin{aligned}(Z \cap D_{\alpha+1})/(Z \cap D_\alpha) &= (Z \cap D_{\alpha+1})/(Z \cap D_{\alpha+1} \cap D_\alpha) \\ &\simeq_G (Z \cap D_{\alpha+1})D_\alpha/D_\alpha,\end{aligned}$$

which shows that

$$C_G(D_{\alpha+1}/D_\alpha) \leqslant C_G((Z \cap D_{\alpha+1})/(Z \cap D_\alpha)).$$

In particular, $Z \leqslant C_G((Z \cap D_{\alpha+1})/(Z \cap D_\alpha))$ for each $\alpha < \eta$. This implies that Z is hypercentral. □

If G is a group, we will stand $\delta_n(G)$ to denote the nth term of the derived series of G.

Corollary 3.3 *Let k be a positive integer and G be a locally k–radical group. Then there exists a positive integer t such that $\delta_t(G)$ is locally nilpotent.*

PROOF — Let $t = \zeta(k)$. If M is a finite subset of $D := \delta_t(G)$, then there exists a finitely generated subgroup F such that $M \subseteq \delta_t(F)$. By Proposition 3.2, $\delta_t(F)$ is hypercentral. We recall that a hypercentral group is locally nilpotent [13]. Therefore $\langle M \rangle$ is nilpotent. It follows that D is locally nilpotent. □

Corollary 3.4 *Let k be a positive integer and G be a locally k–radical group. If every abelian subgroup of G has finite special rank, then G has finite special rank.*

Proof — By Corollary 3.3, G is radical group, so that the result follows from [1, Theorem 6.3]. □

Proof of Theorem B — Let L be an arbitrary locally radical subgroup of G. Then L is locally k–radical. By Corollary 3.4, L has finite special rank. Application of Theorem 2.2 shows that G also has finite special rank. □

Proof of Corollary B1 — Let

$$\langle 1 \rangle = K_0 \leqslant K_1 \leqslant \cdots \leqslant K_\alpha \leqslant K_{\alpha+1} \leqslant \cdots K_\gamma = G$$

be an ascending series whose factors are locally k–generalized radical groups. Since K_1 is locally k–generalized radical group, Theorem B implies that K_1 has finite special rank. By Theorem SR, K_1 is almost hyperabelian. Then K_1 includes a normal subgroup which is either finite or abelian. Consider the factor K_2/K_1 and let A/K_1 be an arbitrary abelian subgroup of it. Then the subgroup A is generalized radical. By Corollary 2.4, A has finite special rank. Thus every abelian subgroup of K_2/K_1 has finite special rank, and hence K_2/K_1 has finite special rank by Theorem B. By Theorem SR, K_2/K_1 is almost hyperabelian.

Proceeding in the same way and applying transfinite induction, we obtain that G is a generalized radical group. Then it suffices to apply Corollary 2.4. □

4 Proofs of Theorems C and D

We need the following auxiliary result.

Proposition 4.1 *Let* G *be a nearly radical group. If every abelian subgroup of* G *has finite section rank, then* G *has finite section rank.*

Proof — By [1, Theorem 8.1], G includes a radical normal subgroup R of finite index and every elementary abelian section of G is finite. Applying the description of radical groups of finite abelian subgroup rank given in [1, Theorem 6.1], we deduce that G has finite section rank, as required. □

We also will need the following notion.

Let G be a locally finite group and p be a prime. A Sylow p–subgroup P of G is said to be a *Wehrfritz* p*–subgroup* if P includes an isomorphic copy of every p–subgroup of G [6, Definition 2.5.2]. We remark that *a locally finite group whose Sylow* p*–subgroups are Chernikov always contains Wehrfritz* p*–subgroups* [19]

Proof of Theorem C — Let F be a finitely generated subgroup of G. Then F is a nearly radical subgroup. Let A be an arbitrary abelian subgroup of F. Being locally minimax, A has finite section rank. By Proposition 4.1, F has finite section rank. Moreover, F includes a radical normal subgroup R of finite index, so that R is finitely generated too. It follows that R is minimax by [11, Theorem 5.10]. Hence, every finitely generated subgroup of G is soluble–by–finite minimax. In particular, it has finite 0–rank.

Suppose that the 0–ranks of finitely generated subgroups of G are unbounded. Then there is a family $\{F_n \mid n \geqslant 1\}$ of finitely generated subgroups such that

$$r_0(F_1) < r_0(F_2) < \cdots < r_0(F_n) < \cdots .$$

Let $K_1 = F_1$, $K_2 = \langle F_1, F_2 \rangle$, $K_n = \langle F_1, \cdots, F_n \rangle$ $(n \geqslant 1)$. As we saw above, being finitely generated, each K_n includes a soluble minimax normal subgroup R_n having finite index in K_n. Reasoning as we did in the proof of Theorem 2.2, we obtain a contradiction that shows that there is a positive integer k such that $r_0(F) \leqslant k$ for every finitely generated subgroup F of G.

By [8, Proposition 2], G has finite 0–rank. By [8, Theorem A], G has normal subgroups

$$T \leqslant L \leqslant K \leqslant G$$

such that T is locally finite, L/T is torsion-free nilpotent, K/L is torsion-free finitely generated abelian, $r_0(K/T)$ is finite, and G/K is finite. For every prime p, a p–subgroup P of G is locally (soluble minimax). Therefore, it has finite section rank. It follows that every abelian subgroup of P is Chernikov. Then P itself is Chernikov [5]. Thus, for every prime p, the Sylow p–subgroups of T are Chernikov. As we noted above, G has a Sylow p–subgroup S such that every p–subgroup of G is isomorphic to some subgroup of S. It follows that every elementary abelian p–section of T is isomorphic to some elementary abelian section of S. Hence $sr_p(T) \leqslant sr_p(S)$. In particular, $sr_p(T)$ is finite. Since this holds for every prime p, we conclude that T has finite section rank. Since G/T has finite special rank, G has finite section rank.

The fact that G is almost radical could be established in the same way as in the proof of Theorem 2.2. □

Now we need to establish a connection between the 0–rank of a soluble–by–finite minimax group and the 0–rank of its abelian subgroups. Let G be a group and A be an infinite abelian normal subgroup of G. Then we say that A is G–*quasifinite* if every proper G–invariant subgroup of A is finite.

Lemma 4.2 *Let* G *be a group and* D *be a divisible Chernikov normal subgroup. Suppose that there exists a positive integer* k *such that* $r_0(A) \leqslant k$ *for every abelian subgroup* A *of* G*. Then* $r_0(C/D) \leqslant k$ *for every torsion-free abelian subgroup* C/D *of* G/D*.*

Proof — Suppose for a contradiction that G/D contains a free abelian subgroup B/D such that $r_0(B/D) > k$. Being Chernikov, D has a finite series of B–invariant subgroups

$$\langle 1 \rangle = D_0 \leqslant D_1 \leqslant \cdots \leqslant D_t = D,$$

whose factors are B–quasifinite. We proceed by induction on t. Suppose first that $t = 1$ so that D is B–quasifinite. Let

$$B/D = \langle b_1 D \rangle \times \cdots \times \langle b_m D \rangle.$$

The subgroups $[b_j, D]$ are B–invariant, so that either $[b_j, D]$ is finite or $[b_j, D] = D$.

If all the subgroups $[b_j, D]$ are finite, since $[b_j, D] \simeq D/C_D(b_j)$, every factor-group $C_D(b_j)$ is infinite, and then $C_D(b_j) = D$ since the latter is B–invariant. Therefore $D \leqslant \zeta(B)$ and B is nilpotent. Put

$$K = \langle b_1, \cdots, b_m \rangle.$$

Since K is a finitely generated nilpotent group, $\mathrm{Tor}(K)$ is finite. There exists a positive integer s such that K^s is torsion-free. The fact that K is finitely generated implies that K/K^s is finite, so that $r_0(K) = r_0(K^s)$. Since

$$K^s \cap \mathrm{Tor}(K) = \langle 1 \rangle,$$

K is torsion-free abelian. However in this case,

$$r_0(B/D) = r_0(K) = r_0(K^s) \leqslant k,$$

and we obtain a contradiction.

Suppose now that there exists a number n such that $D = [b_n, D]$. Pick $x \in B$. Since B/D is abelian, $x^{-1}b_n x = b_n d$ for some $d \in D$. The equation $D = [b_n, D]$ implies that there exists some $c \in D$ such that $d = [b_n, c]$. Then

$$x^{-1}b_n x = b_n d = b_n[b_n, c] = b_n{}^c,$$

and so $xc^{-1} \in C_B(b_n)$. It follows that $B = DC_B(b_n)$. The choice of b_n yields that $D \cap C_B(b_n) = C_D(b_n)$ is finite and then $L := C_G(b_n)$ is a finite–by–abelian group. In particular, it is finitely generated. It is not hard to prove that there exists a positive integer r such that L^r is torsion-free. Since L is finitely generated, L/L^r is finite, and so $r_0(L) = r_0(L^r)$. Since

$$L^r \cap D = \langle 1 \rangle,$$

L is torsion-free abelian. But in this case

$$r_0(B/D) = r_0(L) = r_0(L^r) \leqslant k,$$

and we obtain again a contradiction.

Suppose now that $t > 1$. Proceeding in the same way, we obtain that G contains a subgroup V such that $D_{t-1} \leqslant V \leqslant B$, $|B : VD|$ is finite and $V \cap D = D_{t-1}$. Therefore $k = r_0(B/D) = r_0(V/D_{t-1})$. Applying the inductive hypothesis, we obtain that $r_0(V/D_{t-1}) \leqslant k$. This contradiction proves the result. □

Proposition 4.3 *Let G be a soluble–by–finite minimax group. Suppose that there exists a positive integer k such that $r_0(A) \leqslant k$ for every abelian subgroup A of G. Then $r_0(G) \leqslant k(k+1)$.*

Proof — The group G has a series of normal subgroups

$$D \leqslant L \leqslant K \leqslant G$$

such that D is divisible, L/D is torsion-free nilpotent (provided it is non-trivial), K/L is finitely generated free abelian (provided it is non-trivial), and G/K is finite (see [20] for example). If A/D is an arbitrary abelian subgroup of G/D, by Lemma 4.2, $r_0(A/D) \leqslant k$. Therefore, without loss of generality we can assume that $D = \langle 1 \rangle$. Suppose that G includes a torsion-free nilpotent normal subgroup V of finite index. In particular, $r_0(V) = r_0(G)$. For the subgroup V we have $r_0(V) \leqslant \frac{k(k+1)}{2}$ (N.F. Sesekin [18]). If G is not nilpotent–by–finite, then G

includes a torsion-free nilpotent subgroup U such that LU has finite index in G [22]. Then $r_0(LU) = r_0(G)$. For the subgroups L and U we have $r_0(L),\ r_0(U) \leqslant \frac{k(k+1)}{2}$ [18], and hence $r_0(LU) \leqslant k(k+1)$. Then the result follows. □

Proof of Theorem D — Let F be an arbitrary finitely generated subgroup of G. By Proposition 4.1, F has finite section rank. Moreover, F includes a radical normal subgroup R of finite index, so that R is finitely generated too. It follows that R is minimax by [11, Theorem 5.10]. Hence every finitely generated subgroup of G is soluble–by–finite minimax. By Proposition 4.3, every finitely generated subgroup of G has 0–rank at most $k(k+1)$. Applying [8, Proposition 2], we obtain that $r_0(G) \leqslant k(k+1)$.

By [8, Theorem A], G has normal subgroups

$$T \leqslant L \leqslant K \leqslant G$$

such that T is locally finite, L/T is torsion-free nilpotent, K/L is finitely generated torsion-free abelian, $r_0(K/T)$ is finite, and G/K is finite. For every prime p, an abelian p–subgroup of G is Chernikov. Then every p–subgroup (and hence every Sylow p–subgroup) of G is Chernikov [5]. Proceeding as we did in the proof of Theorem C, we can prove that T has finite section rank. Since G/T has finite special rank, G has finite section rank too, as required. □

Proof of Corollary D2 — As above, every finitely generated subgroup of G is soluble–by–finite minimax and, in particular, it is nearly radical. Applying Theorem D, we obtain that G has finite section rank and $r_0(G) \leqslant k(k+1)$.

Also by [8, Theorem A], G has a locally finite normal subgroup T such that G/T has finite special rank. Since T has finite section rank, T has Chernikov Sylow p–subgroups for every prime p. Then T includes a locally soluble normal subgroup S of finite index [2]. Since every abelian subgroup of S has finite special rank, S has finite special rank [9]. Since T/S is finite, T has finite special rank. It follows that G has finite special rank, as required. □

As a consequence, we obtain the result of Dixon, Evans and Smith [7] mentioned in the Introduction.

Corollary 4.4 *Let G be a locally (soluble–by–finite) group. Suppose that there exists a positive integer k such that $r_0(A) \leqslant k$ for every abelian* *subgroup* A *of* G. *If every abelian subgroup of* G *has finite special rank, then* G *has finite special rank.*

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